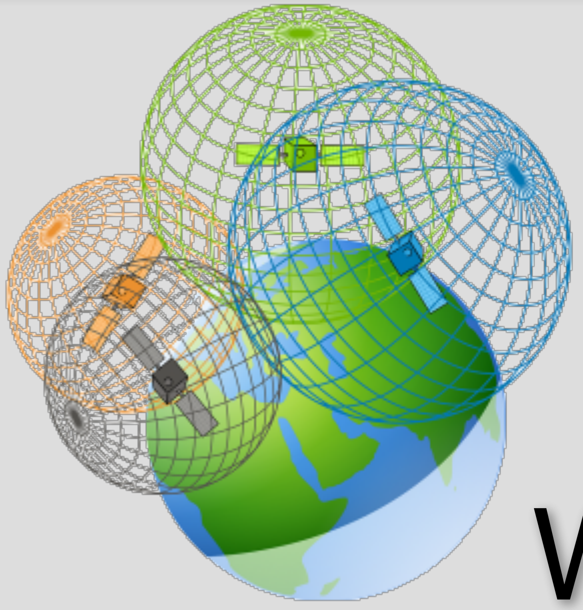
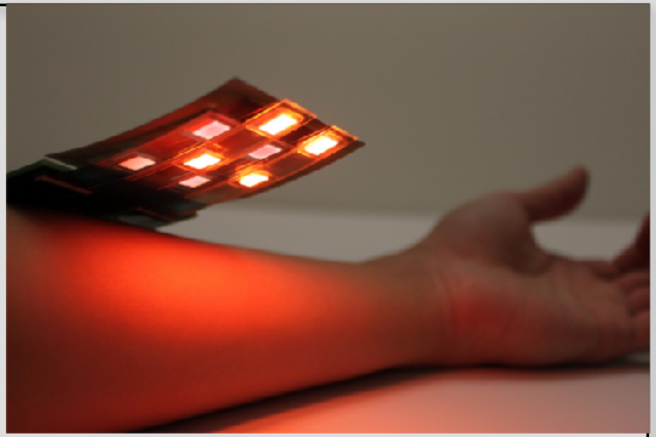




Welcome to EECS 16A!

Designing Information Devices and Systems I

Ana Arias and Miki Lustig

Fall 2021

Lecture 11A

Module 3: GPS, APS, inner products and norms

Announcements

Learning Goals

Not a survey class — rigorous and deep

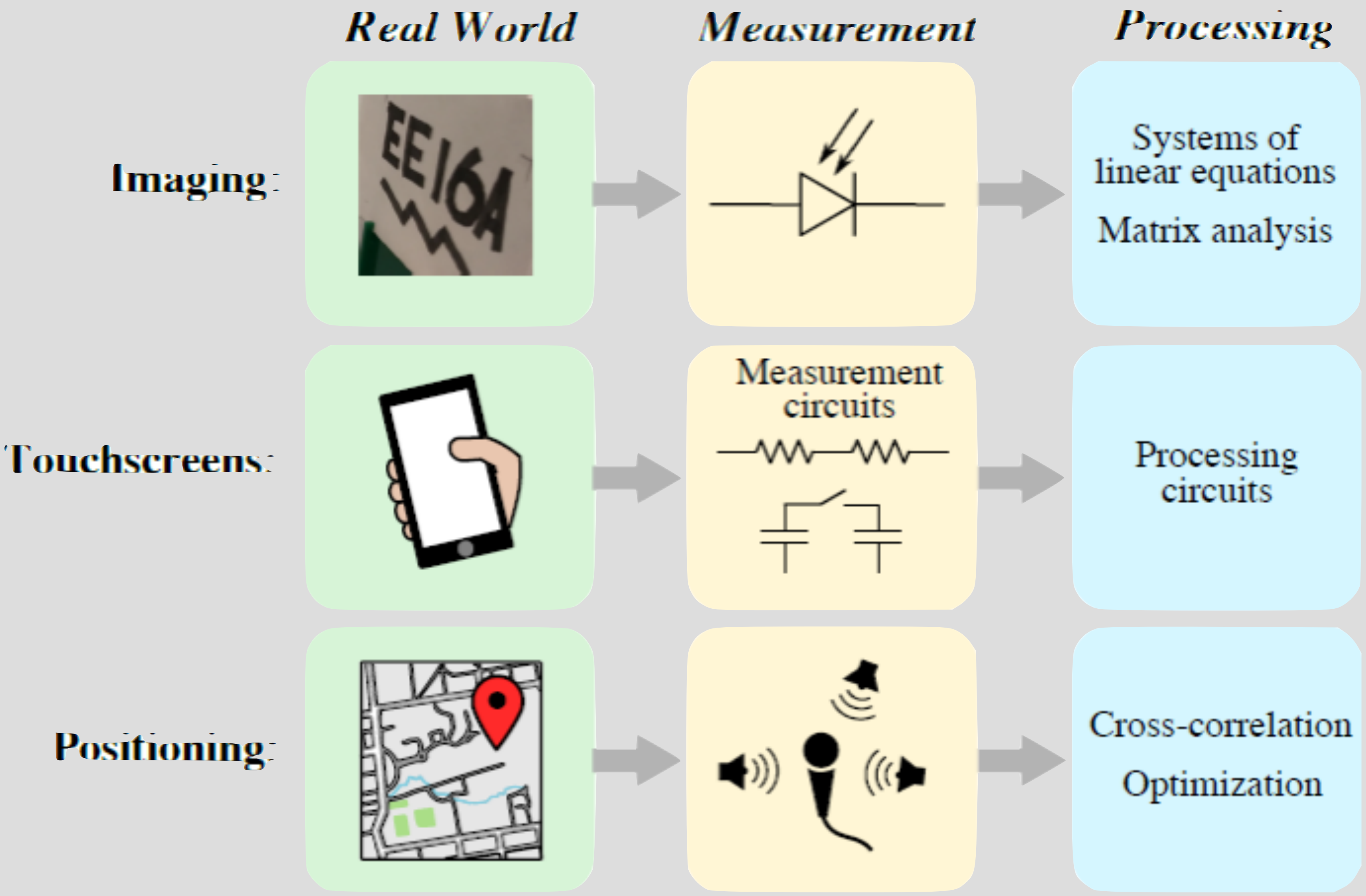
EECS 16A

- Module 1: Introduction to systems
  - How do we collect data? build a model?
- Module 2: Introduction to circuits and design
  - How do we use a model to solve a problem
- Module 3: Introduction Signal Processing and Machine Learning
  - How do we “learn” models from data, and make predictions?

EECS 16B

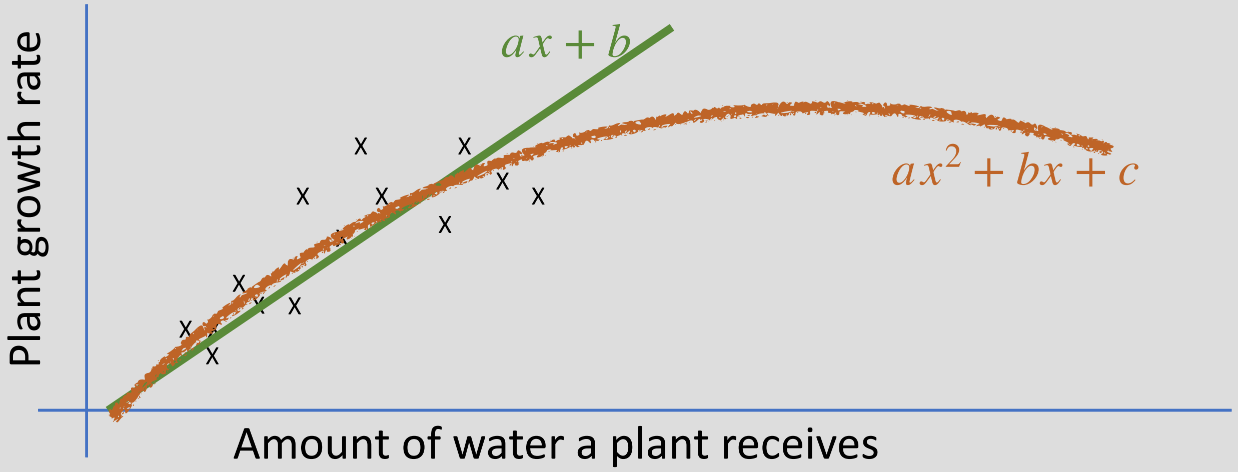
- Module 4: Advanced circuit design / analysis
- Module 5: Introduction to control and robotics
- Module 6: Introduction to data analysis and signal processing

