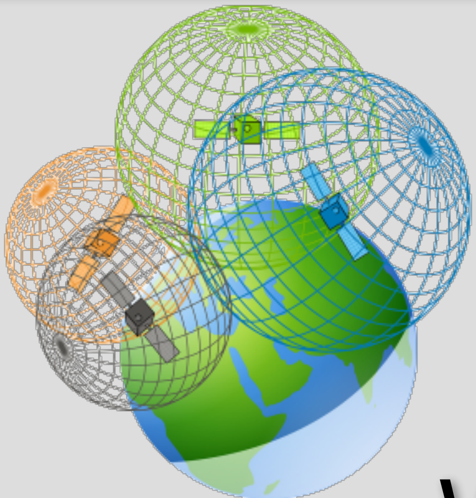


Welcome to EECS 16A!

Designing Information Devices and Systems I

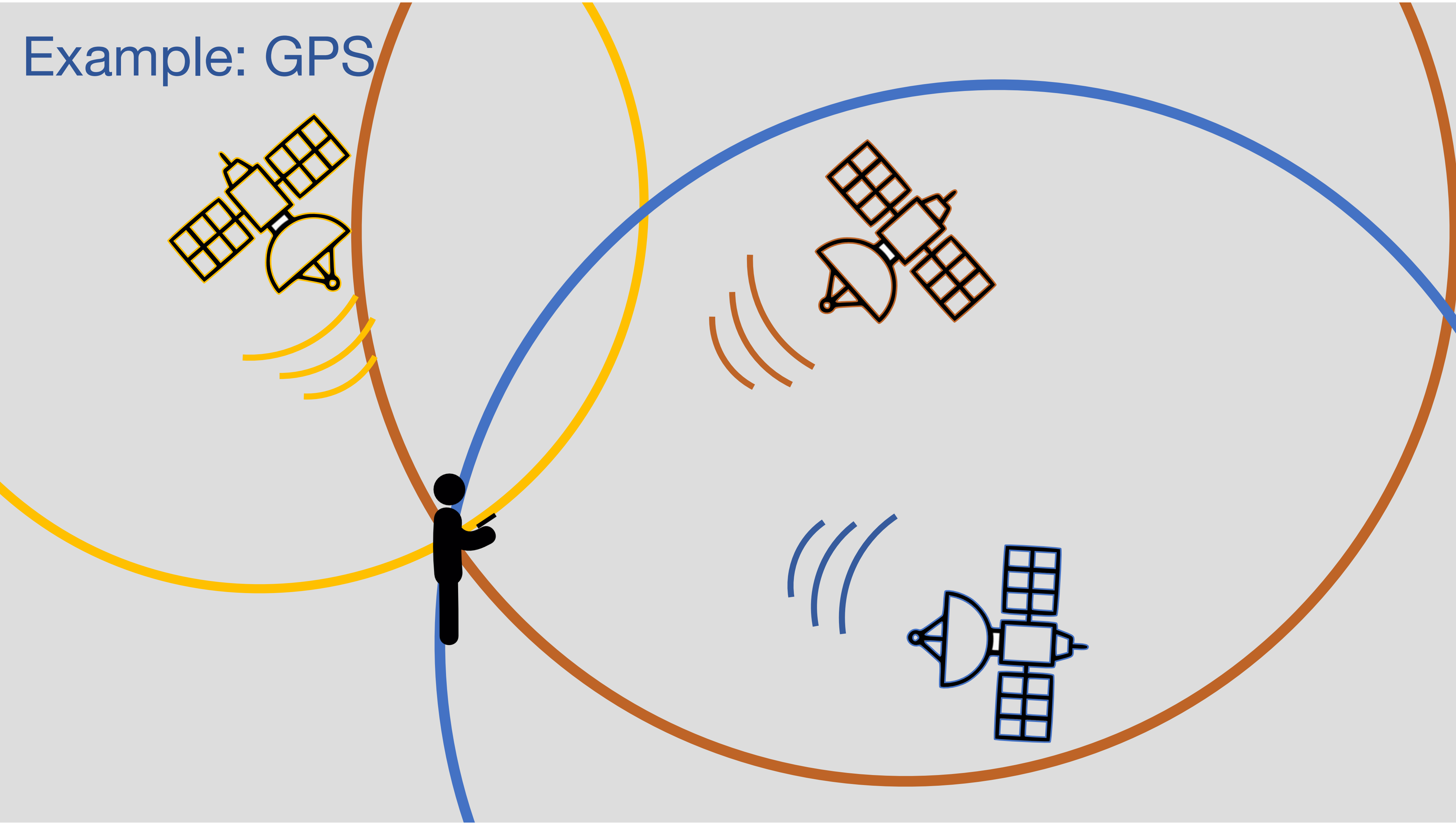
Ana Arias and Miki Lustig

Fall 2021

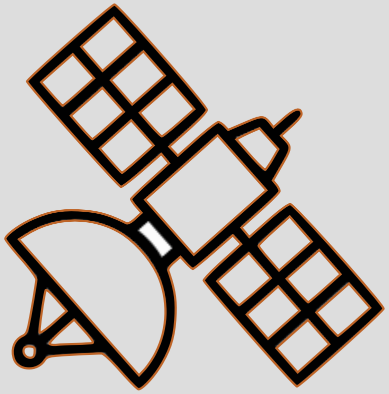
Lecture 12B

Orthogonal Projections and Least squares

Example: GPS



GPS



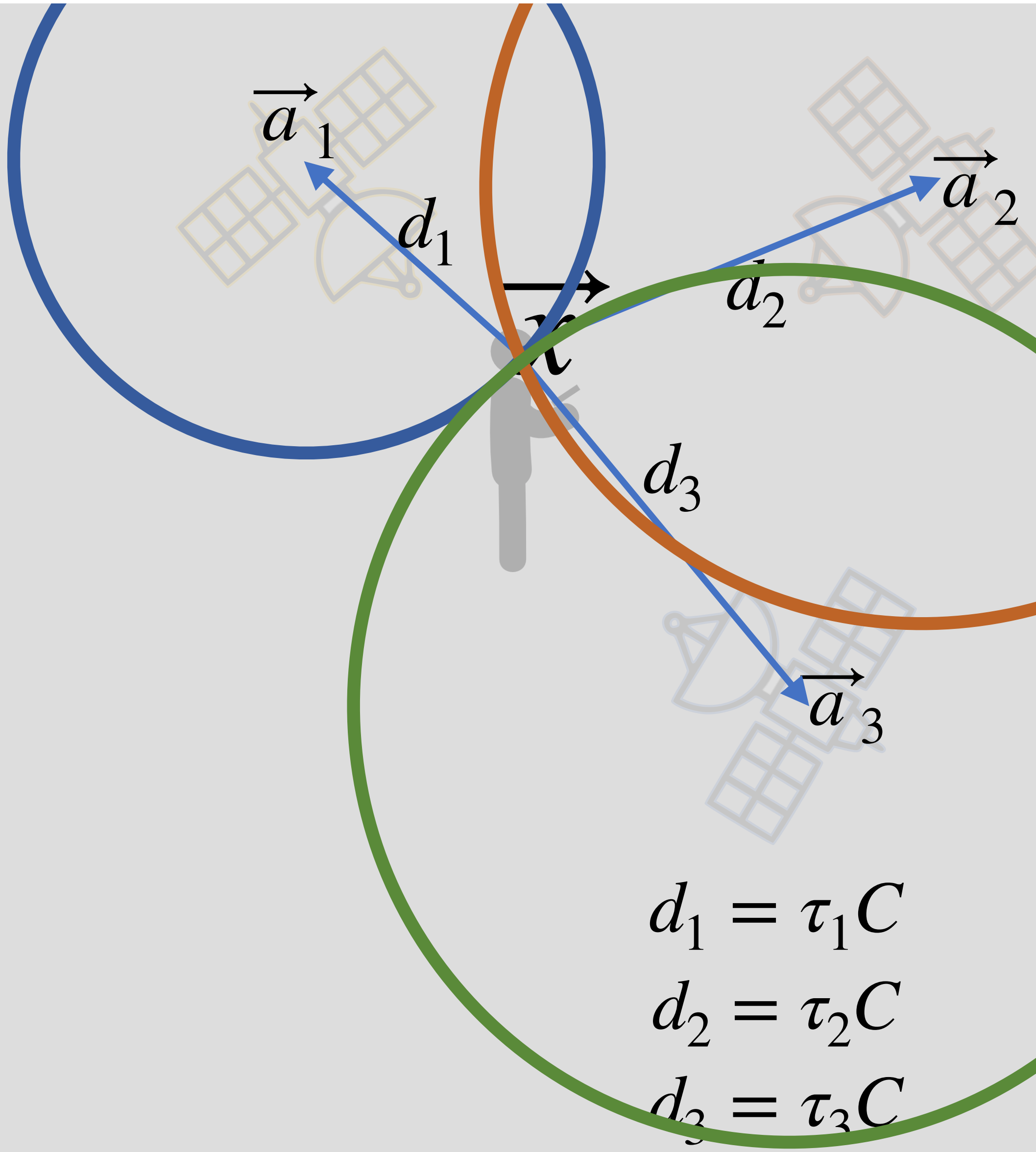
- 24 satellites
  - Known position
  - Time synchronized
  - 8 usually visible
- Problem:
  - Classify which satellite is transmitting
  - Estimate distance to GPS
  - Estimate position from noisy data
- Tools:
  - Inner product
  - Cross correlation
  - Least Squares

Trilateration

(1) $$\|\overrightarrow{x} - \overrightarrow{a}_1\|^2 = d_1^2$$

(2) $$\|\overrightarrow{x} - \overrightarrow{a}_2\|^2 = d_2^2$$

(3) $$\|\overrightarrow{x} - \overrightarrow{a}_3\|^2 = d_3^2$$



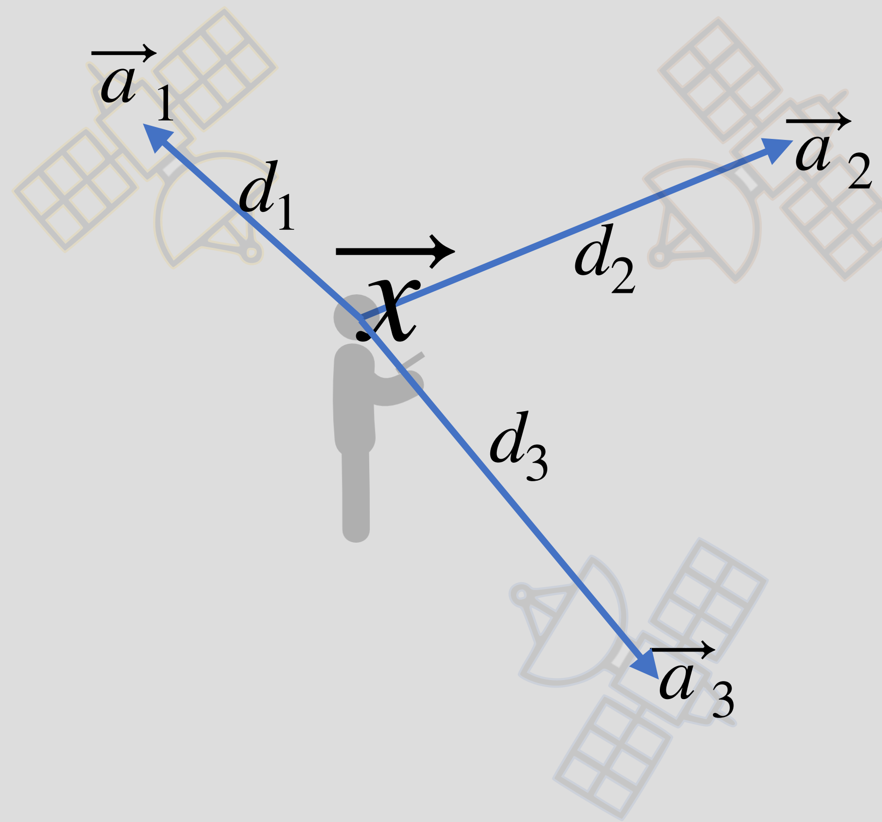
Trilateration

(1) $\|\vec{x}\|^2 - 2\vec{a}_1^T\vec{x} + \|\vec{a}_1\|^2 = C^2\tau_1^2$

(2) $\|\vec{x}\|^2 - 2\vec{a}_2^T\vec{x} + \|\vec{a}_2\|^2 = C^2\tau_2^2$

(3) $\|\vec{x}\|^2 - 2\vec{a}_3^T\vec{x} + \|\vec{a}_3\|^2 = C^2\tau_3^2$

(2) - (1)

