

Welcome to EECS 16A!

Designing Information Devices and Systems I



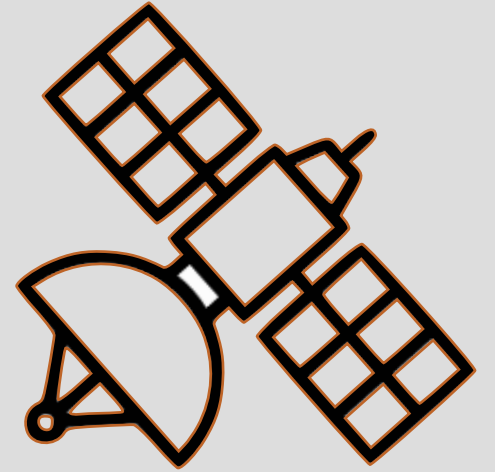
Ana Arias and Miki Lustig
Fall 2021

Lecture 14A
Even More Least squares



GPS

- 24 satellites
 - Known position
 - Time synchronized
 - 8 usually visible
- Problem:
 - Classify which satellite is transmitting
 - Estimate distance to GPS
 - Estimate position from noisy data
- Tools:
 - Inner product
 - Cross correlation
 - Least Squares



Least Squares

$$\begin{bmatrix} - & \vec{a}_1^T & - \\ - & \vec{a}_2^T & - \\ & \vdots & \\ - & \vec{a}_N^T & - \end{bmatrix} \begin{bmatrix} | \\ \vec{b} - \hat{\vec{b}} \\ | \end{bmatrix} = 0$$

$$A^T (\vec{b} - A\hat{x}) = 0$$

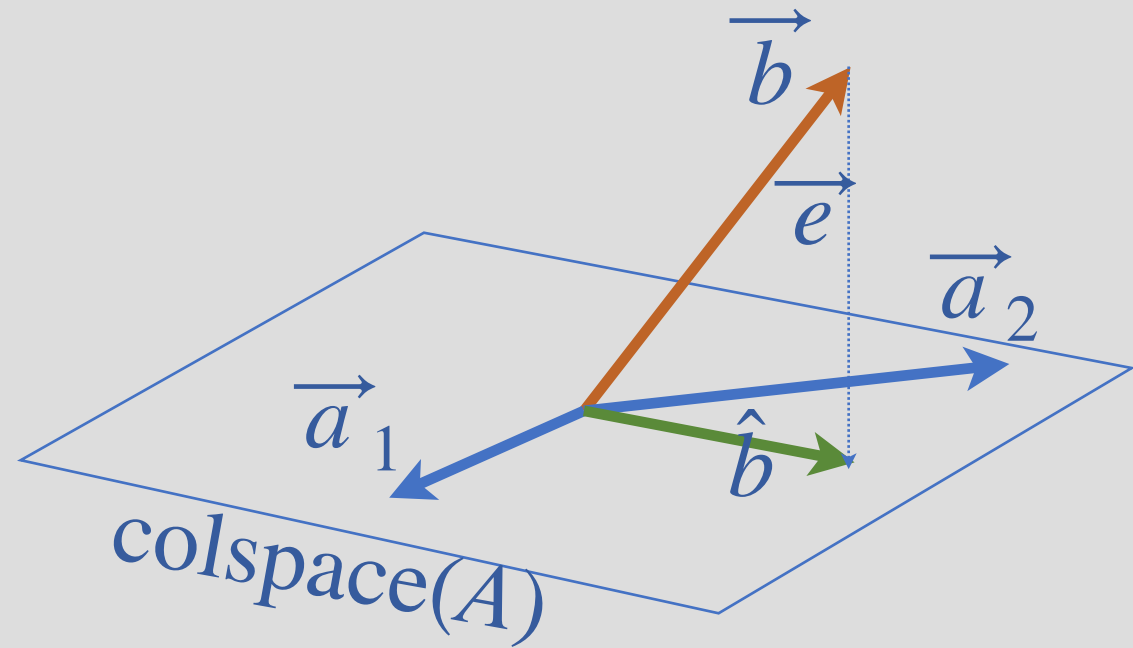
$$A^T \vec{b} - A^T A \hat{x} = 0$$

$$A^T A \hat{x} = A^T \vec{b}$$

If A is full Rank, then $A^T A$ is invertible

$$\hat{x} = (A^T A)^{-1} A^T \vec{b}$$

$$\hat{\vec{b}} = A(A^T A)^{-1} A^T \vec{b}$$



$$A = \begin{bmatrix} | & | & \dots & | \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_N \\ | & | & & | \end{bmatrix}$$