16A Lab Examples

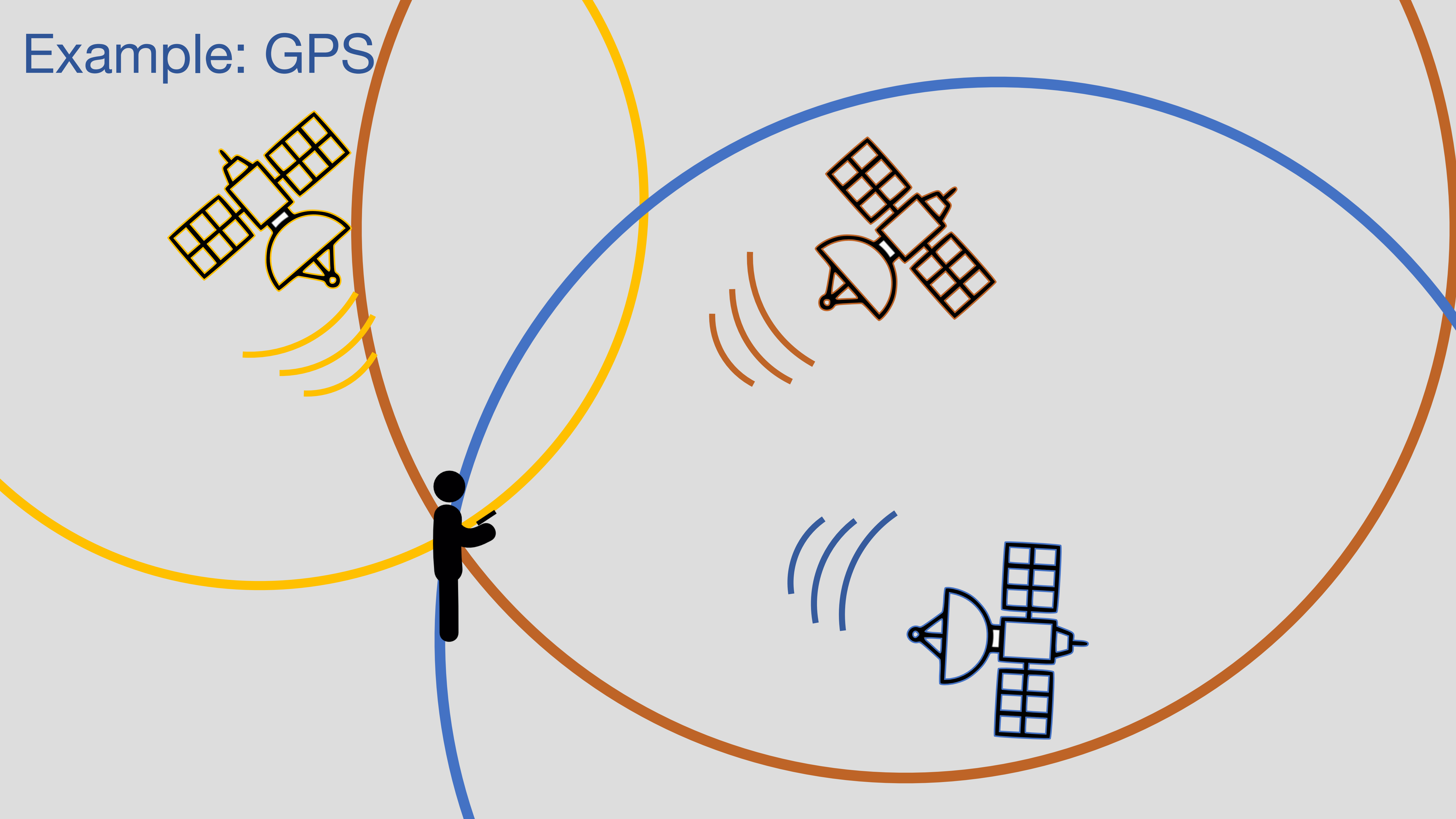


This module

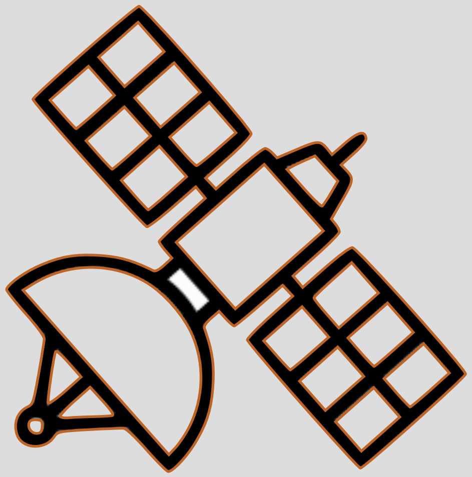
- Classification
  - Example: How can you tell if a picture is Miki or Ana
- Estimation
  - For example, how to estimate model parameters from data
- Prediction
  - How to predict stocks value tomorrow based on [past performance



Example: GPS



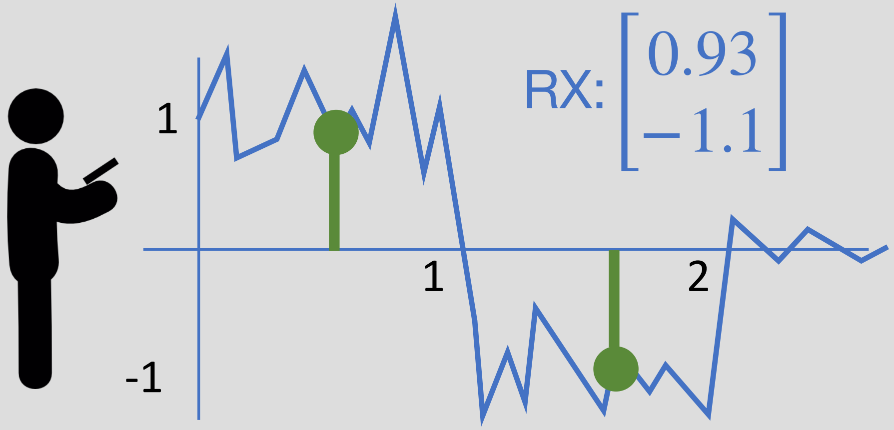
GPS

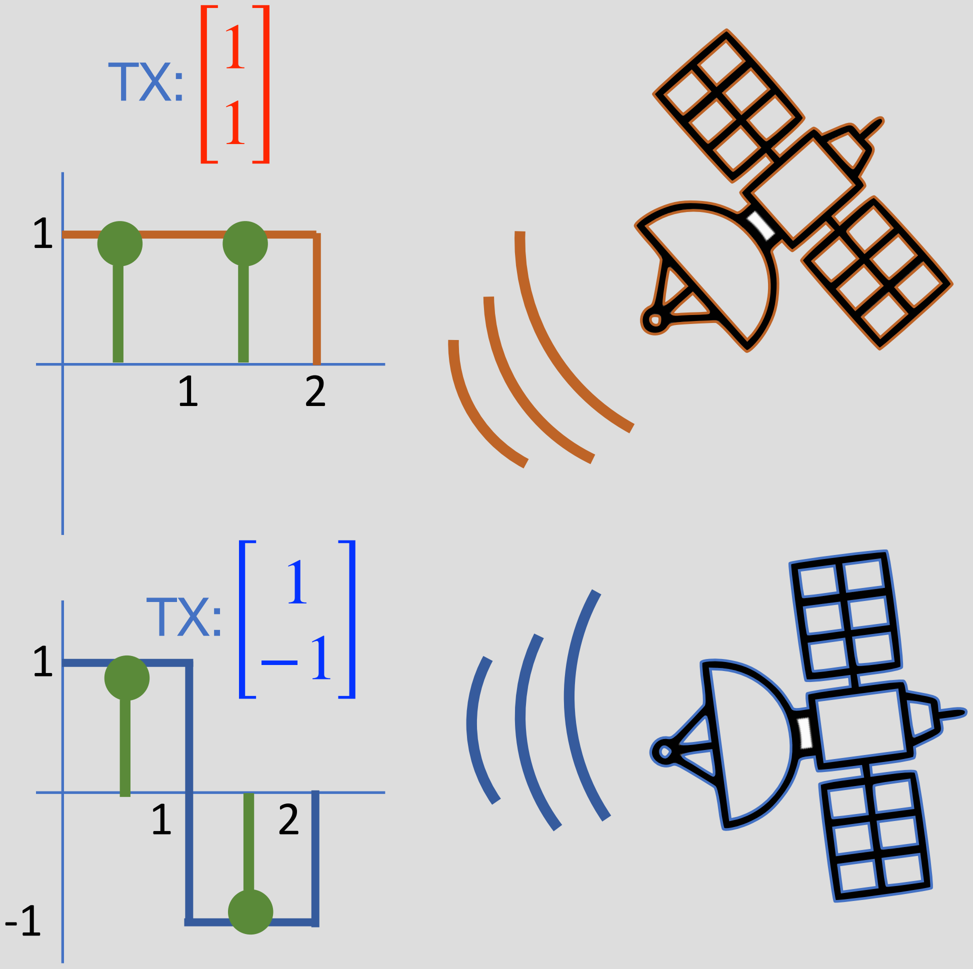


- 24 satellites
  - Known position
  - Time synchronized
  - 8 usually visible
- Problem:
  - Classify which satellite is transmitting
  - Estimate distance to GPS
  - Estimate position from noisy data
- Tools:
  - Inner product
  - Cross correlation

Problem 1: Classification

- Satellites transmit a unique code
  - Radio signal
- Signal is received and digitized by a receiver





Q: Which satellite was received?

Inner Product

- Provide a measure of "similarity" between vectors

- Definition: For a **real-valued** vector space, $\mathbb{V}$, the mapping

$$\vec{u}, \vec{v} \in \mathbb{V} \quad \rightarrow \quad < \vec{u}, \vec{v} > \in \mathbb{R}$$

is called an inner product if it satisfies:

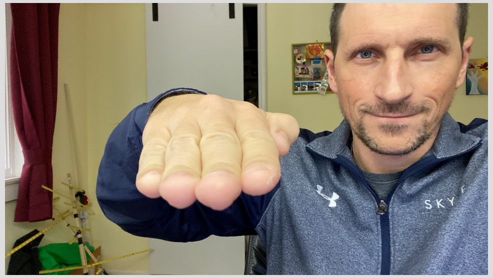
1. Symmetry: $< \vec{u}, \vec{v} > = < \vec{v}, \vec{u} >$ (not true for $\mathbb{V} \in \mathbb{C}^N$)
2. Linearity: $< \alpha \vec{u}, \vec{v} > = \alpha < \vec{u}, \vec{v} > \qquad \alpha \in \mathbb{R}$
   $$< \vec{u} + \vec{w}, \vec{v} > = < \vec{u}, \vec{v} > + < \vec{w}, \vec{v} >$$
3. Positive-definitness:
   $$< \vec{v}, \vec{v} > \geq 0,$$
   iff $< \vec{v}, \vec{v} > = 0 \quad \Leftrightarrow \quad \vec{v} = 0$

Inner Products

Example 1: Euclidean inner product (or dot product)

$$\vec{x}, \vec{y} \in \mathbb{R}^N, \qquad < \vec{x}, \vec{y} > = \vec{x}^T \vec{y}$$

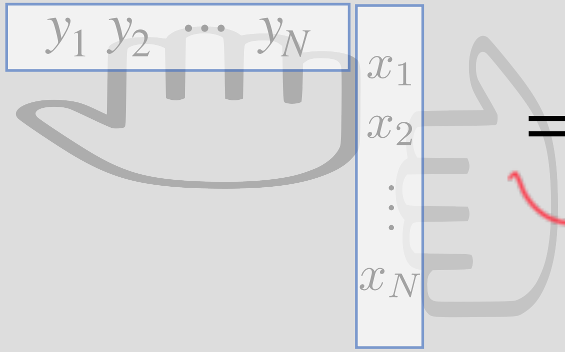
Like this....



and like that!