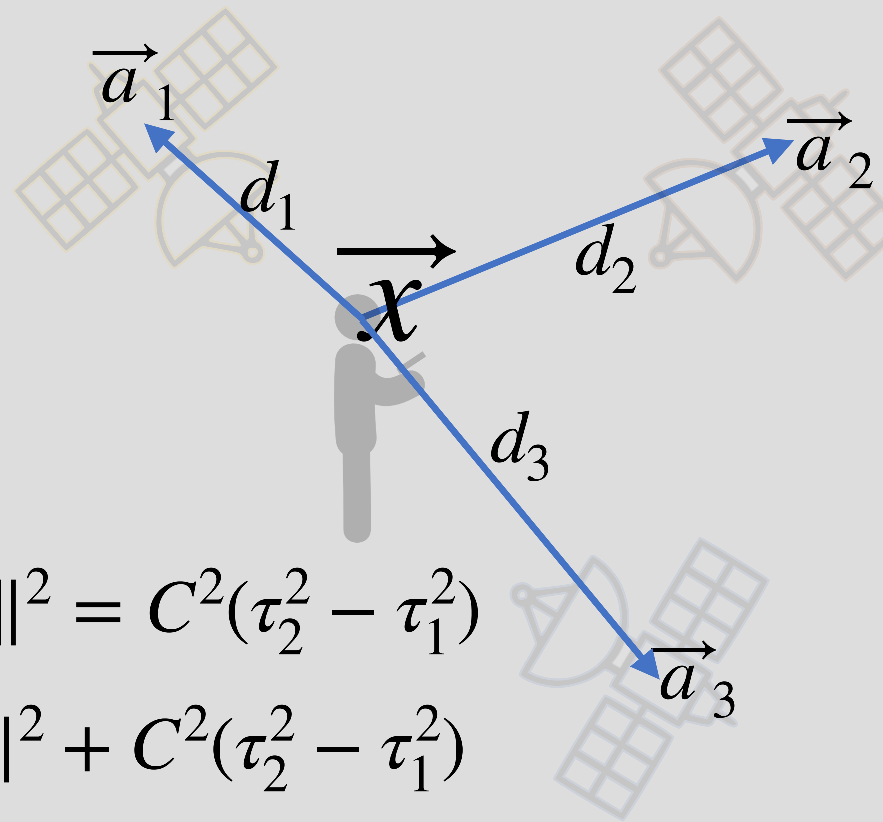


Trilateration

(1) $\|\vec{x}\|^2 - 2\vec{a}_1^T\vec{x} + \|\vec{a}_1\|^2 = C^2\tau_1^2$

(2) $\|\vec{x}\|^2 - 2\vec{a}_2^T\vec{x} + \|\vec{a}_2\|^2 = C^2\tau_2^2$

(3) $\|\vec{x}\|^2 - 2\vec{a}_3^T\vec{x} + \|\vec{a}_3\|^2 = C^2\tau_3^2$



(2) - (1)

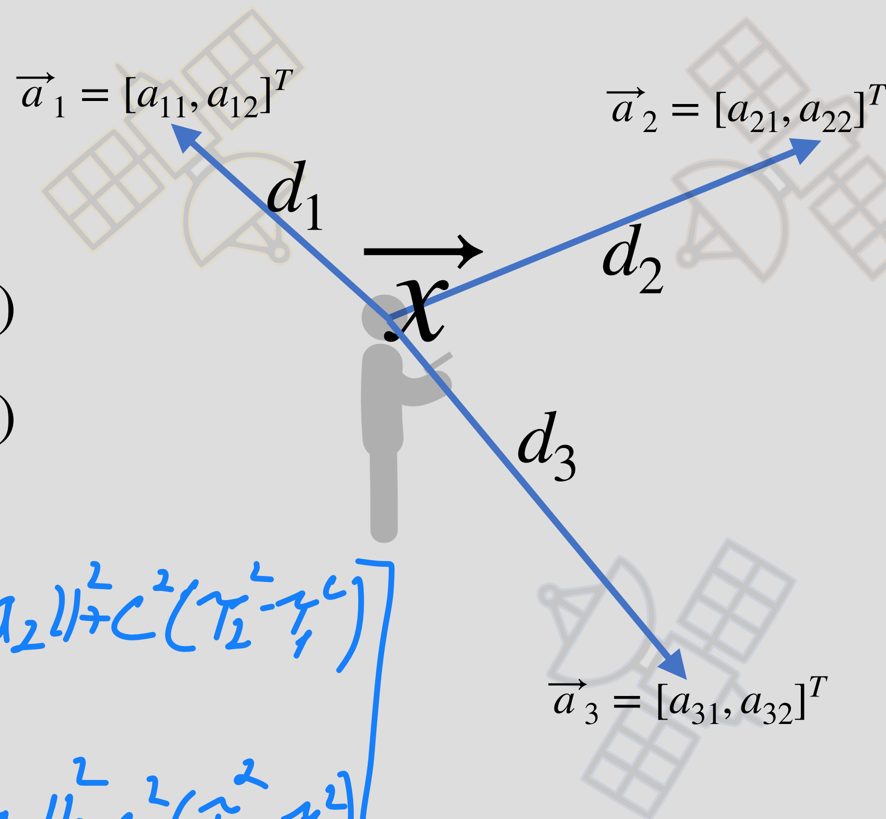
$$-2\vec{a}_2^T\vec{x} + 2\vec{a}_1^T\vec{x} + \|\vec{a}_2\|^2 - \|\vec{a}_1\|^2 = C^2(\tau_2^2 - \tau_1^2)$$

$$2(\vec{a}_1 - \vec{a}_2)^T\vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\tau_2^2 - \tau_1^2)$$

(3) - (1)

$$2(\vec{a}_1 - \vec{a}_3)^T\vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + C^2(\tau_3^2 - \tau_1^2)$$

Trilateration



$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\tau_2^2 - \tau_1^2)$$

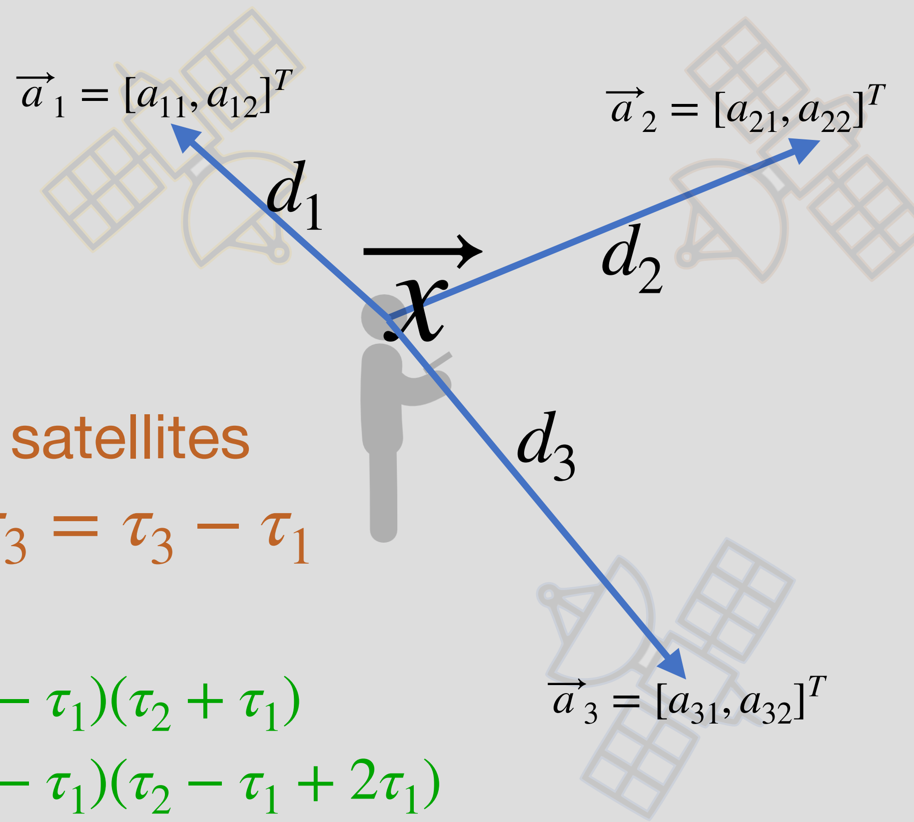
$$2(\vec{a}_1 - \vec{a}_3)^T \vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + C^2(\tau_3^2 - \tau_1^2)$$

$$2\begin{bmatrix} a_{11} - a_{21} & a_{12} - a_{22} \\ a_{11} - a_{31} & a_{12} - a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \|a_1\|^2 - \|a_2\|^2 + c^2(\tau_2^2 - \tau_1^2) \\ \|a_1\|^2 - \|a_3\|^2 + c^2(\tau_3^2 - \tau_1^2) \end{bmatrix}$$

Solve via gaussian elimination!