$A\vec{x} \in \text{colspace}(A)$
Find $\hat{\vec{b}} = A\hat{x}$

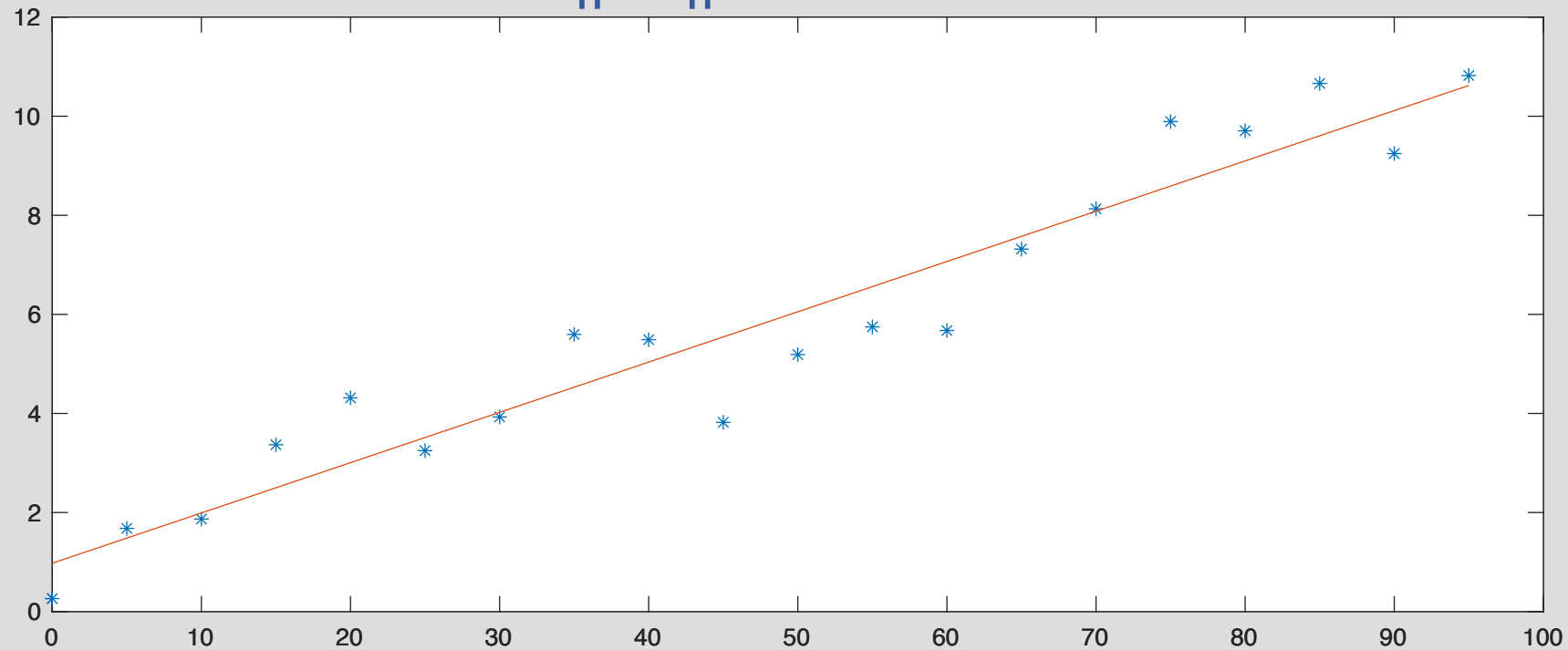
Example 5: Over Fitting

- Consider noisy measurements of $y = 0.1x + 1$:

Model: $y = ax + b$

$$\vec{p} = [0.1015 \quad 0.9757]^T$$

$$\|\vec{e}\| = 3.85$$

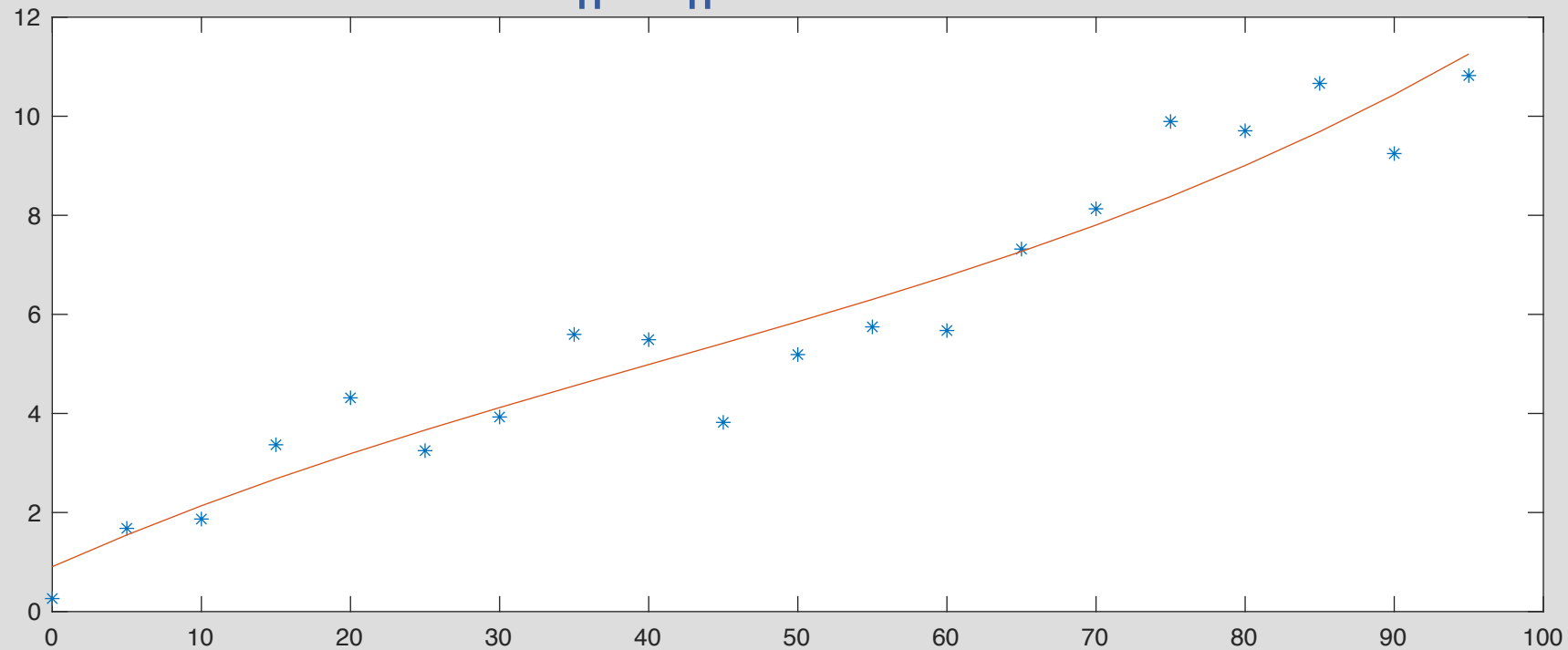


Example 5: Over Fitting

- Consider noisy measurements of $y = 0.1x + 1$:

Model: $y = ax^3 + bx^2 + cx + d$

$$\|\vec{e}\| = 3.71$$

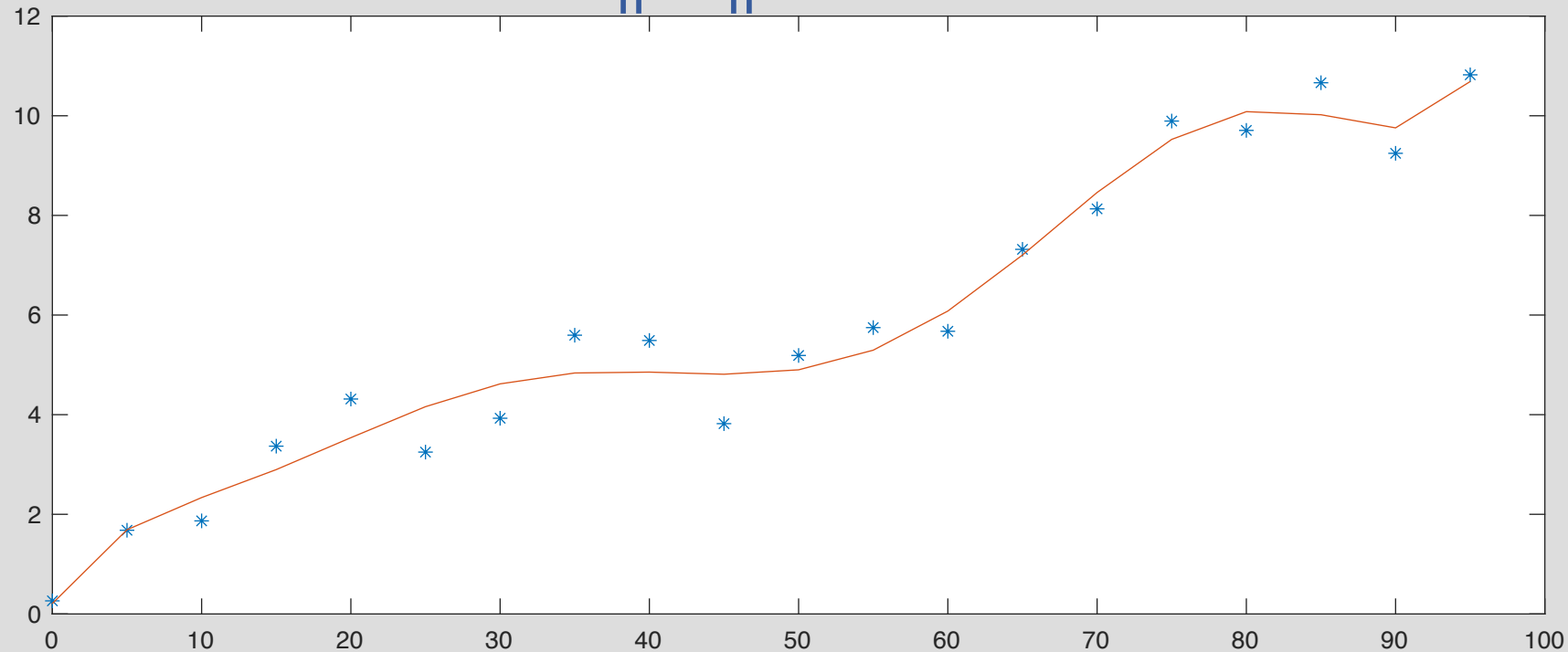


Example 5: Over Fitting

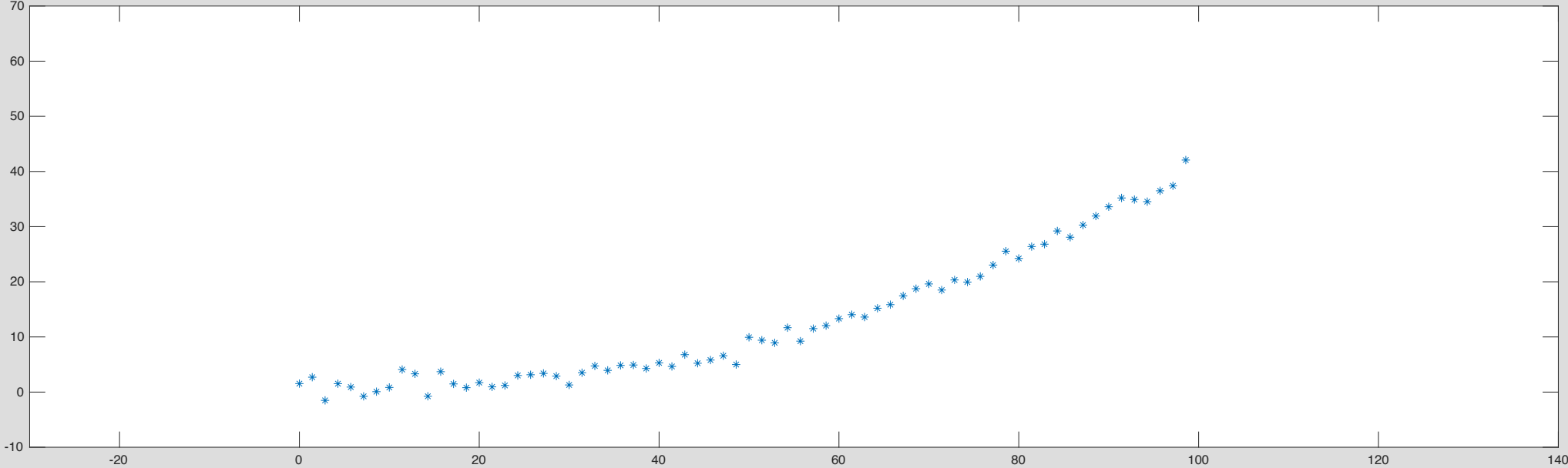
- Consider noisy measurements of $y = 0.1x + 1$:

Model: $y = ax^7 + bx^6 + cx^5 + dx^4 + ex^3 + fx^2 + gx + h$

$$\|\vec{e}\| = 2.42$$

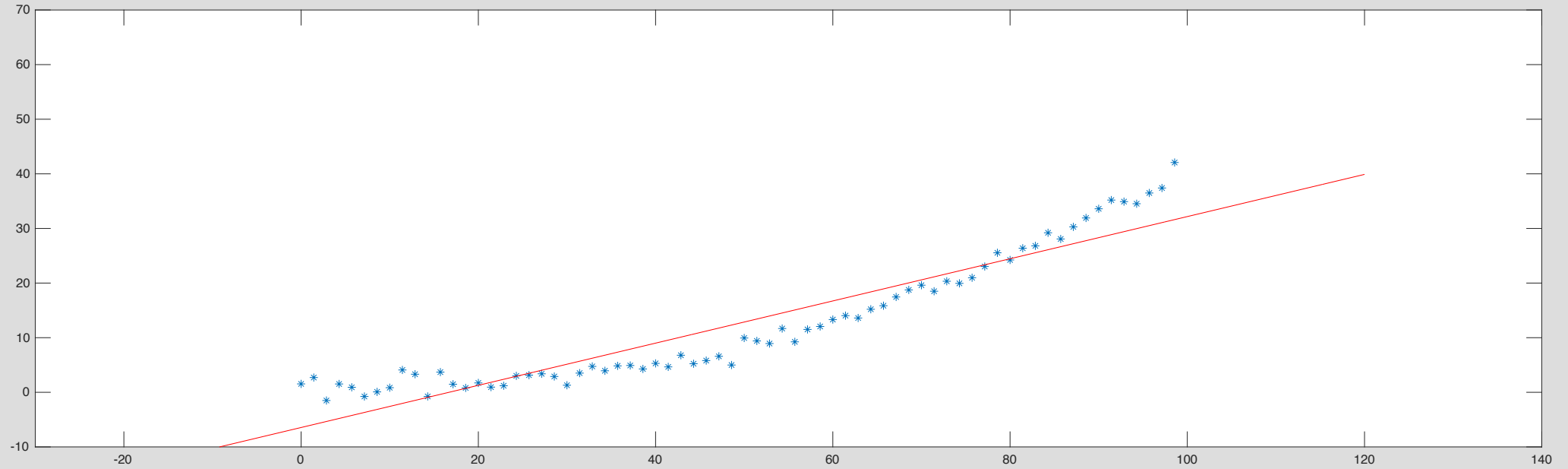


Example 6: Model Order Selection



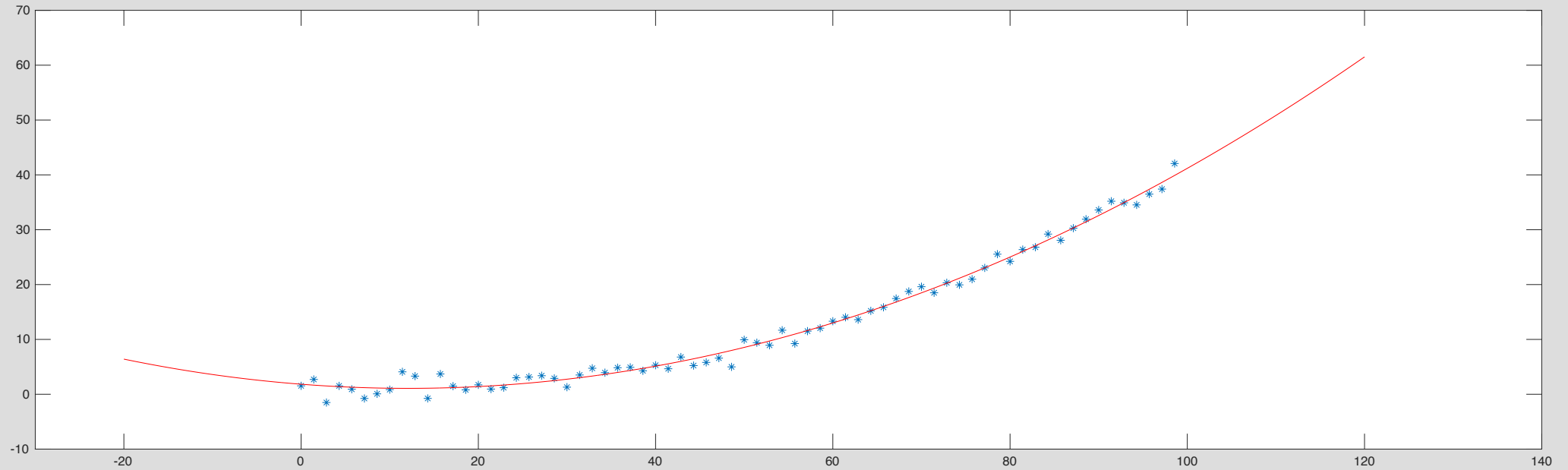
Example 6: Model Order Selection

Model: $y = ax + b$