$1 \times N$

$\vec{y}^T \vec{x} =$



$N \times 1$

$$= y_1 x_1 + y_2 x_2 + y_3 x_3 + \cdots + y_N x_N = \square$$

scalar 1×1

1×1

Example 1: Euclidean inner product

$$\overrightarrow{x}, \overrightarrow{y} \in \mathbb{R}^N, \qquad < \overrightarrow{x}, \overrightarrow{y} > = \overrightarrow{x}^T \overrightarrow{y}$$

Test:

Symmetry:

$$< \overrightarrow{x}, \overrightarrow{y} > = \overrightarrow{x}^T \overrightarrow{y}$$
$$< \overrightarrow{y}, \overrightarrow{x} > = \overrightarrow{y}^T \overrightarrow{x}$$

Linearity

$$< a\overrightarrow{x}, \overrightarrow{y} > = (a\overrightarrow{x})^T \overrightarrow{y} = a\overrightarrow{x}^T \overrightarrow{y}$$
$$< \overrightarrow{x} + \vec{z}, \overrightarrow{y} > = (\overrightarrow{x} + \vec{z})^T \overrightarrow{y} = \overrightarrow{x}^T \overrightarrow{y} + \vec{z}^T \overrightarrow{y}$$

Positive Definitness

$$< \overrightarrow{x}, \overrightarrow{x} > = \overrightarrow{x}^T \overrightarrow{x} = x_1^2 + x_2^2 + \cdots + x_N^2 \geq 0$$

Example 2: Weighted Inner Product

$\vec{x}, \vec{y} \in \mathbb{R}^N, \;\; Q \in \mathbb{R}^{N\times N}$ symmetric with positive eigenvalues

Define:

$$< \vec{x}, \vec{y} > = \vec{x}^T Q \vec{y}$$

Specific example:

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \qquad \vec{x}, \vec{y} \in \mathbb{R}^2$$

Symmetry:

$$\vec{x}^T Q \vec{y} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 & 3x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1 y_1 + 3 x_2 y_2$$

$$\vec{y}^T Q \vec{x} = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 & 3y_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 y_1 + 3 x_2 y_2$$

😍

Example 2: Weighted Inner Product

Specific example: $Q = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ $\vec{x}, \vec{y} \in \mathbb{R}^2$

Symmetry:

$$\vec{x}^T Q \vec{y} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 & 3x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1 y_1 + 3 x_2 y_2$$

$$\vec{y}^T Q \vec{x} = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 & 3y_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 y_1 + 3 x_2 y_2$$

😍

Linearity: obvious! 😍

Positive Definitness:

$$\vec{x}^T Q \vec{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + 3 x_2^2 \geq 0$$

😍

Norms

- For each inner product there's an associated norm
  - A measure of **<u>a length</u>** of elements in the vector space

$$||\vec{v}|| = \sqrt{<\vec{v}, \vec{v}>}$$

- Properties of norms:
  1. Homogeneity $\|\alpha \vec{v}\| = |\alpha| \|\vec{v}\|$ $\alpha \in \mathbb{R}$
  2. Non-negativity $\|\vec{v}\| \geq 0$
  3. Triangle Inequality $\|\vec{v} + \vec{u}\| \leq \|v\| + \|u\|$

Euclidian Norm

- Euclidean inner-product induces the euclidean norm

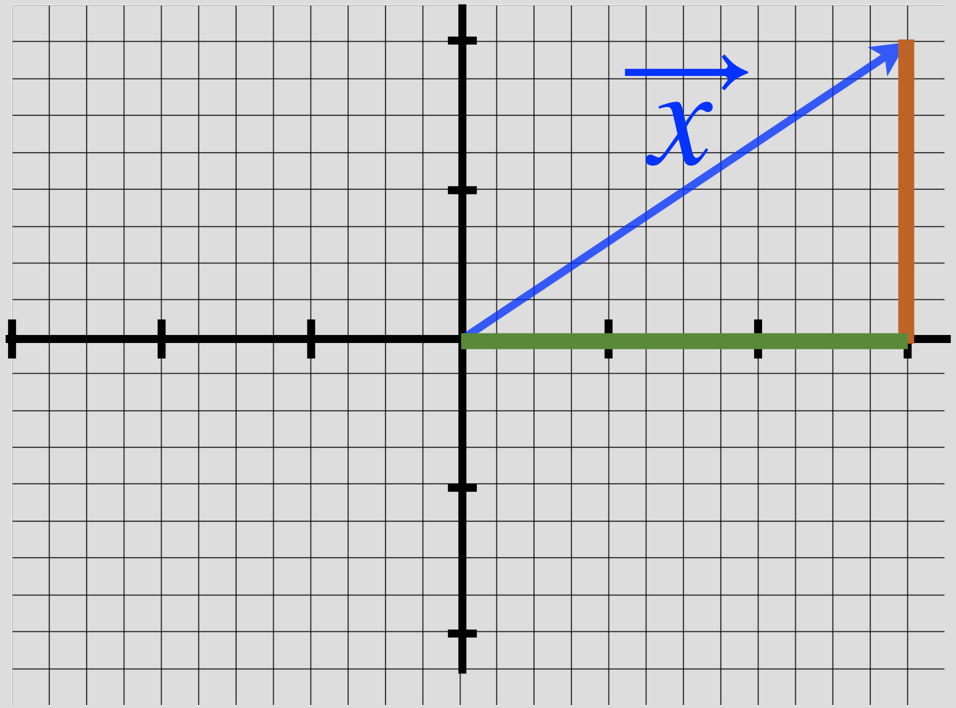
$$\vec{x} \in \mathbb{R}^N, \quad < \vec{x}, \vec{x} > = \vec{x}^T \vec{x}$$

$$\| \vec{x} \| = \sqrt{\vec{x}^T \vec{x}}$$

Specific example:

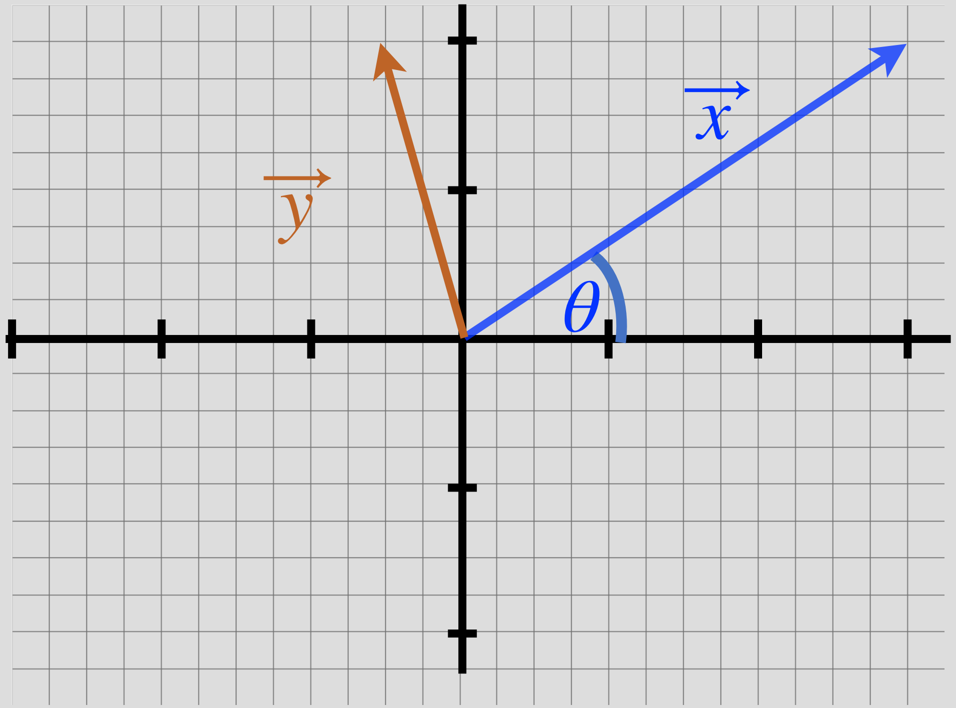
$$\vec{x} \in \mathbb{R}^2$$

$$\| \vec{x} \| = \sqrt{\vec{x}^T \vec{x}} = \sqrt{x_1^2 + x_2^2}$$



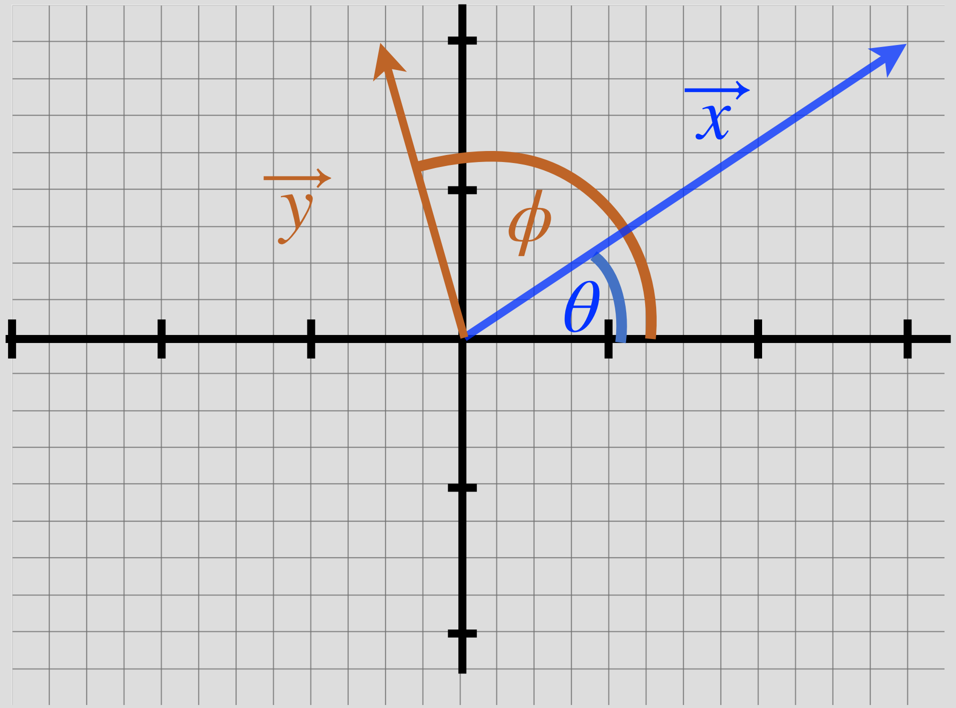
Geometrical Interpretation of Inner Product

$$\overrightarrow{x} = \|\overrightarrow{x}\| \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$