Trilateration

$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\tau_2^2 - \tau_1^2)$$
$$2(\vec{a}_1 - \vec{a}_3)^T \vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + C^2(\tau_3^2 - \tau_1^2)$$

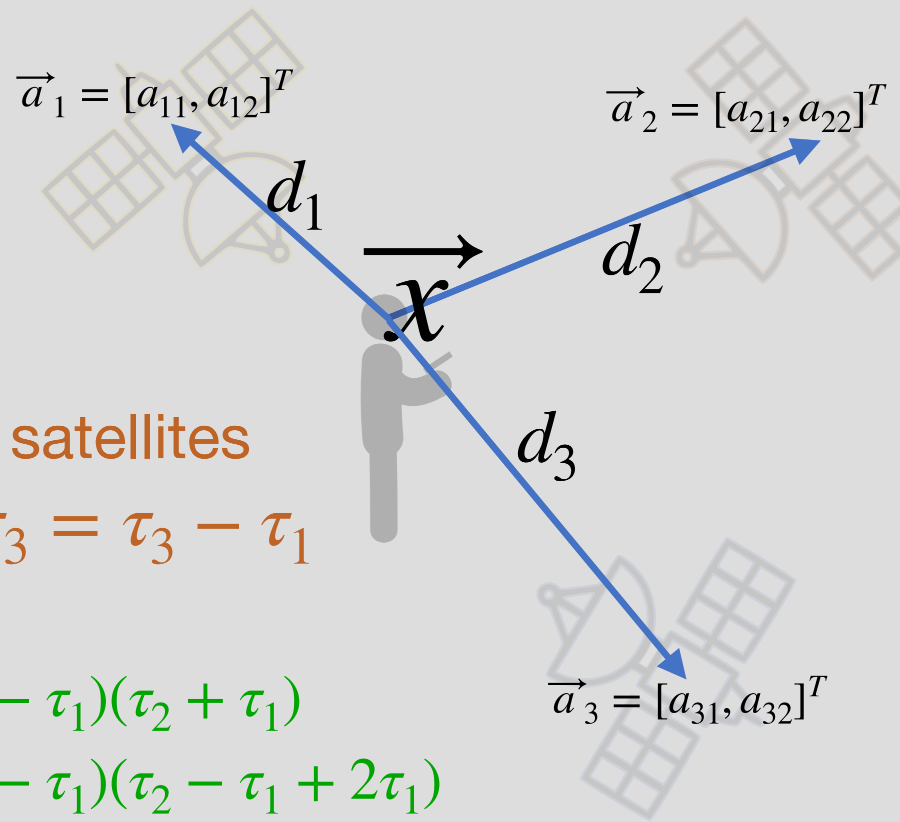


Problem — receiver clock is not synced to satellites

τ_1 is unknown, but $\Delta\tau_2 = \tau_2 - \tau_1$, and $\Delta\tau_3 = \tau_3 - \tau_1$ are known

$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\tau_2 - \tau_1)(\tau_2 + \tau_1)$$
$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\tau_2 - \tau_1)(\tau_2 - \tau_1 + 2\tau_1)$$
$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\Delta\tau_2)(\Delta\tau_2 + 2\tau_1)$$

Trilateration



$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\tau_2^2 - \tau_1^2)$$

$$2(\vec{a}_1 - \vec{a}_3)^T \vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + C^2(\tau_3^2 - \tau_1^2)$$

Problem — receiver clock is not synced to satellites

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$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\tau_2 - \tau_1)(\tau_2 - \tau_1 + 2\tau_1)$$

$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\Delta\tau_2)(\Delta\tau_2 + 2\tau_1)$$

$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} - 2C^2\Delta\tau_2\tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\Delta\tau_2)^2$$

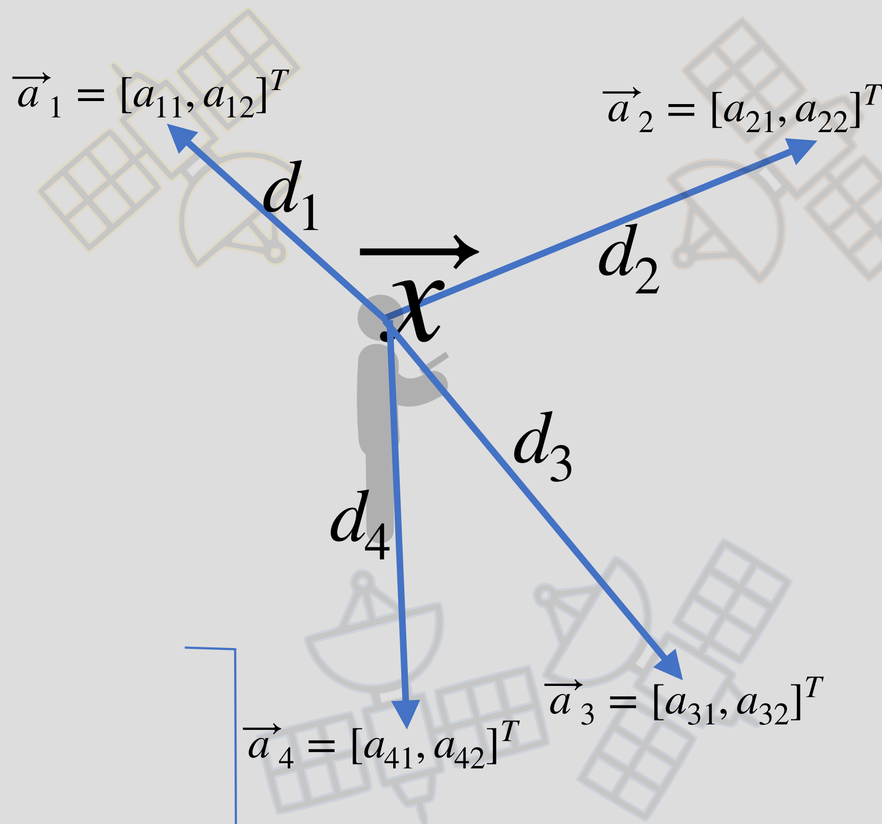
Another variable! Need 1 more equation (satellite)

Trilateration

$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} - 2C^2 \Delta\tau_2 \tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\Delta\tau_2)^2$$

$$2(\vec{a}_1 - \vec{a}_3)^T \vec{x} - 2C^2 \Delta\tau_3 \tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + C^2(\Delta\tau_3)^2$$

$$2(\vec{a}_1 - \vec{a}_4)^T \vec{x} - 2C^2 \Delta\tau_4 \tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_4\|^2 + C^2(\Delta\tau_4)^2$$



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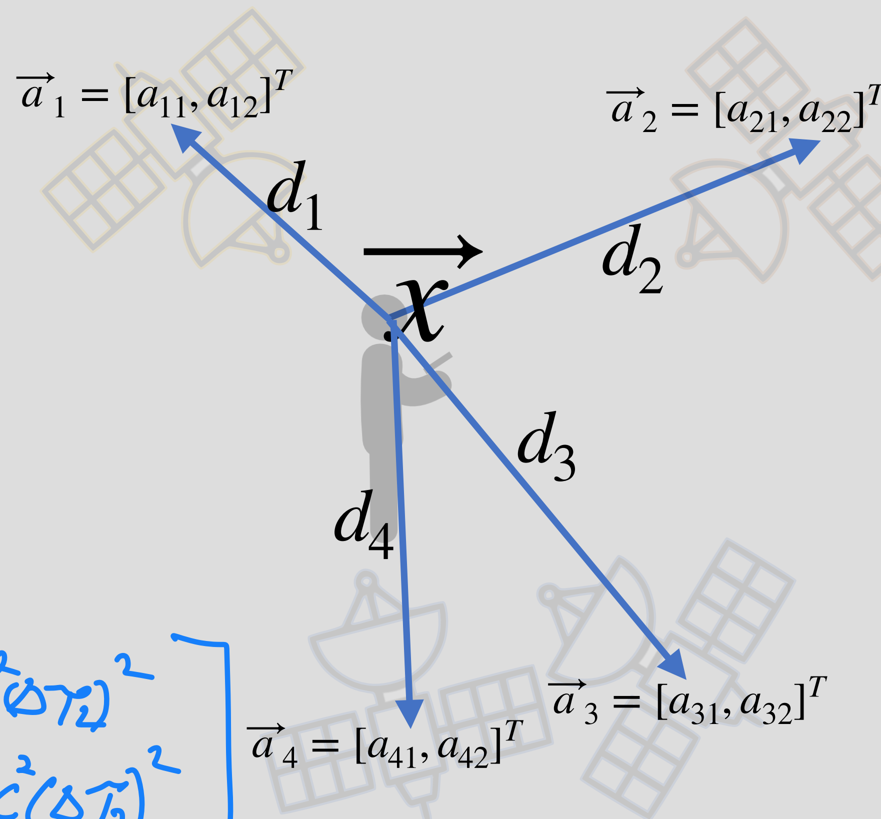
Trilateration

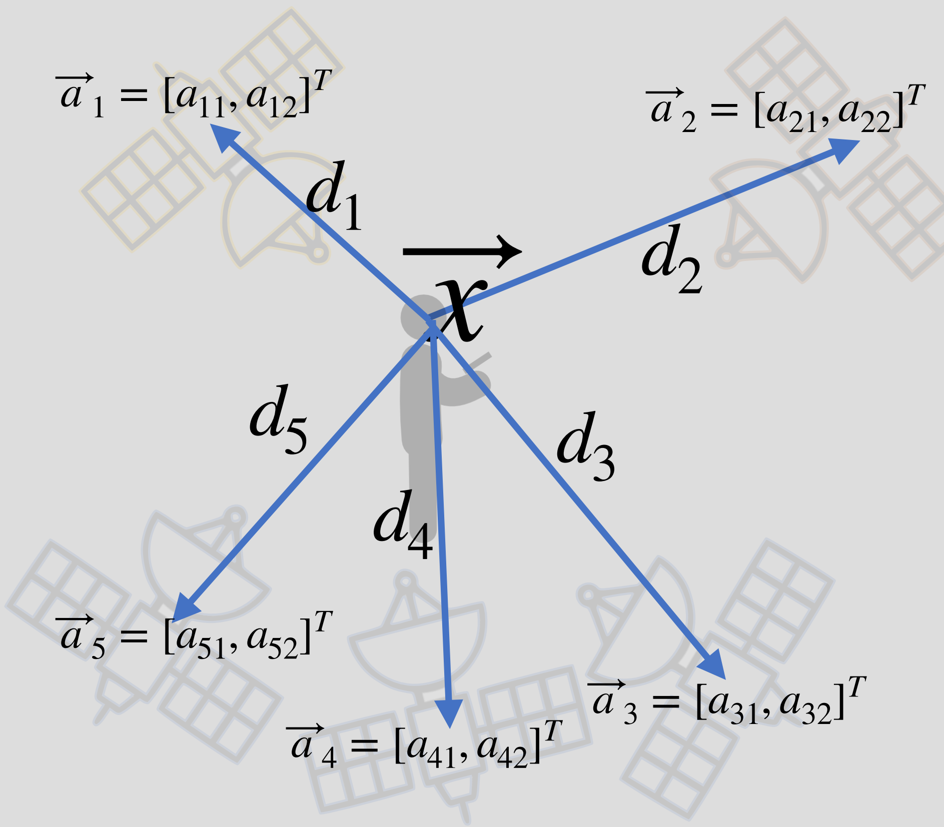
$$2(\vec{a}_1 - \vec{a}_2)^T\vec{x} - 2C^2\Delta\tau_2\tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\Delta\tau_2)^2$$
$$2(\vec{a}_1 - \vec{a}_3)^T\vec{x} - 2C^2\Delta\tau_3\tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + C^2(\Delta\tau_3)^2$$
$$2(\vec{a}_1 - \vec{a}_4)^T\vec{x} - 2C^2\Delta\tau_4\tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_4\|^2 + C^2(\Delta\tau_4)^2$$



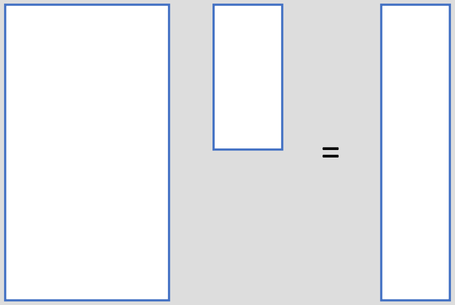
$$2\begin{bmatrix} a_{11}-a_{21} & a_{12}-a_{22} & -C^2\Delta\tau_2 \\ a_{11}-a_{31} & a_{12}-a_{32} & -C^2\Delta\tau_3 \\ a_{11}-a_{41} & a_{12}-a_{42} & -C^2\Delta\tau_4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \tau_1 \end{bmatrix} = \begin{bmatrix} \|a_1\|^2-\|a_2\|^2+C^2(\Delta\tau_2)^2 \\ \|a_1\|^2-\|a_3\|^2+C^2(\Delta\tau_3)^2 \\ \|a_1\|^2-\|a_4\|^2+C^2(\Delta\tau_4)^2 \end{bmatrix}$$

Multi-Lateration

$$2(\vec{a}_1 - \vec{a}_2)^T\vec{x} - 2C^2\Delta\tau_2\tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\Delta\tau_2)^2$$
$$2(\vec{a}_1 - \vec{a}_3)^T\vec{x} - 2C^2\Delta\tau_3\tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + C^2(\Delta\tau_3)^2$$
$$2(\vec{a}_1 - \vec{a}_4)^T\vec{x} - 2C^2\Delta\tau_4\tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_4\|^2 + C^2(\Delta\tau_4)^2$$
$$2(\vec{a}_1 - \vec{a}_5)^T\vec{x} - 2C^2\Delta\tau_5\tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_5\|^2 + C^2(\Delta\tau_5)^2$$



More equations than unknowns



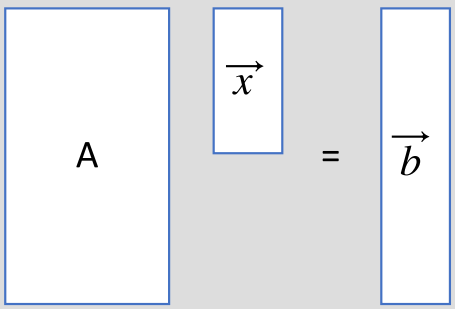
Over-determined — may not have a solution!