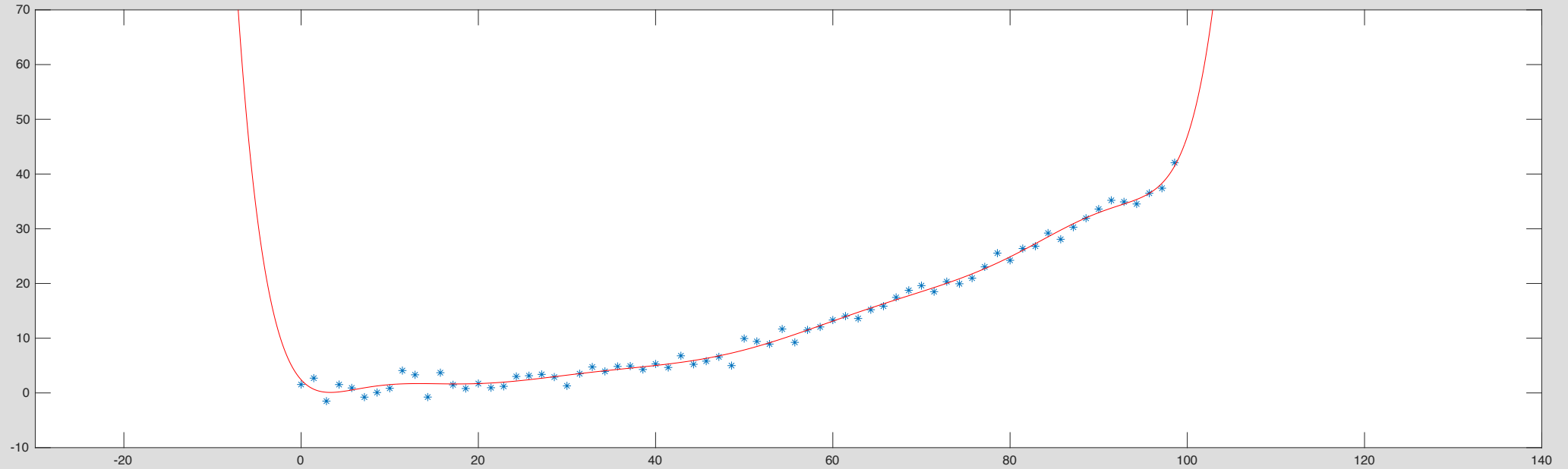
Example 6: Model Order Selection

Model: $y = ax^2 + bx + c$

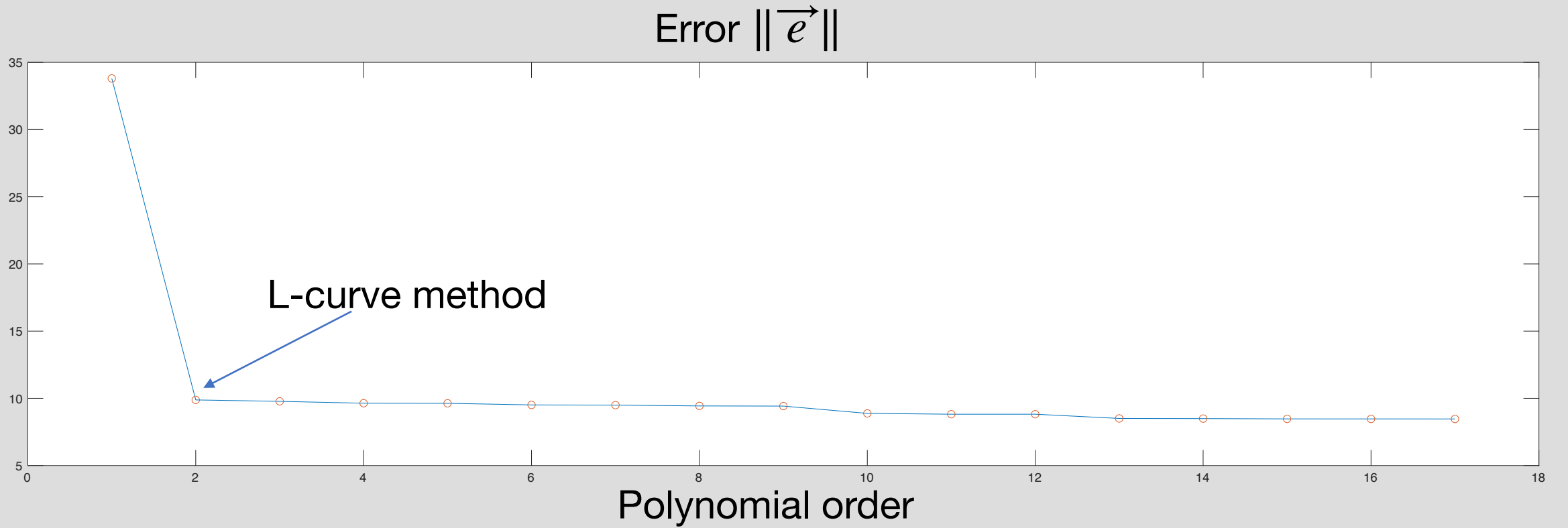


Example 6: Model Order Selection

Model: $y = ax^{10} + bx^9 + cx^8 + dx^7 + ex^6 + fx^5 + gx^4 + hx^3 + ix^2 + jx + k$