Geometrical Interpretation of Inner Product

$$\vec{x} = \|\vec{x}\| \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \qquad \vec{y} = \|\vec{y}\| \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix}$$

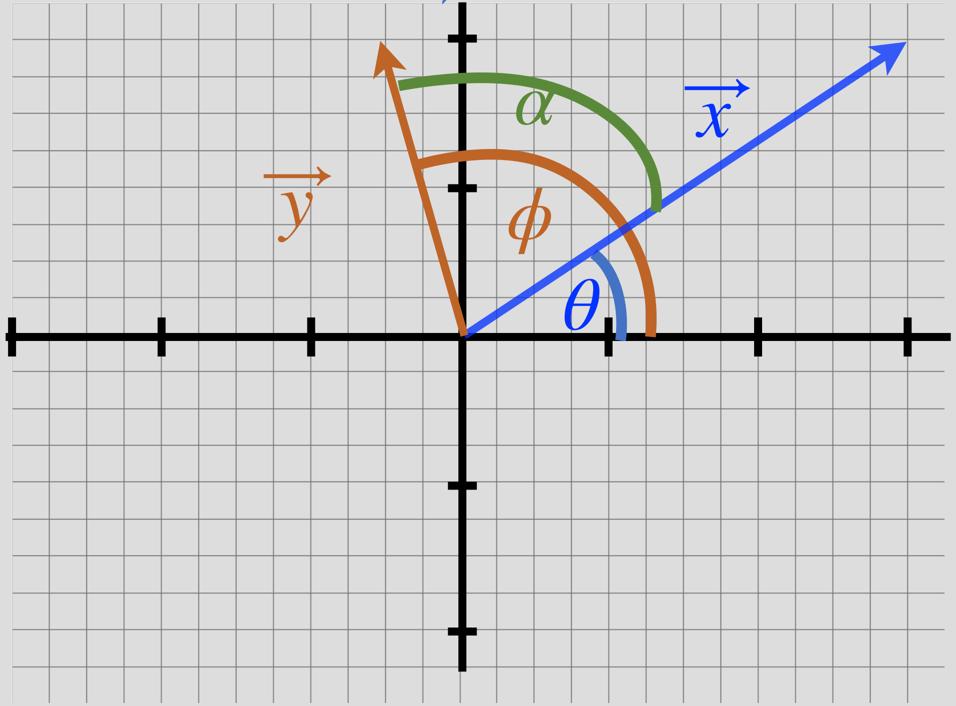


Geometrical Interpretation of Inner Product

$$\vec{x} = \|\vec{x}\| \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \qquad \vec{y} = \|\vec{y}\| \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix}$$

- For Euclidian inner product:

$$\vec{x}^T\vec{y} = \|\vec{x}\|\|\vec{y}\| \left(\cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi)\right)$$
$$= \|\vec{x}\|\|\vec{y}\| \cos(\phi - \theta)$$
$$= \|\vec{x}\|\|\vec{y}\| \cos(\alpha)$$



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- For Euclidian inner product:

$$\vec{x}^T\vec{y} = \|\vec{x}\|\|\vec{y}\| \left(\cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi)\right)$$

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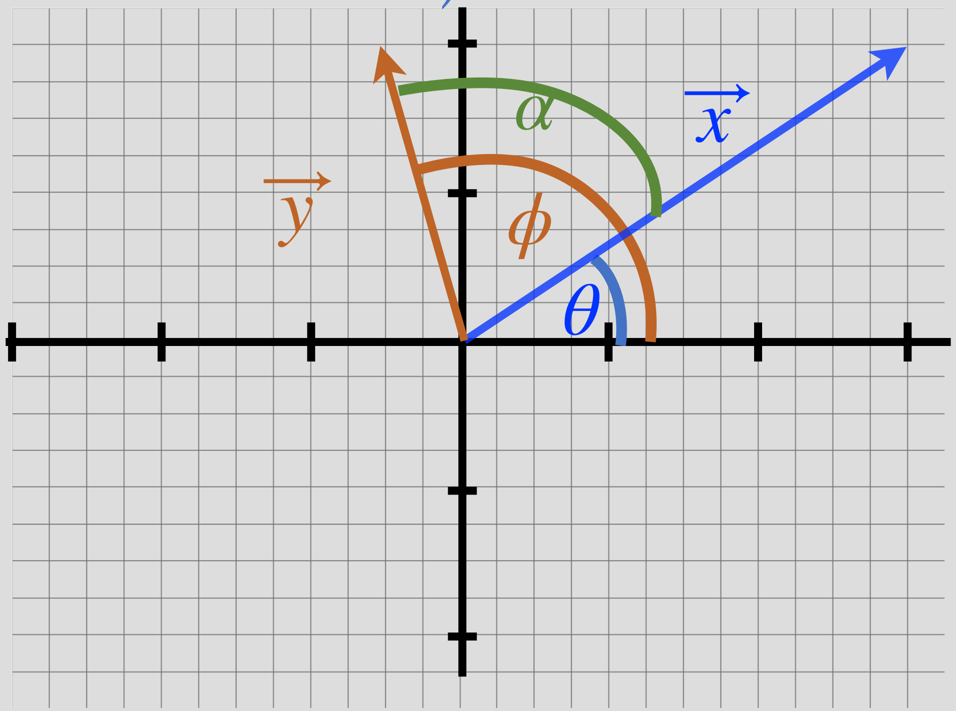
$$= \|\vec{x}\|\|\vec{y}\| \cos(\alpha)$$



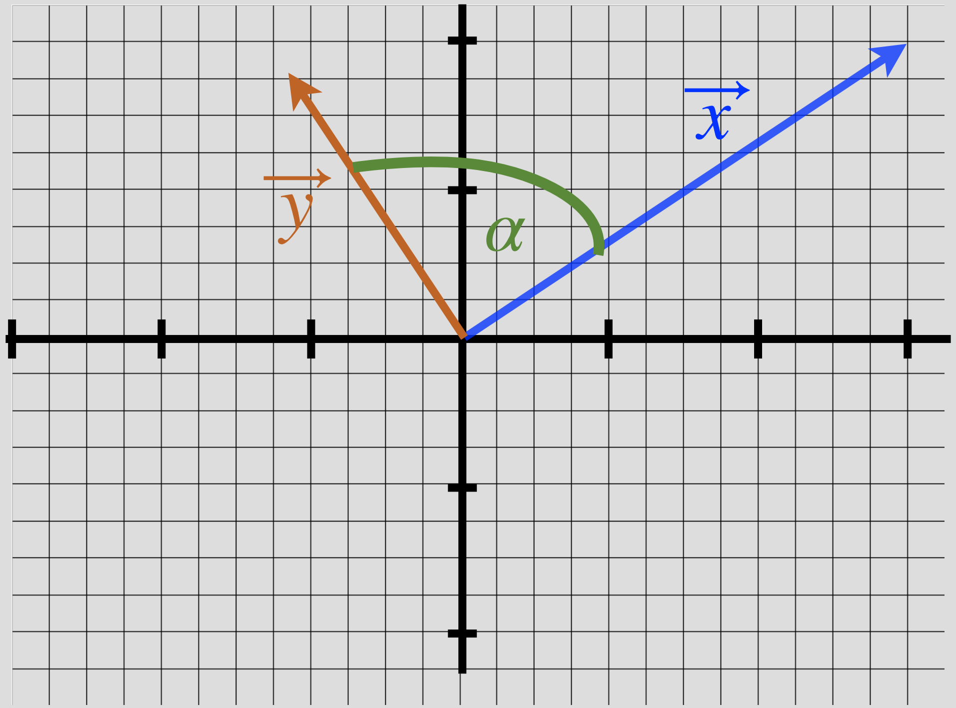
Orthogonality

- For an inner product $< \cdot , \cdot >$, two vectors $\overrightarrow{x}, \overrightarrow{y}$ are said to be orthogonal if $< \overrightarrow{x}, \overrightarrow{y} > = 0$

$$< \overrightarrow{x}, \overrightarrow{y} > = \|\overrightarrow{x}\| \|\overrightarrow{y}\| \cos(\alpha)$$