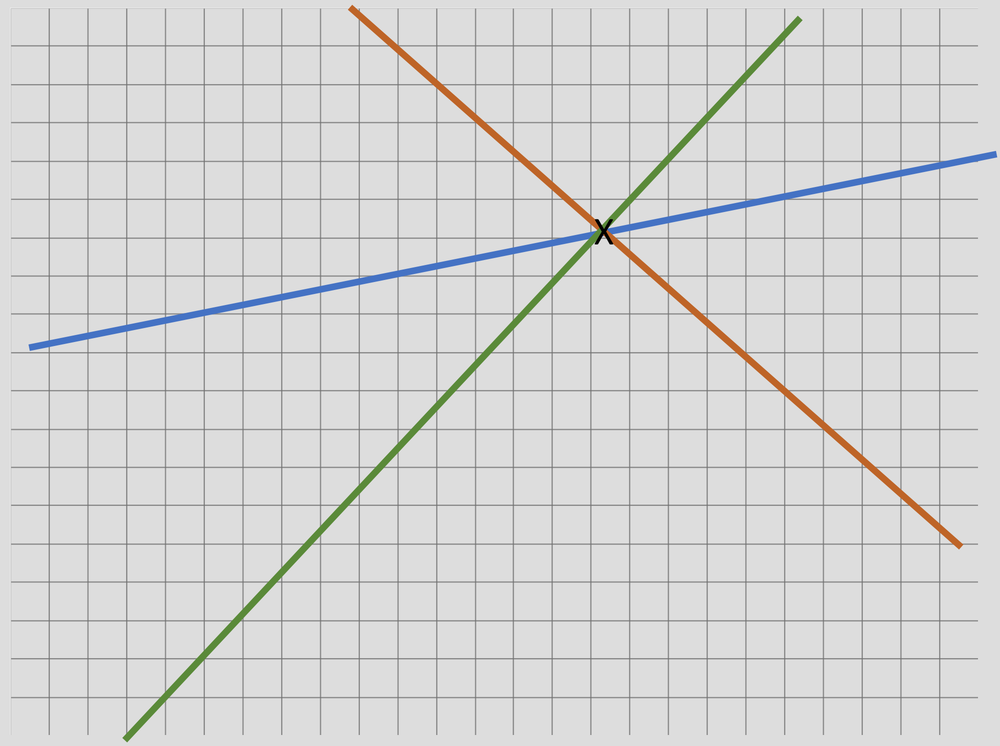
Inconsistent Linear Equations

$a_{11}x_1 + a_{12}x_2 = b_1$

$a_{21}x_1 + a_{22}x_2 = b_2$

$a_{31}x_1 + a_{32}x_2 = b_3$





Q: When is there a solution?

A: When $\vec{b} \in \text{Span}\{\text{cols of } A\}$

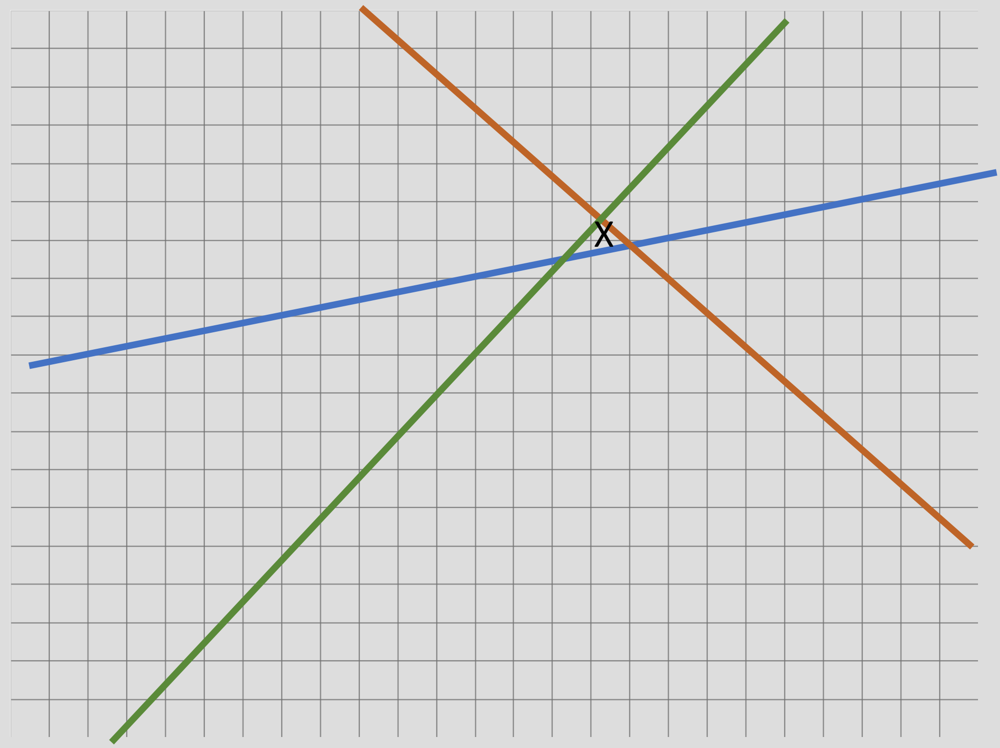
Inconsistent Linear Equations

$$a_{11}x_1 + a_{12}x_2 = b_1 + e_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2 + e_2$$

$$a_{31}x_1 + a_{32}x_2 = b_3 + e_3$$

$$A \vec{x} = \vec{b} + \vec{e}$$



Q: With noise, equations will be inconsistent! - no solution.

Towards the Least Squares Algorithm

Fact:

We have measurements: $\overrightarrow{b}$

We have a model that : $A\overrightarrow{x} = \overrightarrow{b}$

Problem:

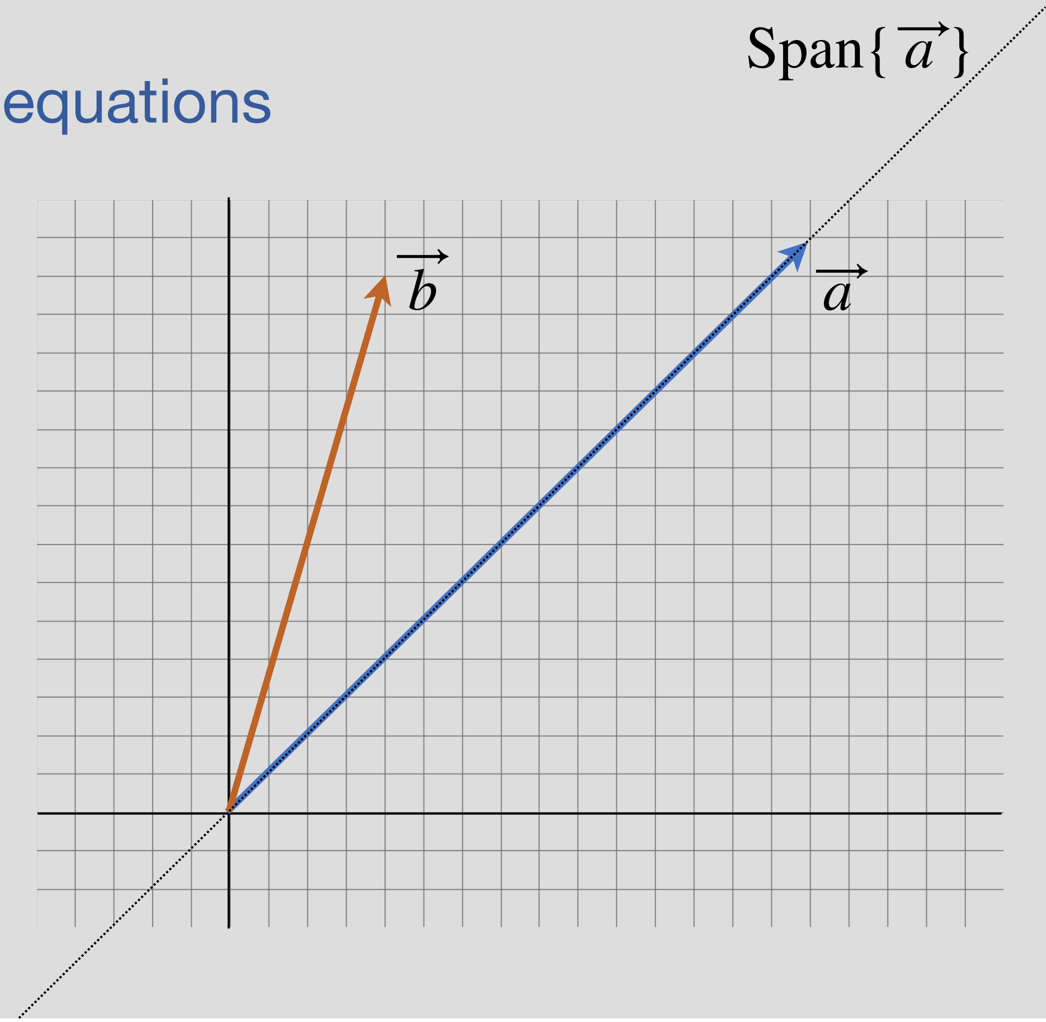
But $A\overrightarrow{x} = \overrightarrow{b}$ does not have a solution!