Example 6: Model Order Selection

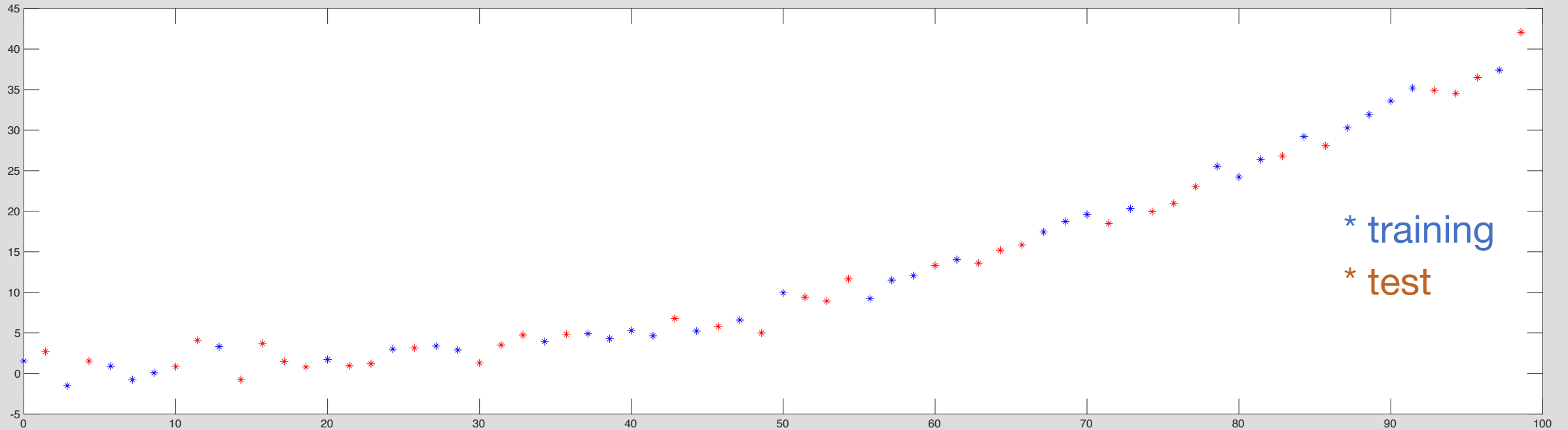


Example 6: Model Order Selection

Split data into training / test sets

Fit model on training set

Evaluate error on test set

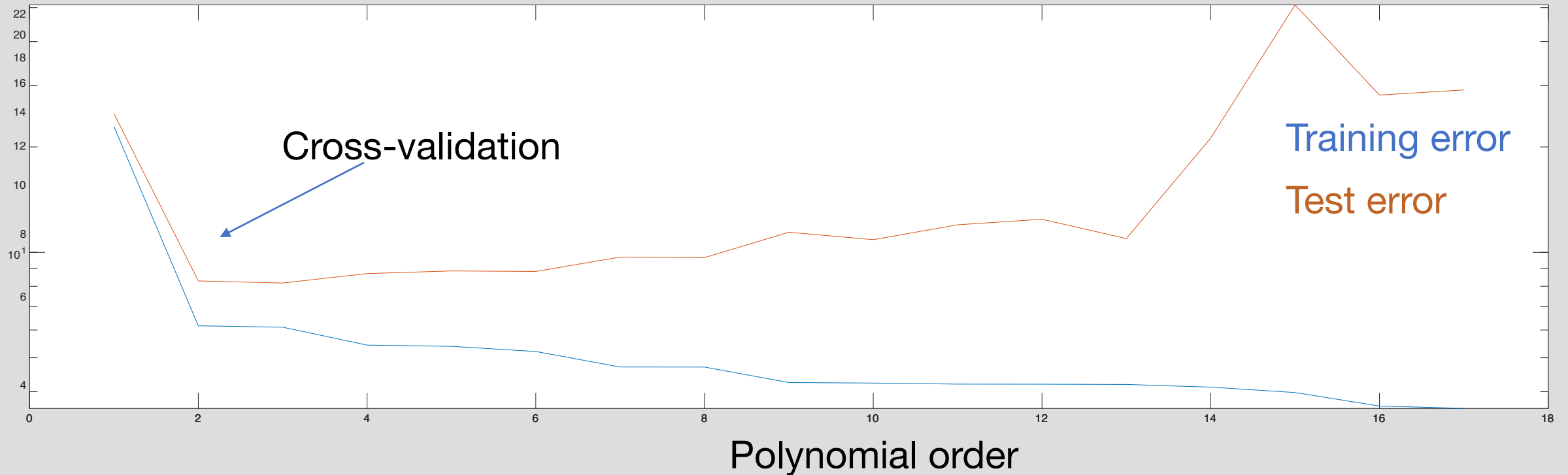


Example 6: Model Order Selection

Split data into training / test sets

Fit model on training set

Evaluate error on test set



Multi-Lateration

$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} - 2C^2 \Delta\tau_2 \tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\Delta\tau_2)^2$$

$$2(\vec{a}_1 - \vec{a}_3)^T \vec{x} - 2C^2 \Delta\tau_3 \tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + C^2(\Delta\tau_3)^2$$

$$2(\vec{a}_1 - \vec{a}_4)^T \vec{x} - 2C^2 \Delta\tau_4 \tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_4\|^2 + C^2(\Delta\tau_4)^2$$

$$2(\vec{a}_1 - \vec{a}_5)^T \vec{x} - 2C^2 \Delta\tau_5 \tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_5\|^2 + C^2(\Delta\tau_5)^2$$

More equations than unknowns

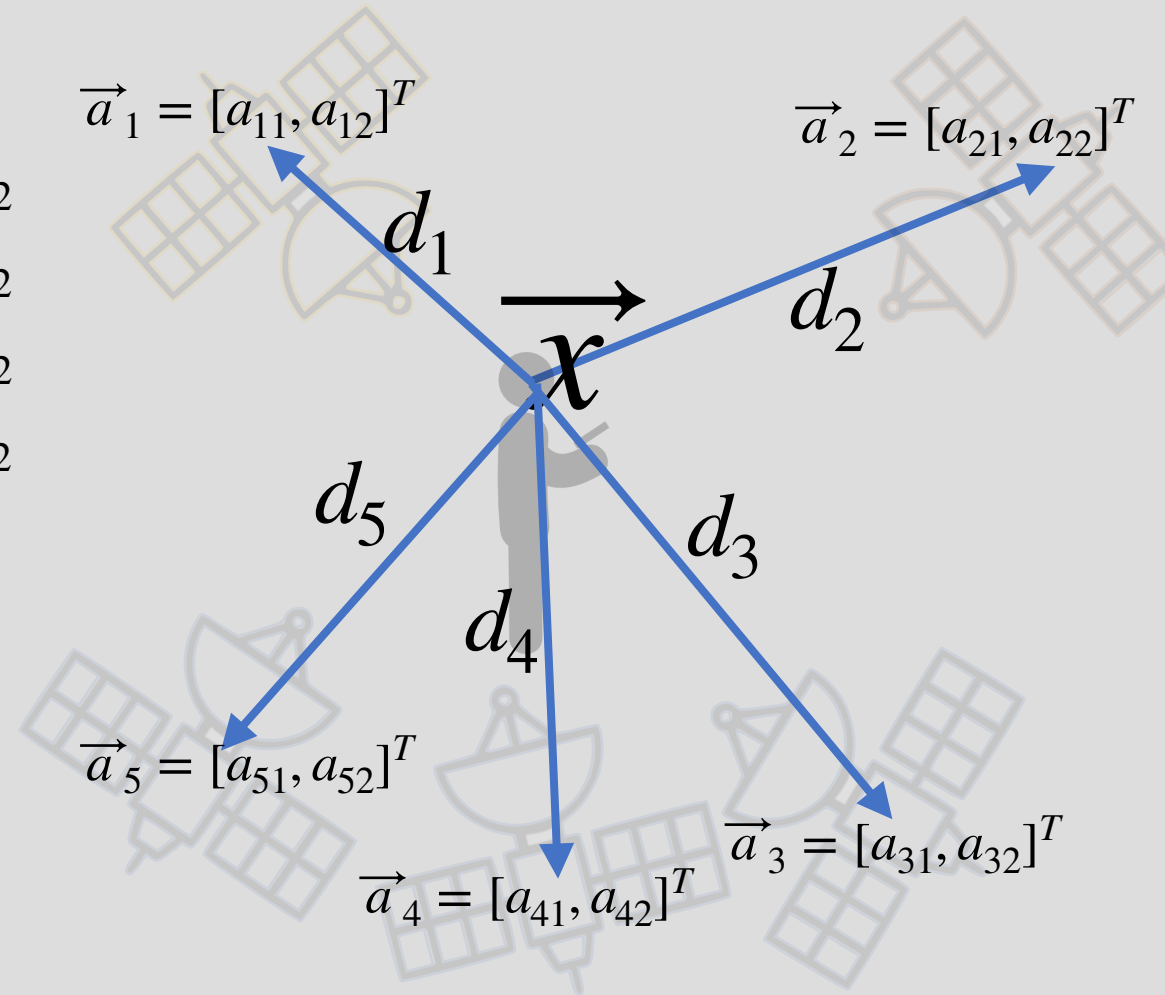
$$\begin{matrix} \boxed{A} \\ \vec{p} \end{matrix} = \begin{matrix} \vec{b} \end{matrix}$$

Over-determined — Solve via Least-Squares

Q: How do we know if $A^T A$ is invertible?