$$\Rightarrow \cos(\alpha) = 0$$

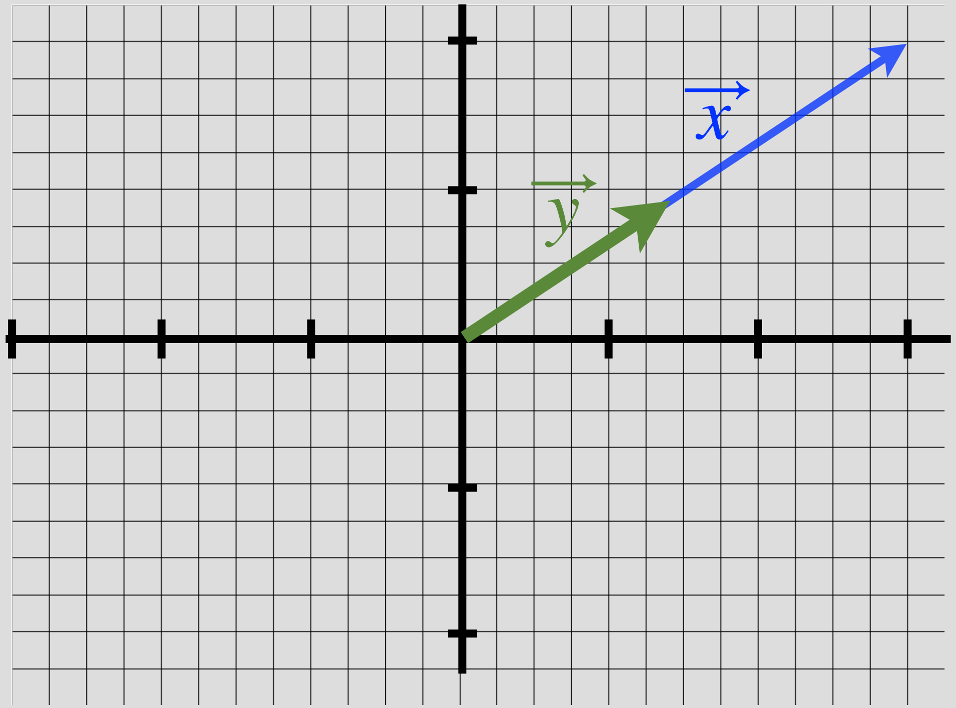
$$\Rightarrow \alpha = \frac{\pi}{2}$$



Cauchy-Schwarz Inequality

- Consider: $| < \overrightarrow{x}, \overrightarrow{y} > |$

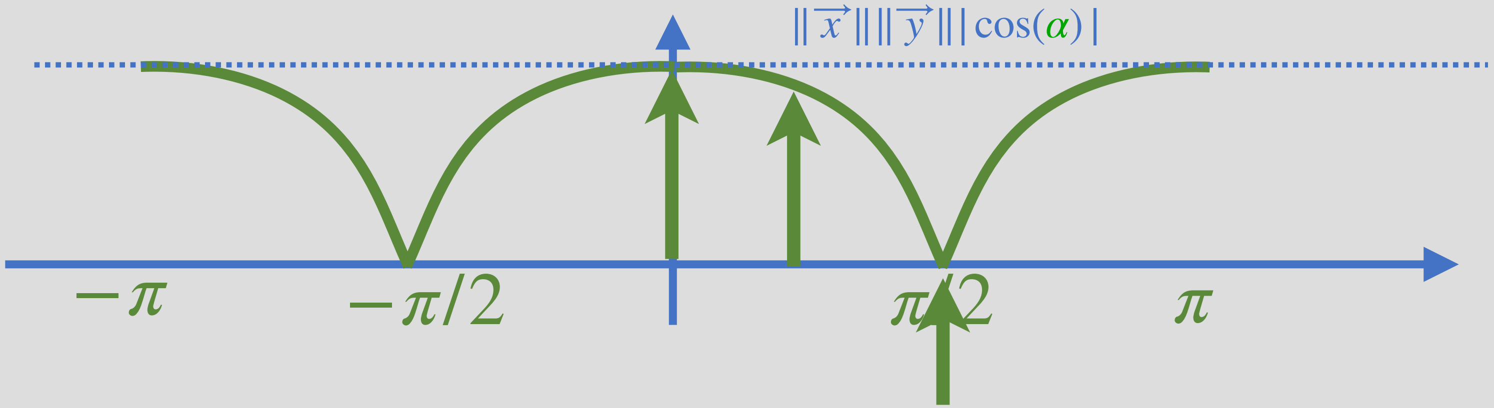
$$| < \overrightarrow{x}, \overrightarrow{y} > | = \| \overrightarrow{x} \| \| \overrightarrow{y} \| | \cos(\alpha) |$$

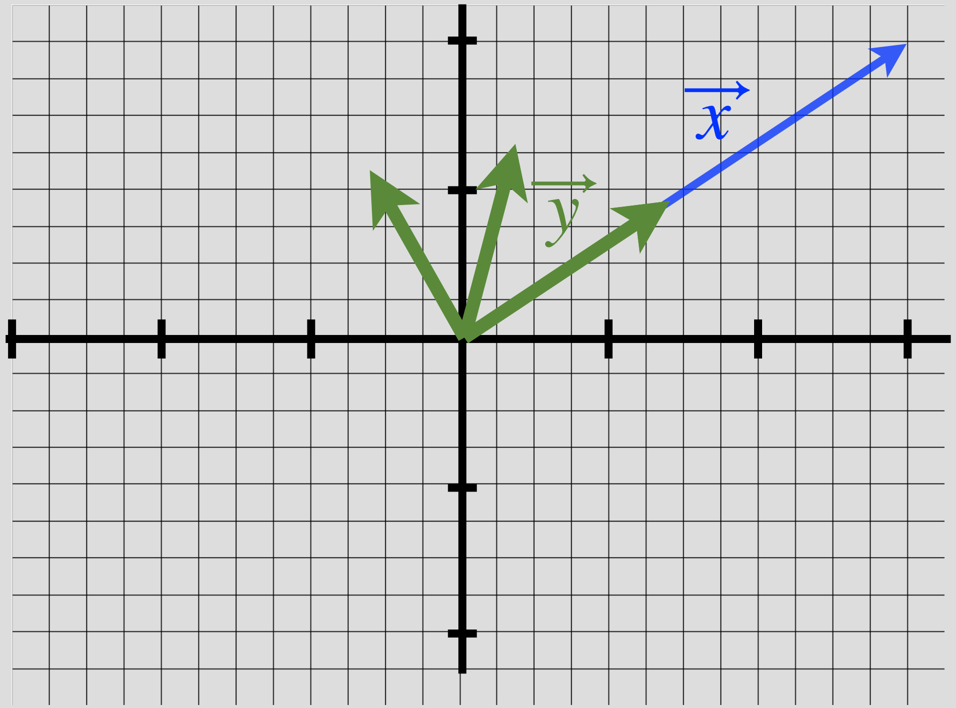


Cauchy-Schwarz Inequality

- Consider: $| < \overrightarrow{x}, \overrightarrow{y} > |$

$$| < \overrightarrow{x}, \overrightarrow{y} > | = ||\overrightarrow{x}|| ||\overrightarrow{y}|| |\cos(\alpha)|$$





Cauchy-Schwarz Inequality

- Consider: $| < \vec{x}, \vec{y} > |$

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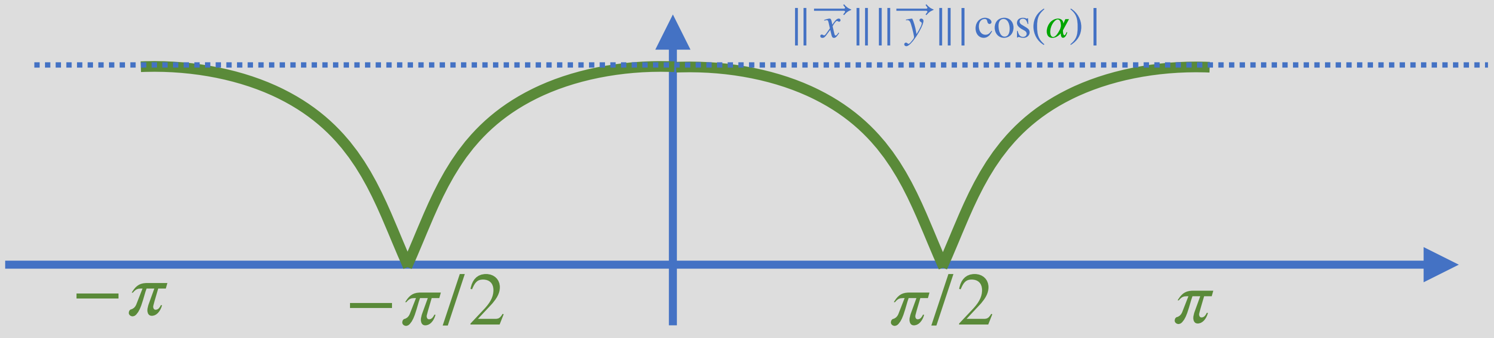
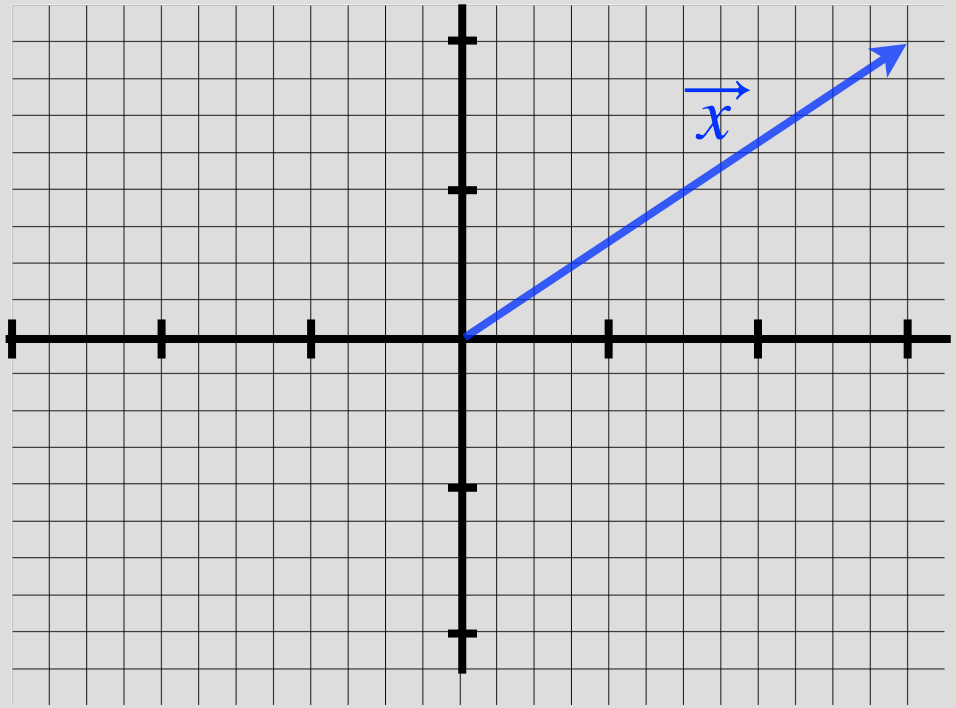
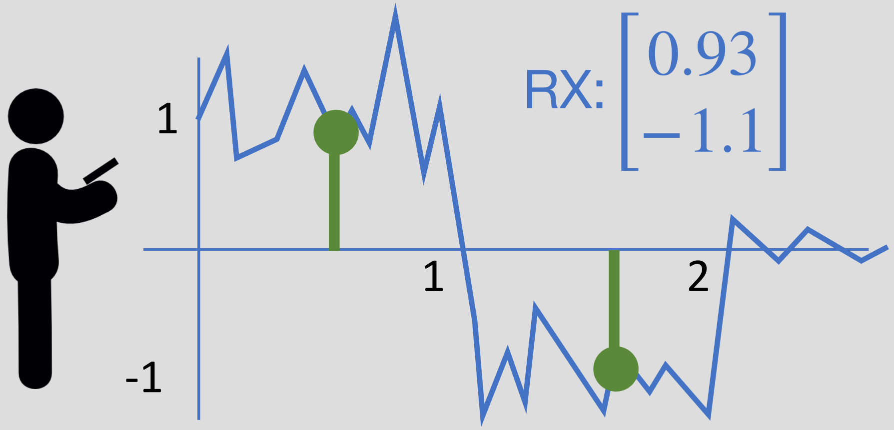