Solution:

Want to find $\hat{x}$, such that $A\hat{x}$ is the closest to $\overrightarrow{b}$

Example: a scalar problem

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$, one unknown, two equations



Solution:

find $\hat{x}$ that has the smallest error

$$\|\vec{e}\| = \|\vec{a}\hat{x} - \vec{b}\|$$

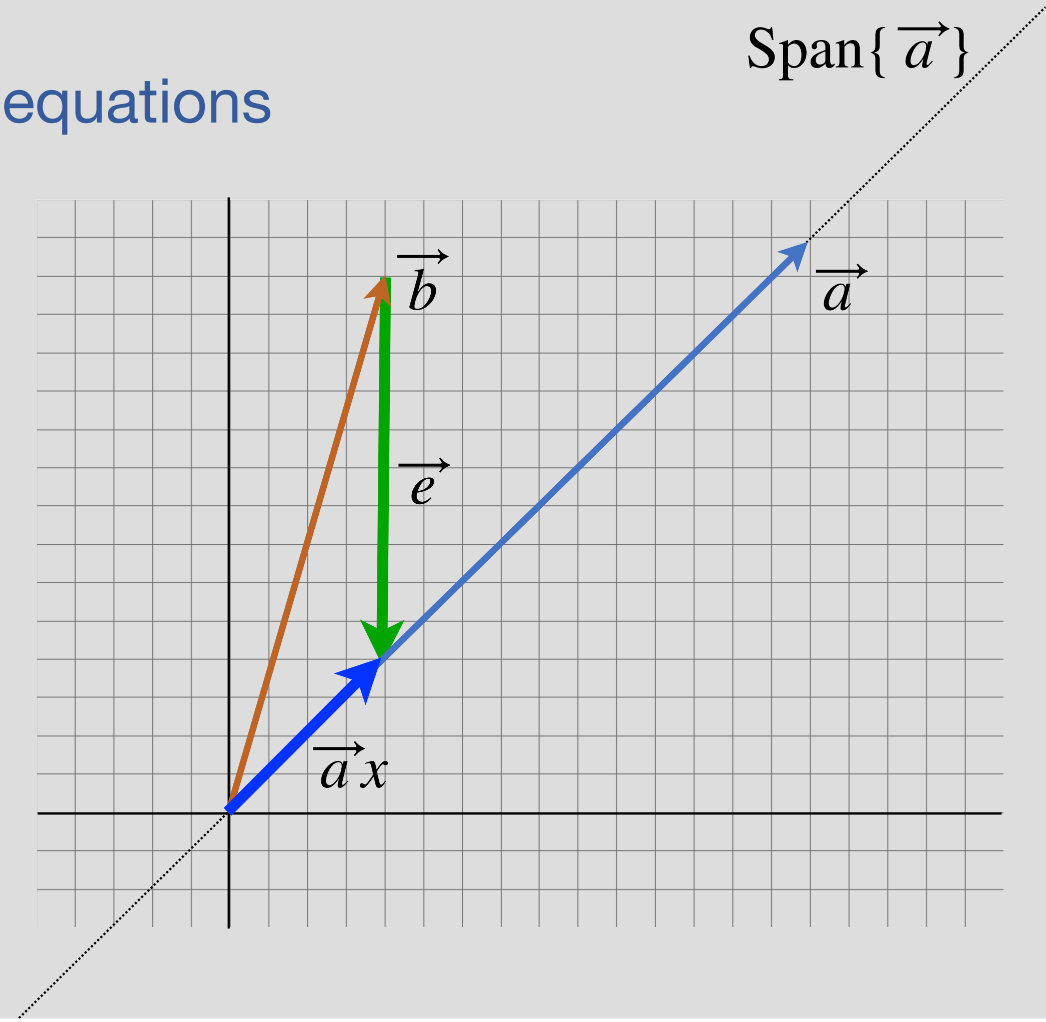
Example: a scalar problem

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$, one unknown, two equations

Solution:

find $\hat{x}$ that has the smallest error

$$\| \vec{e} \| = \| \vec{a}\hat{x} - \vec{b} \| \leq \| \vec{a}x - \vec{b} \|$$



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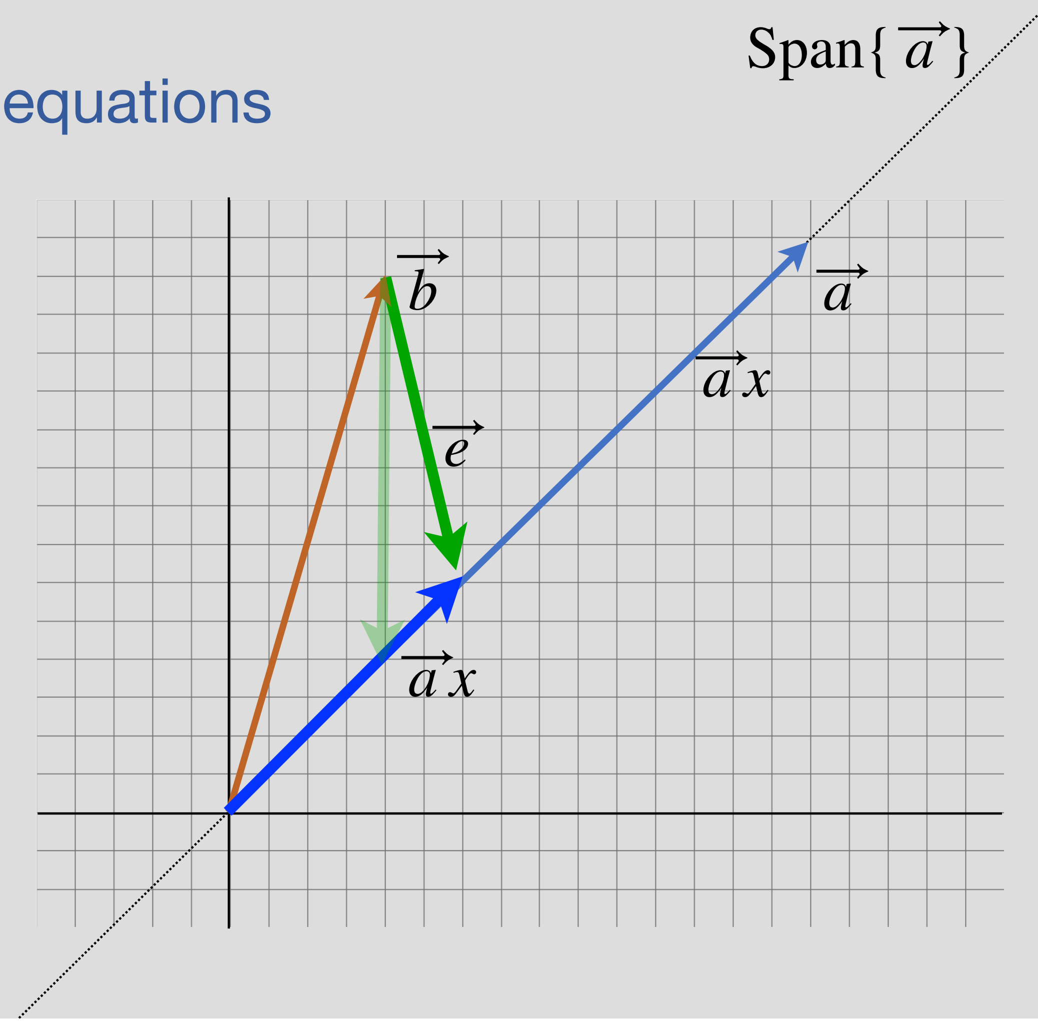

Solution:

find $\hat{x}$ that has the smallest error

$$\| \vec{e} \| = \| \vec{a}\hat{x} - \vec{b} \| \leq \| \vec{a}\,x - \vec{b} \|$$

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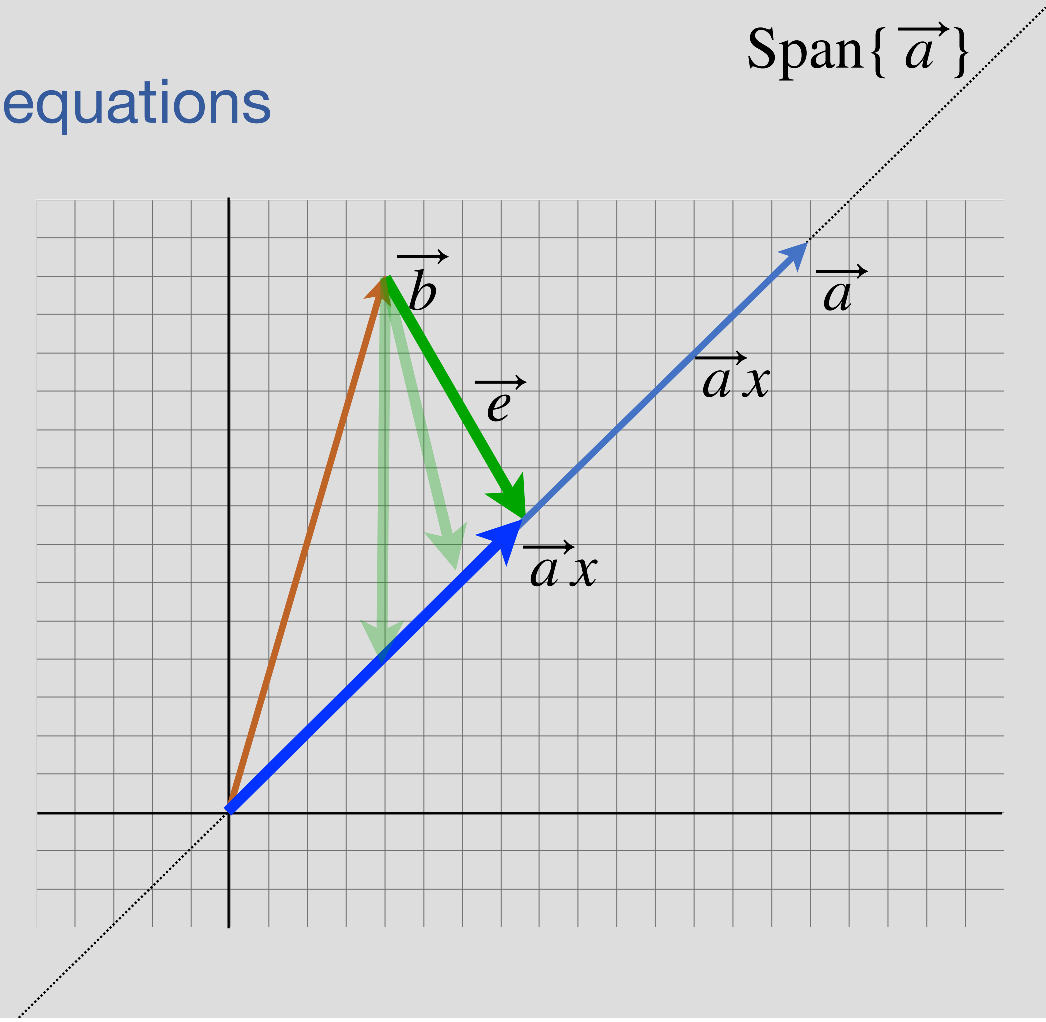

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$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$, one unknown, two equations