A: if A is full rank!?!?

$$\hat{p} = (A^T A)^{-1} A^T \vec{y}$$



Matrix Transposes

$$(AB)^T \left(\begin{array}{|c|c|} \hline \color{red}{M} & \color{red}{L} \\ \hline \color{blue}{N} & \color{blue}{M} \\ \hline \end{array} \right)^T = \left(\begin{array}{|c|} \hline \color{red}{L} \\ \hline \color{blue}{N} \\ \hline \end{array} \right)^T \in \mathbb{R}^{L \times N} \quad \begin{array}{|c|} \hline \color{red}{N} \\ \hline \color{blue}{L} \\ \hline \end{array}$$

$$B^T A^T \begin{array}{|c|c|} \hline \color{red}{L} & \color{red}{M} \\ \hline \color{blue}{M} & \color{blue}{N} \\ \hline \end{array} \in \mathbb{R}^{L \times N} \quad \begin{array}{|c|} \hline \color{red}{N} \\ \hline \color{blue}{L} \\ \hline \end{array}$$

$$(AB)^T = B^T A^T$$

Invertibility of $A^T A$

- Invertible \Rightarrow Trivial null space \Rightarrow Linear independent cols/rows....

The matrix $A^T A$ is invertible iff $\text{Null}(A^T A) = \vec{0}$

Theorem: $\text{Null}(A^T A) = \text{Null}(A)$

Proof: (1) show that if $\vec{w} \in \text{Null}(A)$, then $\vec{w} \in \text{Null}(A^T A)$
(2) show that if $\vec{v} \in \text{Null}(A^T A)$, then $\vec{v} \in \text{Null}(A)$

$$\begin{aligned} (1). \quad & \vec{w} \in \text{Null}(A) \\ & A\vec{w} = \vec{0} \\ & A^T A\vec{w} = A^T \vec{0} \\ & A^T A\vec{w} = \vec{0} \quad \checkmark \end{aligned}$$

$$\begin{aligned} (2). \quad & \vec{v} \in \text{Null}(A^T A) \\ & A^T A\vec{v} = \vec{0} \quad \text{Need to show } A\vec{v} = \vec{0} \\ & \quad \text{Or... } \|A\vec{v}\| = 0 \\ & \|A\vec{v}\|^2 = (A\vec{v})^T (A\vec{v}) \\ & \quad = \vec{v}^T A^T (A\vec{v}) \\ & \quad = \vec{v}^T (A^T A\vec{v}) = 0 \quad \checkmark \end{aligned}$$