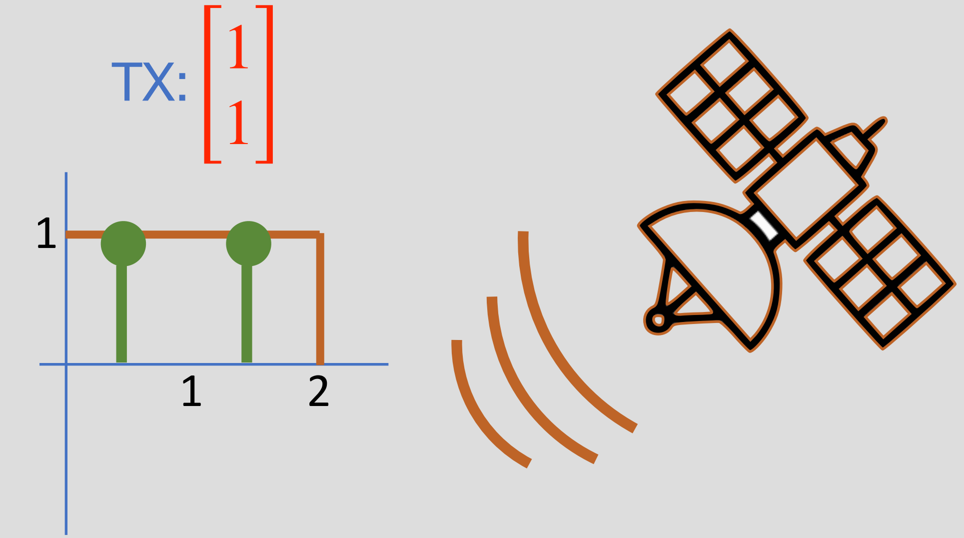


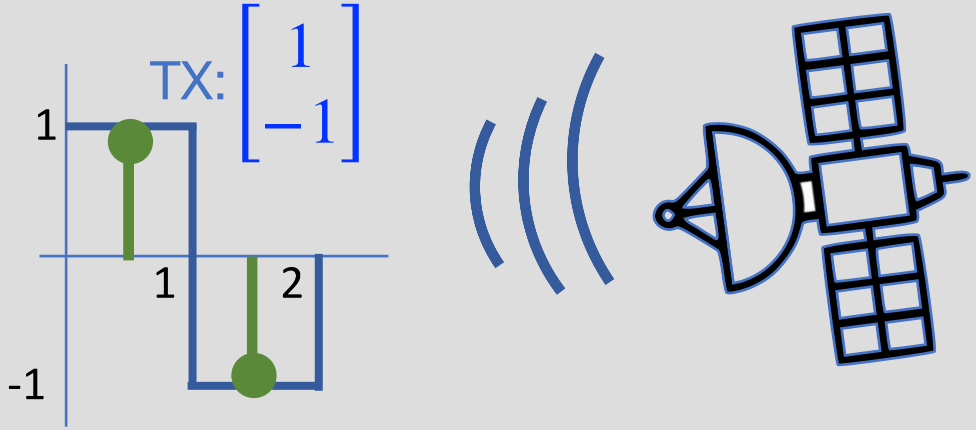
$$| < \vec{x}, \vec{y} > | \leq \|\vec{x}\| \|\vec{y}\|$$

Problem 1: Classification

- Satellites transmit a unique code
  - Radio signal
- Signal is received and digitized by a receiver





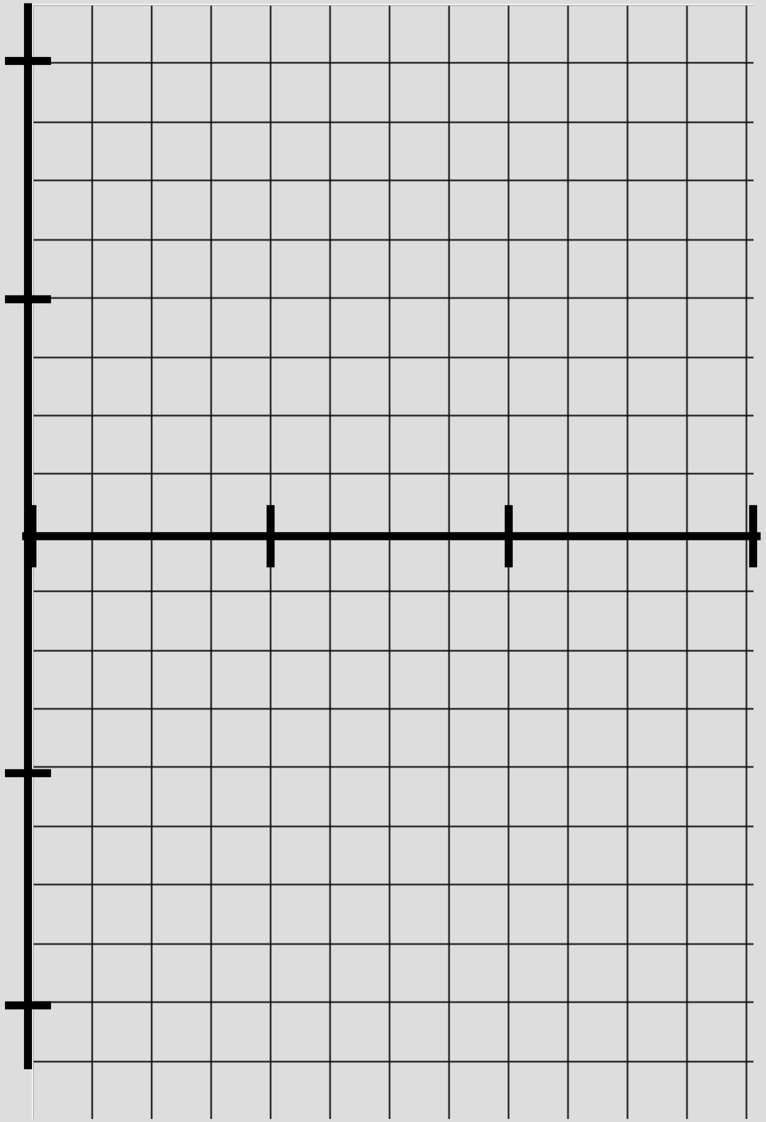


Q: Which satellite was received?

Classification

- Q: How to mathematically formulate the classification problem?

$$\vec{r} = \begin{bmatrix} 0.93 \\ -1.1 \end{bmatrix} \qquad \vec{s}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{Or?} \quad \vec{s}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



Classification

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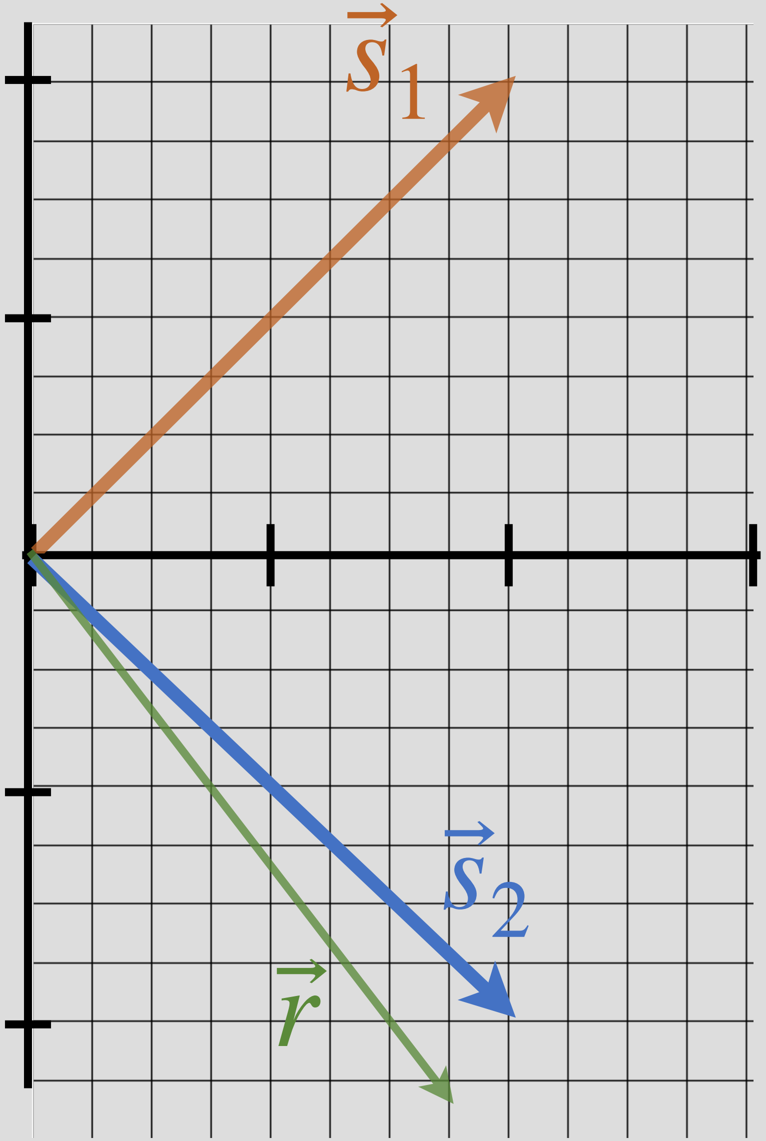