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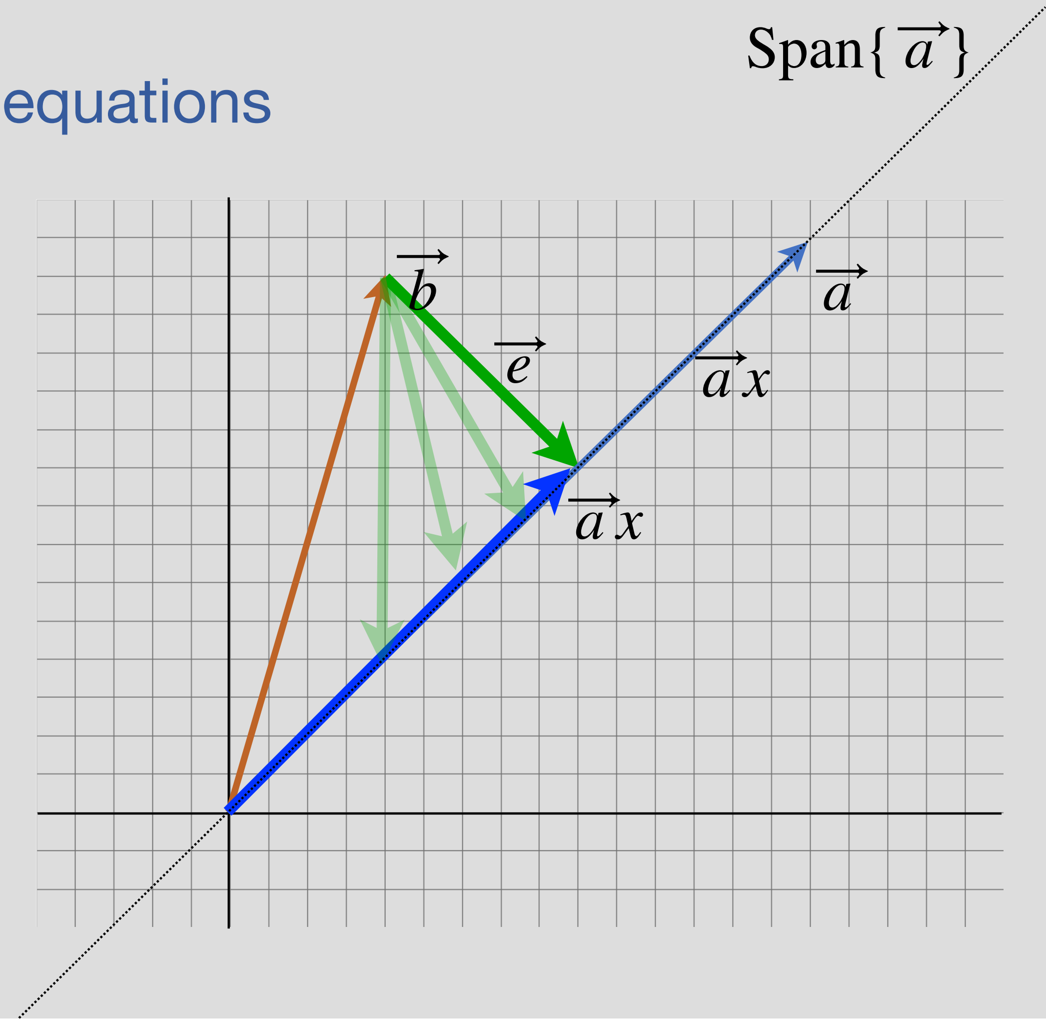
find $\hat{x}$ that has the smallest error

$$\| \vec{e} \| = \| \vec{a}\hat{x} - \vec{b} \| \leq \| \vec{a}x - \vec{b} \|$$



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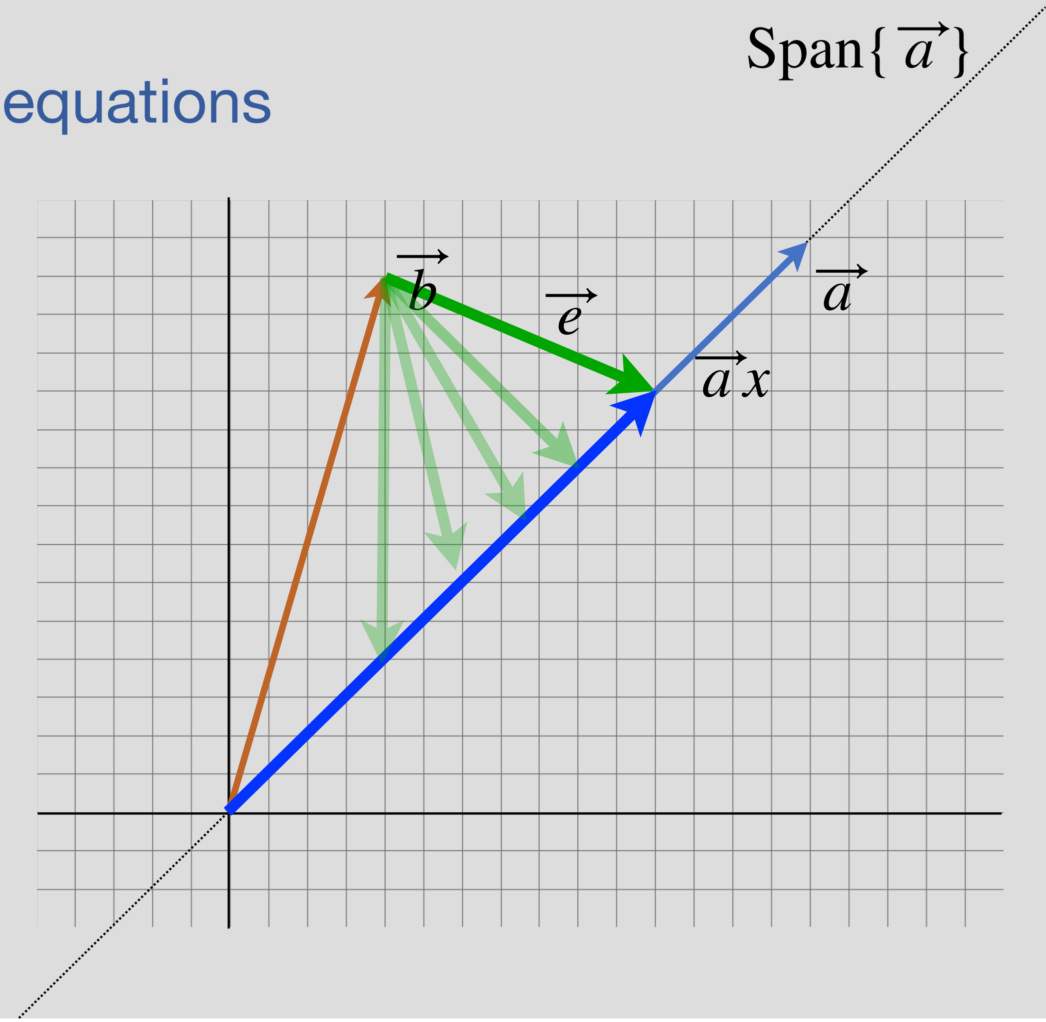
$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$, one unknown, two equations



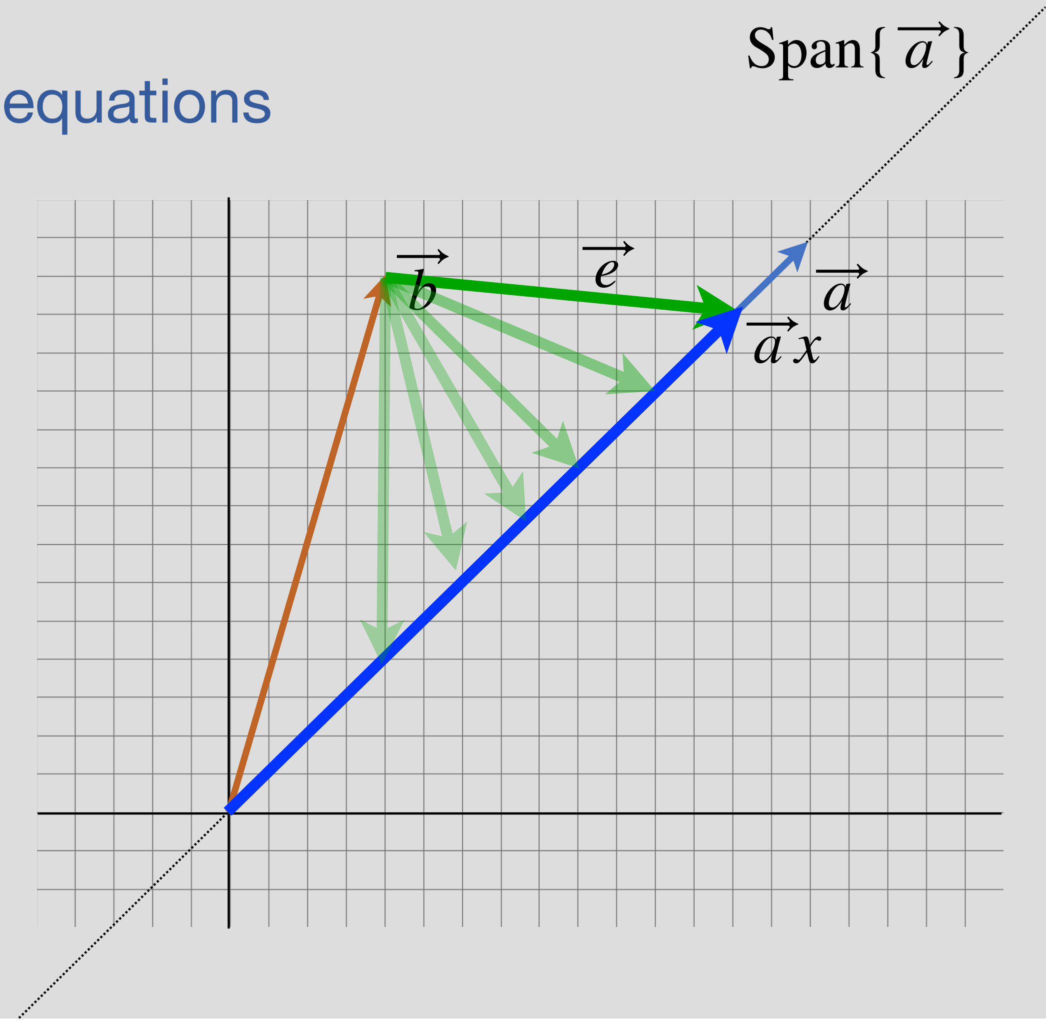
Solution:

find $\hat{x}$ that has the smallest error

$$\|\vec{e}\| = \|\vec{a}\hat{x} - \vec{b}\| \leq \|\vec{a}x - \vec{b}\|$$

Example: a scalar problem

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$, one unknown, two equations



Solution:

find $\hat{x}$ that has the smallest error

$$\|\vec{e}\| = \|\vec{a}\hat{x} - \vec{b}\| \leq \|\vec{a}x - \vec{b}\|$$

Example: a scalar problem

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$, one unknown, two equations

Solution:

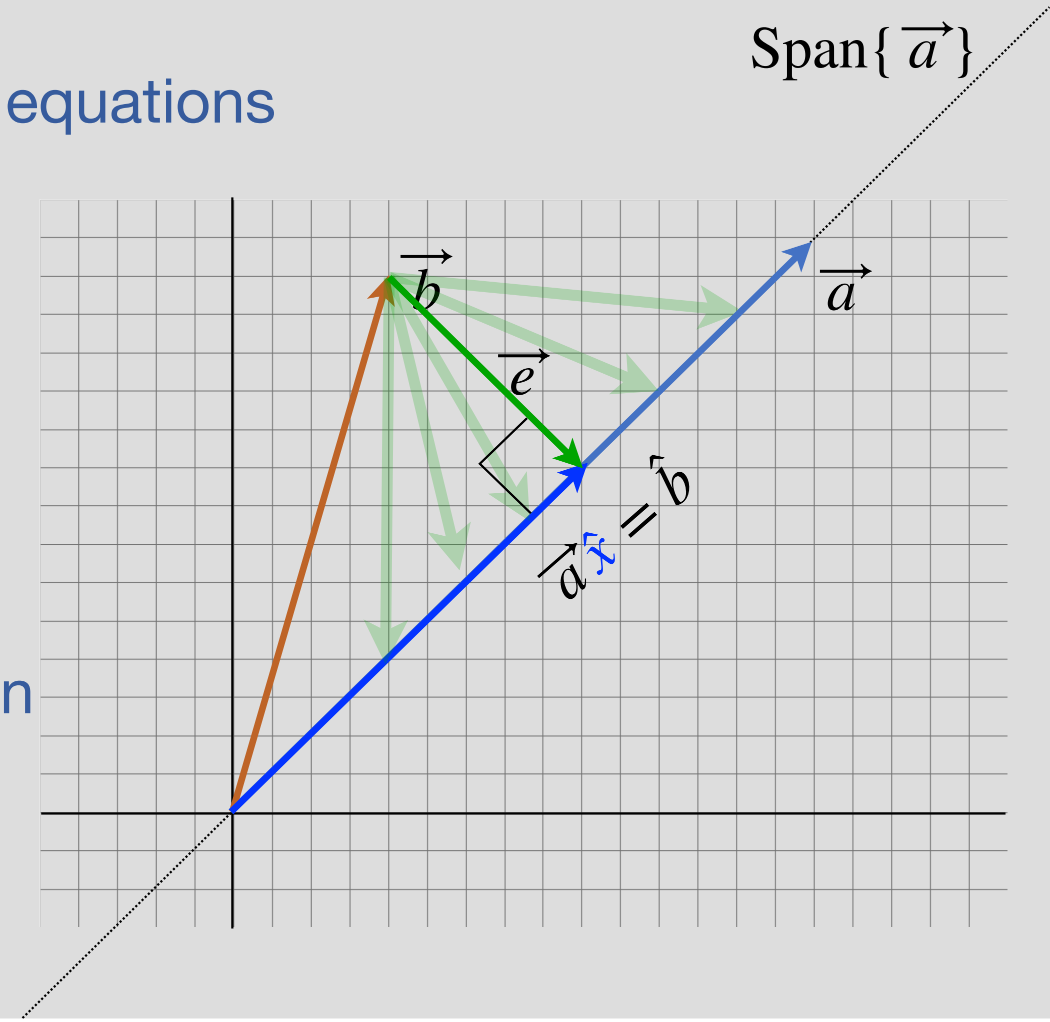
find $\hat{x}$ that has the smallest error

$$\|\vec{e}\| = \|\vec{a}\hat{x} - \vec{b}\| \leq \|\vec{a}x - \vec{b}\|$$

Theorem:

shortest distance between a point an

a line is the orthogonal projection



Projections

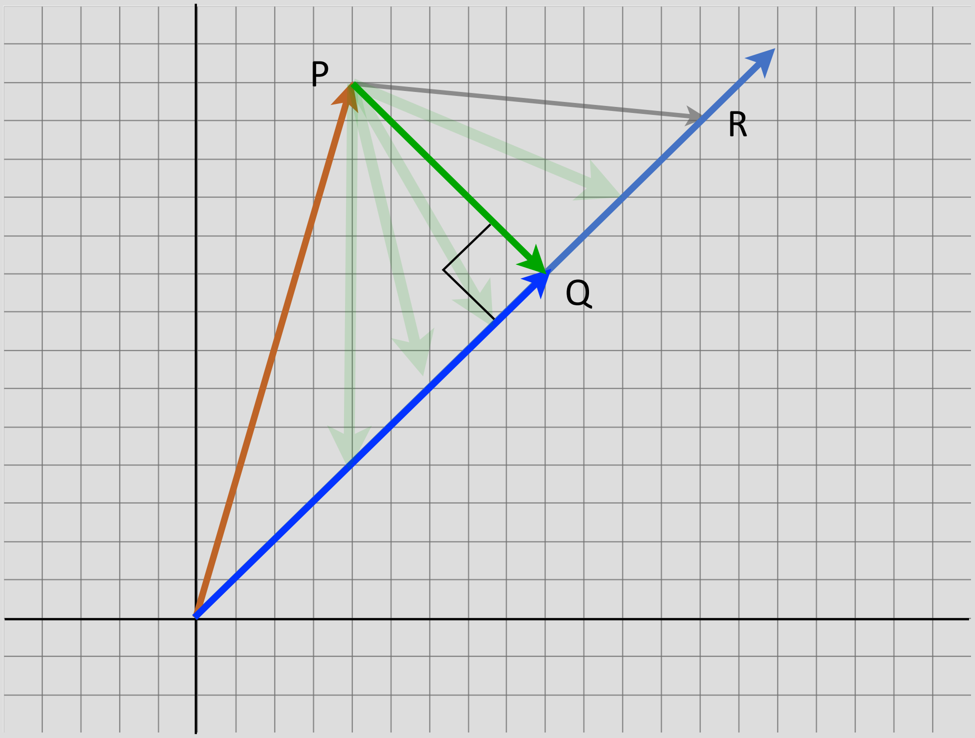
Theorem:
shortest distance between a point an a line is the orthogonal projection

Proof:

Pythagoras: $(PR)^2 = (PQ)^2 + (QR)^2$
> 0

$$(PR)^2 > (PQ)^2$$

$$(PR) > (PQ)$$



Projections

find $\hat{x}$ that has the smallest error

$\|\vec{e}\| = \|\vec{a}\hat{x} - \vec{b}\| \leq \|\vec{a}x - \vec{b}\|$

Need to find the orthogonal projection!

We know: $\vec{e} \perp \hat{b}, \vec{e} \perp \vec{a}$

$$\left\langle \vec{e}, \vec{a} \right\rangle = 0$$

$$\left\langle \vec{b} - \hat{b}, \vec{a} \right\rangle = 0$$

$$\left\langle \vec{b}, \vec{a} \right\rangle - \left\langle \hat{b}, \vec{a} \right\rangle = 0$$

$$\left\langle \vec{b}, \vec{a} \right\rangle = \left\langle \hat{b}, \vec{a} \right\rangle$$

$$\left\langle \vec{b}, \vec{a} \right\rangle = \left\langle \vec{a}\hat{x}, \vec{a} \right\rangle$$

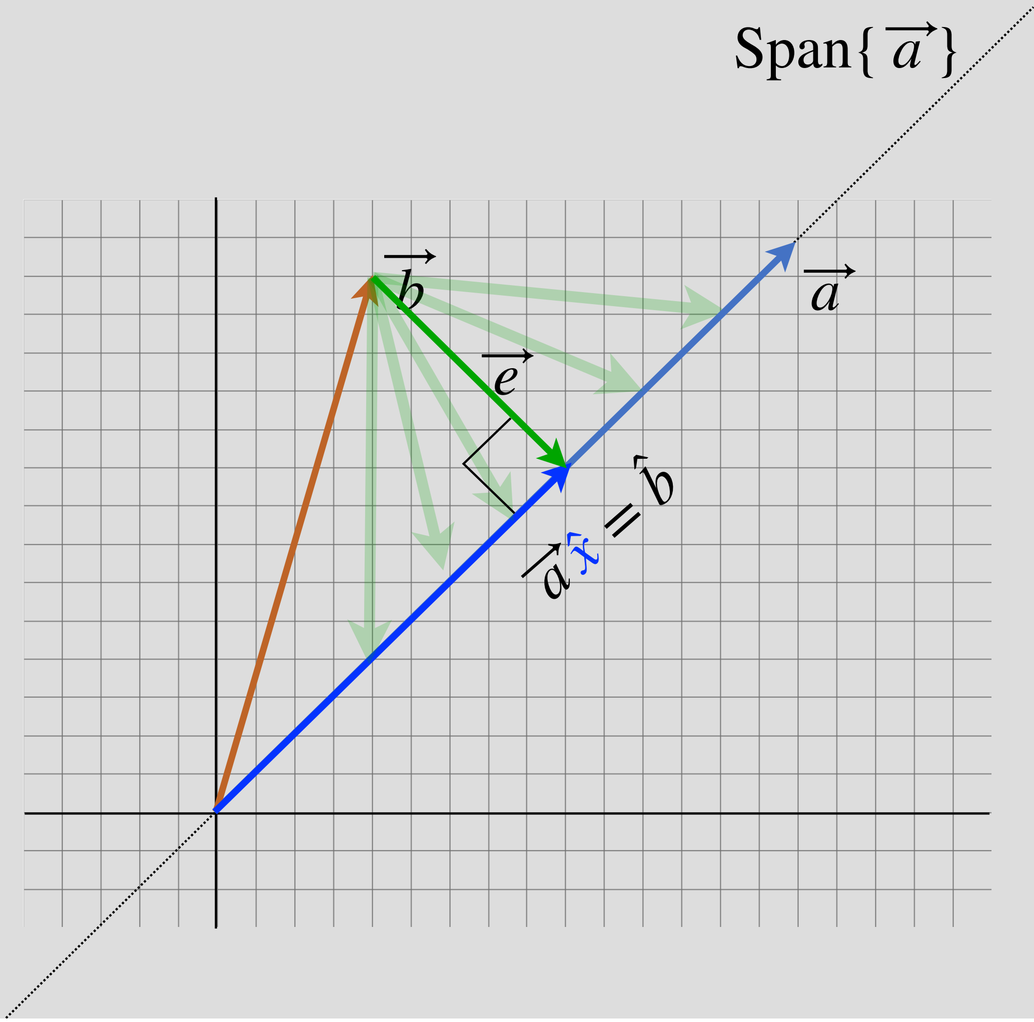
$$\left\langle \vec{b}, \vec{a} \right\rangle = \hat{x} \left\langle \vec{a}, \vec{a} \right\rangle$$

$$\left\langle \vec{b}, \vec{a} \right\rangle = \hat{x}\|\vec{a}\|^2$$

$$\hat{x} = \frac{\left\langle \vec{b}, \vec{a} \right\rangle}{\|\vec{a}\|^2}$$

$$\hat{b} = \frac{\left\langle \vec{b}, \vec{a} \right\rangle}{\|\vec{a}\|^2} \vec{a}$$

$$\hat{b} = \frac{\vec{b}^T \vec{a}}{\vec{a}^T \vec{a}} \vec{a}$$



Orthogonal Projections

Given vectors $\overrightarrow{a}$, $\overrightarrow{b}$, we say that the orthogonal projection of $\overrightarrow{b}$ onto $\overrightarrow{a}$ is:

$$\text{Proj}_{\overrightarrow{b}}(\overrightarrow{a}) = \frac{\overrightarrow{b}^T \overrightarrow{a}}{\|\overrightarrow{a}\|^2} \overrightarrow{a}$$