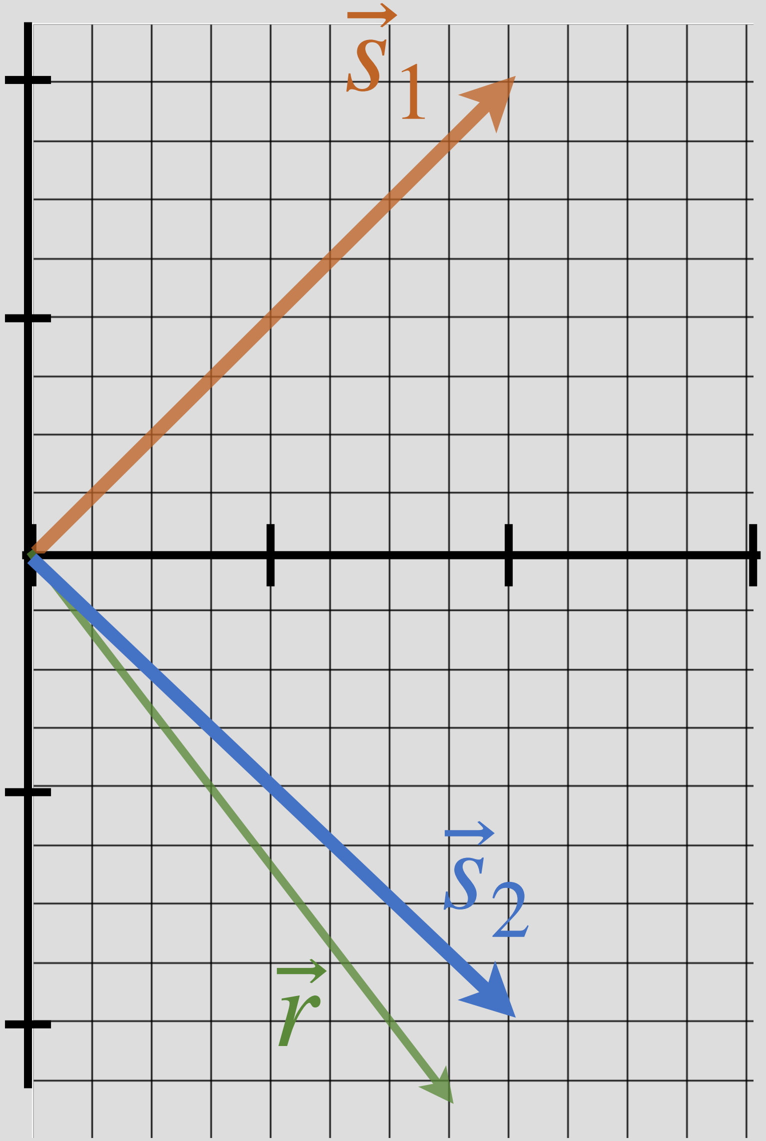
Classification

- Q: How to mathematically formulate the classification problem?

$\vec{r} = \begin{bmatrix} 0.93 \\ -1.1 \end{bmatrix}$ $\vec{s}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ Or? $\vec{s}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

- A: Look at the length of the error vector

$$i^* = \underset{i \in \{1,2\}}{\operatorname{argmin}} \|\vec{r} - \vec{s}_i\|$$



Classification

$$i^* = \underset{i \in \{1,2\}}{\operatorname{argmin}} \|\vec{r} - \vec{s}_i\|^2$$

$$\vec{s}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{s}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{aligned} \|\vec{r} - \vec{s}_i\|^2 &= \langle \vec{r} - \vec{s}_i, \vec{r} - \vec{s}_i \rangle \\ &= \langle \vec{r}, \vec{r} - \vec{s}_i \rangle - \langle \vec{s}_i, \vec{r} - \vec{s}_i \rangle \end{aligned}$$

Classification

$$i^* = \underset{i \in \{1,2\}}{\operatorname{argmin}} \|\vec{r} - \vec{s}_i\|^2$$

$$\vec{s}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{s}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\|\vec{r} - \vec{s}_i\|^2 = \langle \vec{r} - \vec{s}_i, \vec{r} - \vec{s}_i \rangle$$

$$= \langle \vec{r}, \vec{r} - \vec{s}_i \rangle - \langle \vec{s}_i, \vec{r} - \vec{s}_i \rangle$$

$$= \langle \vec{r}, \vec{r} \rangle - \langle \vec{r}, \vec{s}_i \rangle - \langle \vec{s}_i, \vec{r} \rangle + \langle \vec{s}_i, \vec{s}_i \rangle$$

$$= \|\vec{r}\|^2 + \|\vec{s}_i\|^2 - 2 \langle \vec{r}, \vec{s}_i \rangle$$

Fixed! =2

Classification

$$\|\vec{r} - \vec{s}_i\|^2 = \|\vec{r}\|^2 + \|\vec{s}_i\|^2 - 2\langle \vec{r}, \vec{s}_i \rangle$$

Fixed! =2

If $\langle \vec{r}, \vec{s}_i \rangle$ is maximized, then $\|\vec{r} - \vec{s}_i\|^2$ is minimized

Classification procedure:

for $i \in \{1,2\}$
compute $\langle \vec{r}, \vec{s}_i \rangle$

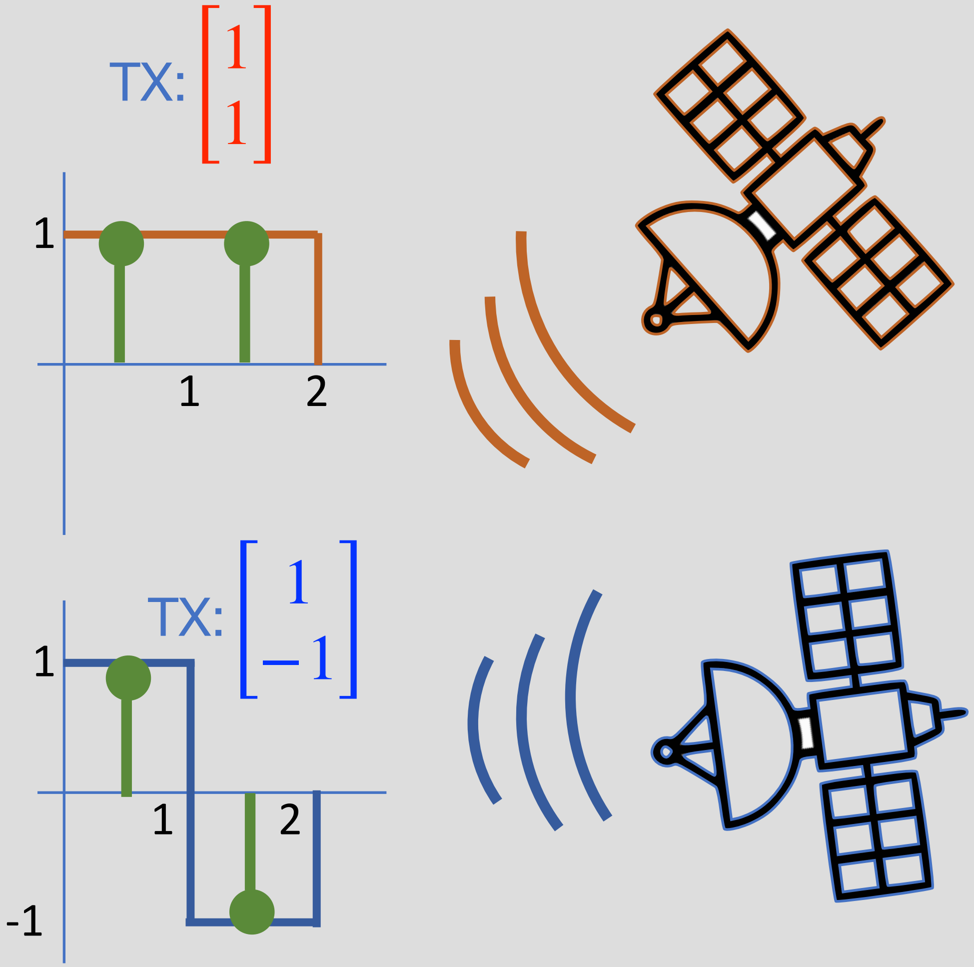
$$\langle \vec{r}, \vec{s}_1 \rangle = -0.17$$

$$\langle \vec{r}, \vec{s}_2 \rangle = 2.03$$

Return index i the maximizes the above $i^* = 2$

Localization

- Satellites transmit a unique code
  - Radio signal
- Signal is received and digitized by a receiver



Two problems:

1. Interference
2. Timing (next week)

Interference

Possibility 1: Both sats are in TX

$$\vec{r} = \vec{s}_1 + \vec{s_2} + \vec{n}$$

Possibility 2: Only S1 is in Tx

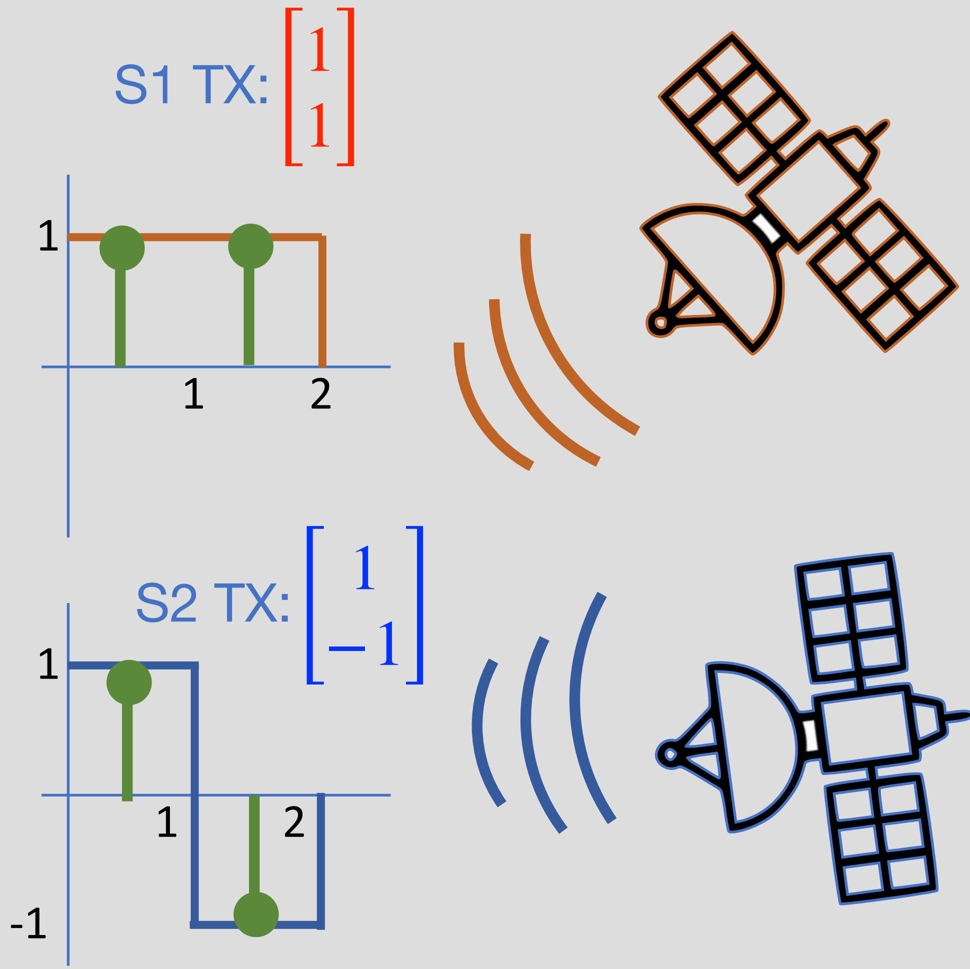
$$\vec{r} = \vec{s}_1 + \vec{n}$$

Possibility 3: Only S2 is in Tx

$$\vec{r} = \vec{s}_2 + \vec{n}$$

Possibility 4: None is in Tx

$$\vec{r} = \vec{n}$$



Interference

Possibility 1: Both sats are in TX

$$\vec{r} = \vec{s}_1 + \vec{s}_2 + \vec{n}$$

$$\langle \vec{r}, \vec{s}_1 \rangle = \langle \vec{s}_1 + \vec{s}_2 + \vec{n}, \vec{s}_1 \rangle$$

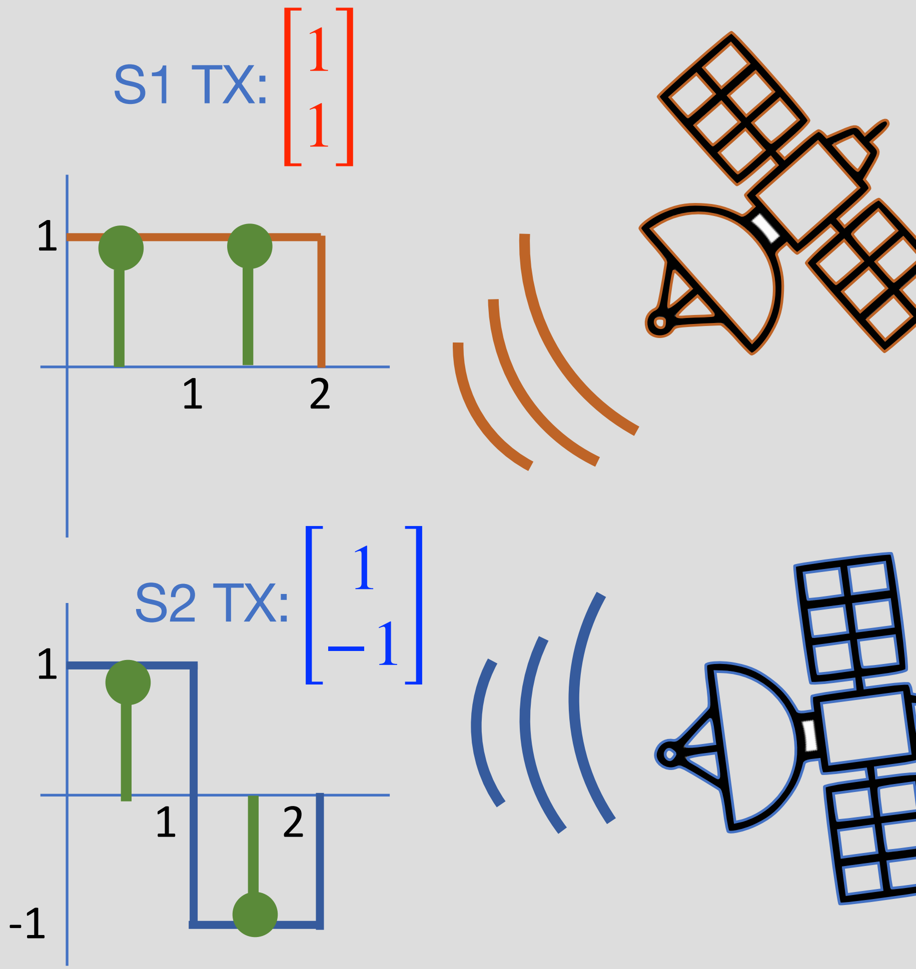
$$= \langle \vec{s}_1, \vec{s}_1 \rangle + \langle \vec{s}_2, \vec{s}_1 \rangle + \langle \vec{n}, \vec{s}_1 \rangle$$

Desired Interference Small

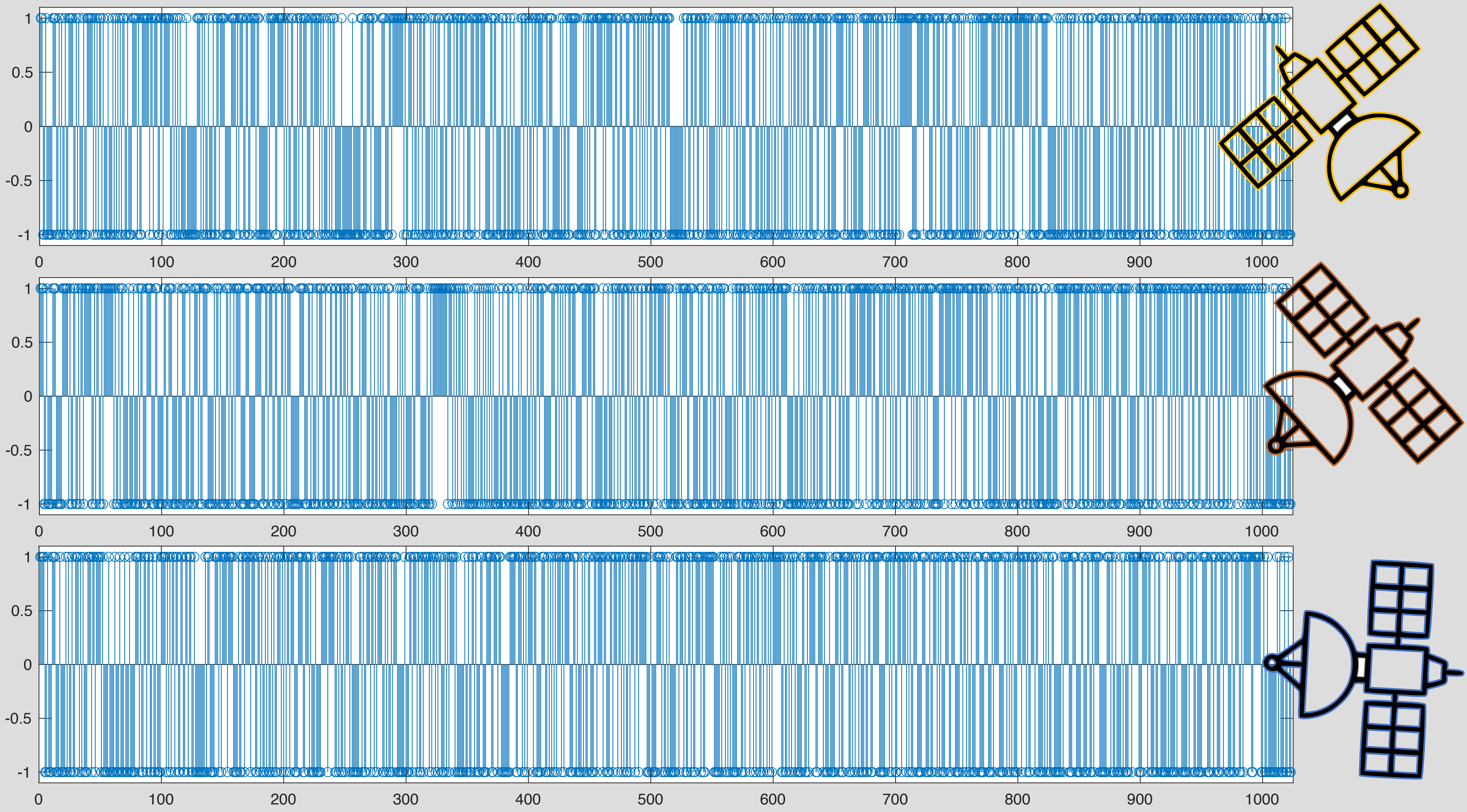
Q: How to design codes that don't interfere?

A: Make them orthogonal!

$$\langle \vec{s}_2, \vec{s}_1 \rangle = 0$$

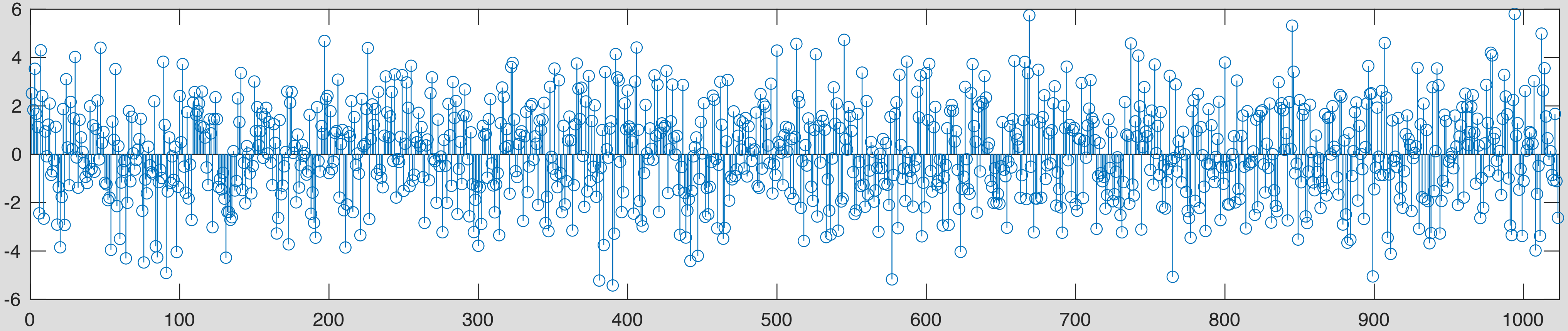


GPS Gold Codes

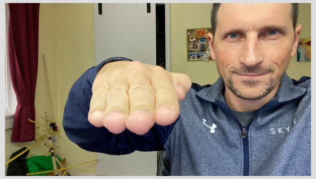


Example:

$$\vec{r} =$$



$$\langle \vec{r}, \vec{s}_i \rangle = \vec{r}^T \vec{s}_i$$



$$\begin{bmatrix} \vec{r}^T \end{bmatrix} \begin{bmatrix} \vec{s}_1 & \vec{s}_2 & \vec{s}_3 & \cdots & \vec{s}_{24} \end{bmatrix} = \begin{bmatrix} \vec{r}^T\vec{s}_1 & \vec{r}^T\vec{s}_2 & \cdots & \vec{r}^T\vec{s}_{24} \end{bmatrix}$$