Example 2D

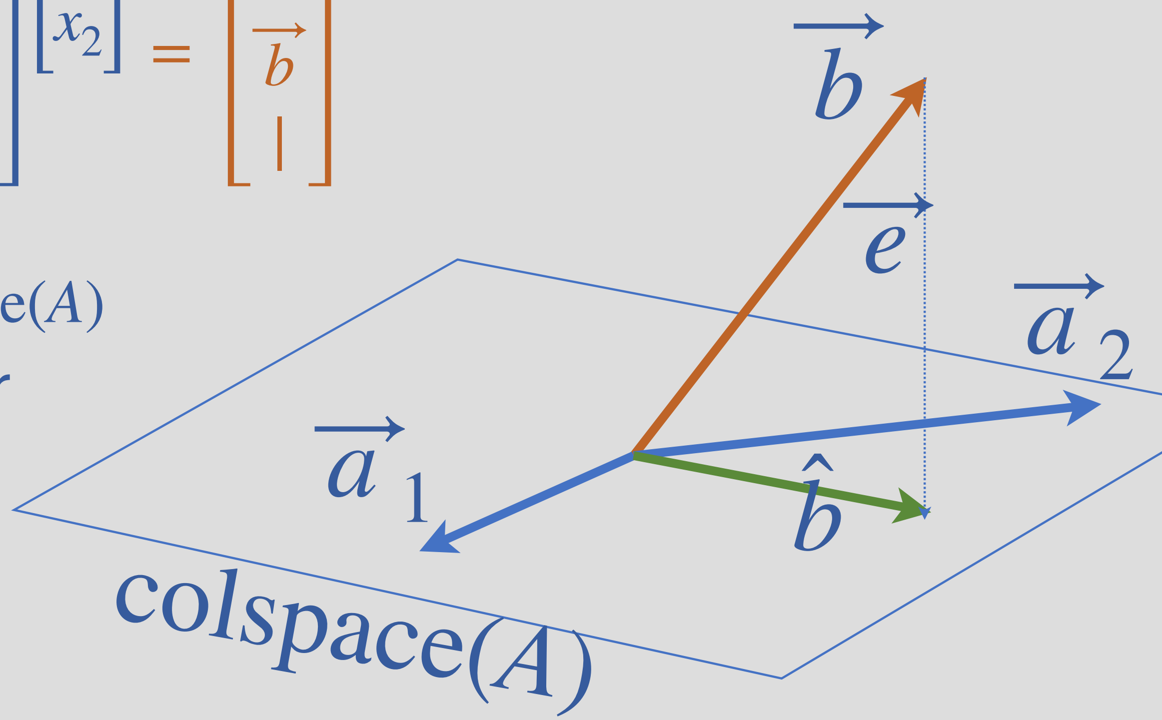
3 equations 2 unknowns:

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}} \qquad \begin{bmatrix} | & | \\ \vec{a}_1 & \vec{a}_2 \\ | & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} | \\ \vec{b} \\ | \end{bmatrix}$$

No solution means: $\vec{b} \notin \text{colspace}(A)$

Find $\hat{x}$ that has the smallest error

$$\|\vec{e}\| = \|A\hat{x} - \vec{b}\| \leq \|Ax - \vec{b}\|$$



Theorem: Consider matrix A, and $\overrightarrow{y} \in \text{colspace}(A)$

If $\exists \vec{z}$, such that $\langle \vec{z}, \overrightarrow{a}_i \rangle = 0$, then $\langle \vec{z}, \overrightarrow{y} \rangle = 0$.

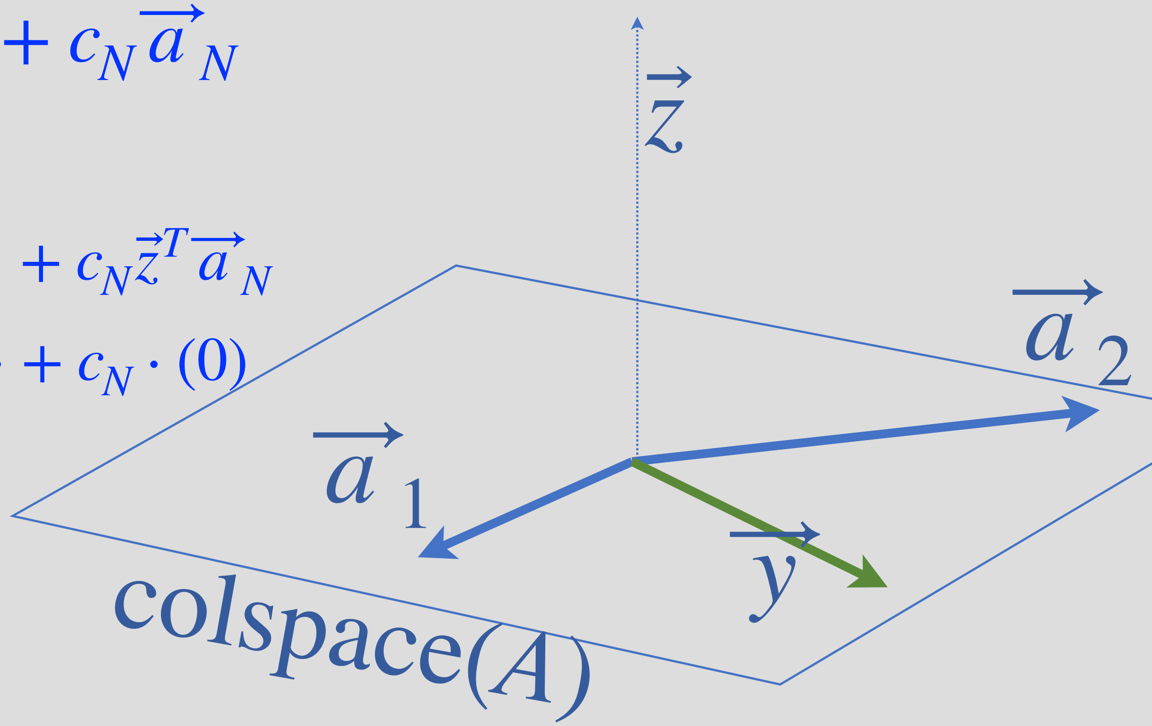
$$A = \begin{bmatrix} | & | & & | \\ \overrightarrow{a}_1 & \overrightarrow{a}_2 & \cdots & \overrightarrow{a}_N \\ | & | & & | \end{bmatrix}$$

Proof:

Know: $\overrightarrow{y} = c_1 \overrightarrow{a}_1 + c_2 \overrightarrow{a}_2 + \cdots + c_N \overrightarrow{a}_N$

Show: $\langle \vec{z}, \overrightarrow{y} \rangle = 0$

$$\begin{aligned} \langle \vec{z}, c_1 \overrightarrow{a}_1 + \cdots + c_N \overrightarrow{a}_N \rangle &= c_1 \vec{z}^T \overrightarrow{a_1} + \cdots + c_N \vec{z}^T \overrightarrow{a}_N \\ &= c_1 \cdot (0) + \cdots + c_N \cdot (0) \\ &= 0 \end{aligned}$$



Least Squares

$$\|\vec{e}\| = \|A\vec{x} - \vec{b}\|$$

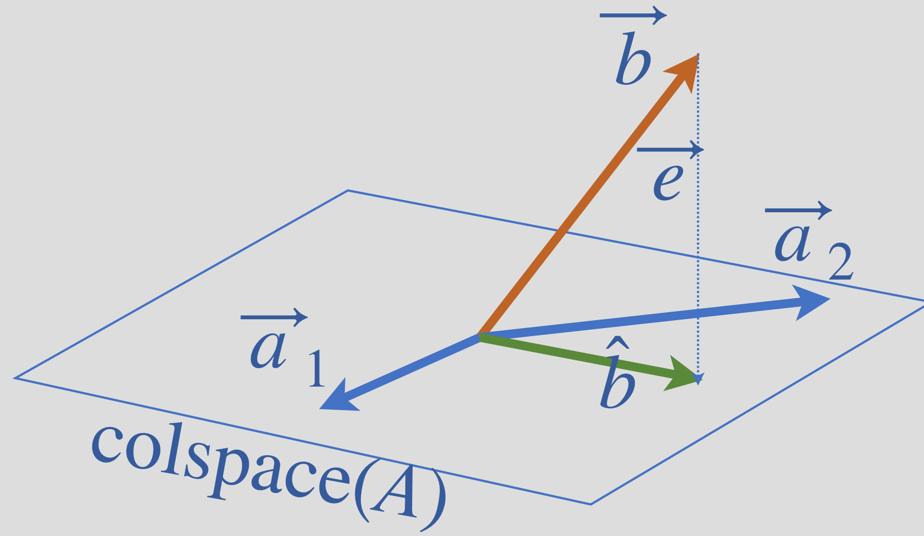
$$\vec{e} = \vec{b} - \hat{b}$$

Since $\vec{e} \perp \text{col}(A)$, $\left\langle \vec{a}_i, \vec{e} \right\rangle = 0$

$$\left\langle \vec{a}_i, \vec{b} - \hat{b} \right\rangle = 0$$

$$\vec{a}_i^T(\vec{b} - \hat{b}) = 0$$

$$\begin{bmatrix} - & \vec{a}_1^T & - \\ - & \vec{a}_2^T & - \\ & \vdots & \\ - & \vec{a}_N^T & - \end{bmatrix} \begin{bmatrix} | \\ \vec{b} - \hat{b} \\ | \end{bmatrix} = 0$$



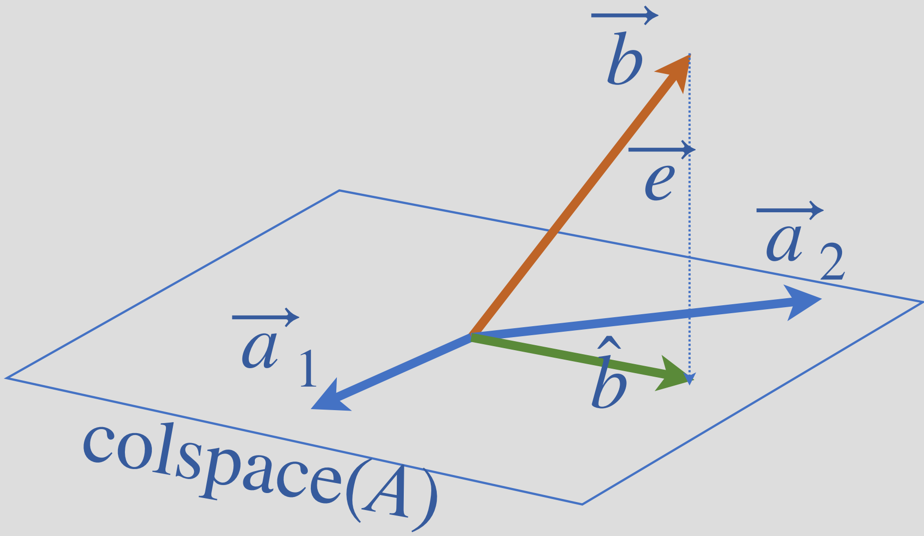
$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_N \\ | & | & & | \end{bmatrix}$$

$A\vec{x} \in \text{colspace}(A)$

Find $\hat{b} = A\hat{x}$

Least Squares

$$\begin{bmatrix} - & \vec{a}_1^T & - \\ - & \vec{a}_2^T & - \\ & \vdots & \\ - & \vec{a}_N^T & - \end{bmatrix} \begin{bmatrix} | \\ \vec{b} - \hat{b} \\ | \end{bmatrix} = 0$$



$$A^T\left(\vec{b} - A\hat{x}\right) = 0$$

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_N \\ | & | & & | \end{bmatrix}$$

$$A^T\vec{b} - A^T A\hat{x} = 0$$

$$A^T A\hat{x} = A^T\vec{b}$$

$A\vec{x} \in \text{colspace}(A)$

→ Find $\hat{b} = A\hat{x}$

If A is full Rank, then $A^T A$ is invertible

$$\hat{x} = (A^T A)^{-1} A^T \vec{b}$$

$$\hat{b} = A(A^T A)^{-1} A^T \vec{b}$$

$$\hat{x} = (A^T A)^{-1} A^T \vec{b}$$