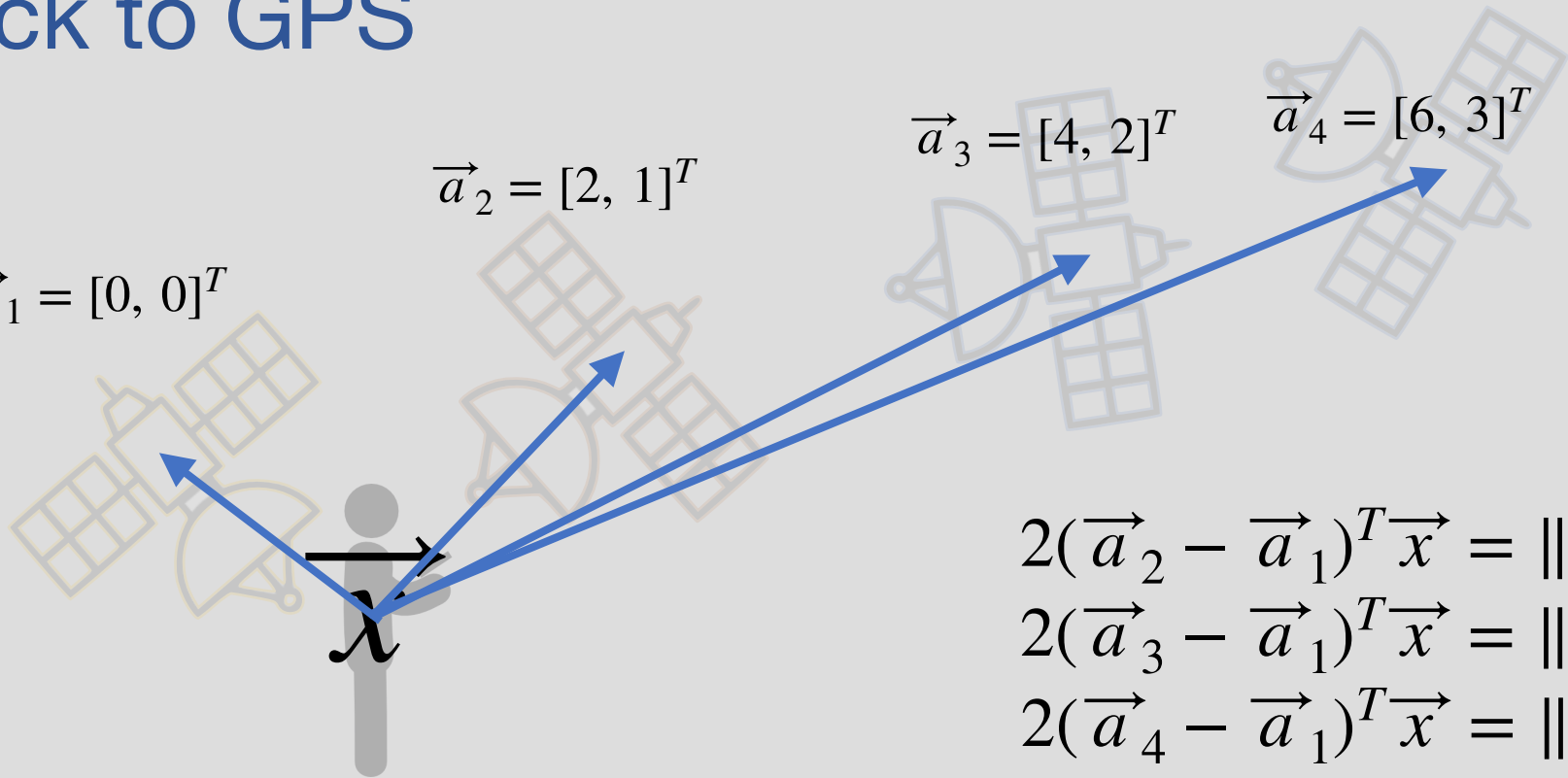
Back to GPS

$$\vec{a}_1 = [0, 0]^T$$

$$\vec{a}_2 = [2, 1]^T$$

$$\vec{a}_3 = [4, 2]^T$$

$$\vec{a}_4 = [6, 3]^T$$



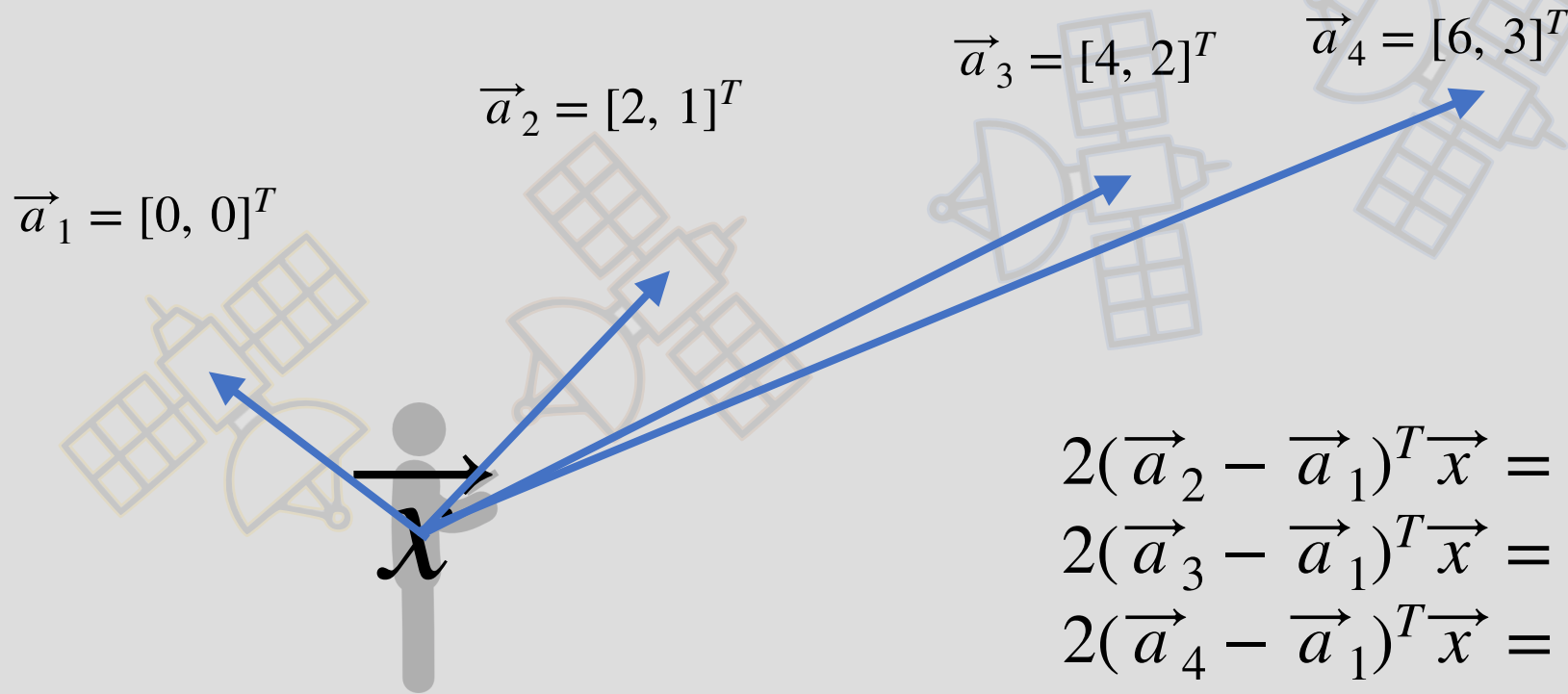
$$2(\vec{a}_2 - \vec{a}_1)^T \vec{x} = \|\vec{a}_2\|^2 - \|\vec{a}_1\|^2 + d_1^2 - d_2^2$$

$$2(\vec{a}_3 - \vec{a}_1)^T \vec{x} = \|\vec{a}_3\|^2 - \|\vec{a}_1\|^2 + d_1^2 - d_3^2$$

$$2(\vec{a}_4 - \vec{a}_1)^T \vec{x} = \|\vec{a}_4\|^2 - \|\vec{a}_1\|^2 + d_1^2 - d_4^2$$

$$\begin{bmatrix} 2(\vec{a}_2 - \vec{a}_1)^T \\ 2(\vec{a}_3 - \vec{a}_1)^T \\ 2(\vec{a}_4 - \vec{a}_1)^T \end{bmatrix} \vec{x} = \begin{bmatrix} \|\vec{a}_2\|^2 - \|\vec{a}_1\|^2 + d_1^2 - d_2^2 \\ \|\vec{a}_3\|^2 - \|\vec{a}_1\|^2 + d_1^2 - d_3^2 \\ \|\vec{a}_4\|^2 - \|\vec{a}_1\|^2 + d_1^2 - d_4^2 \end{bmatrix}$$

Back to GPS



$$2(\vec{a}_2 - \vec{a}_1)^T \vec{x} = \|\vec{a}_2\|^2 - \|\vec{a}_1\|^2 + d_1^2 - d_2^2$$

$$2(\vec{a}_3 - \vec{a}_1)^T \vec{x} = \|\vec{a}_3\|^2 - \|\vec{a}_1\|^2 + d_1^2 - d_3^2$$

$$2(\vec{a}_4 - \vec{a}_1)^T \vec{x} = \|\vec{a}_4\|^2 - \|\vec{a}_1\|^2 + d_1^2 - d_4^2$$

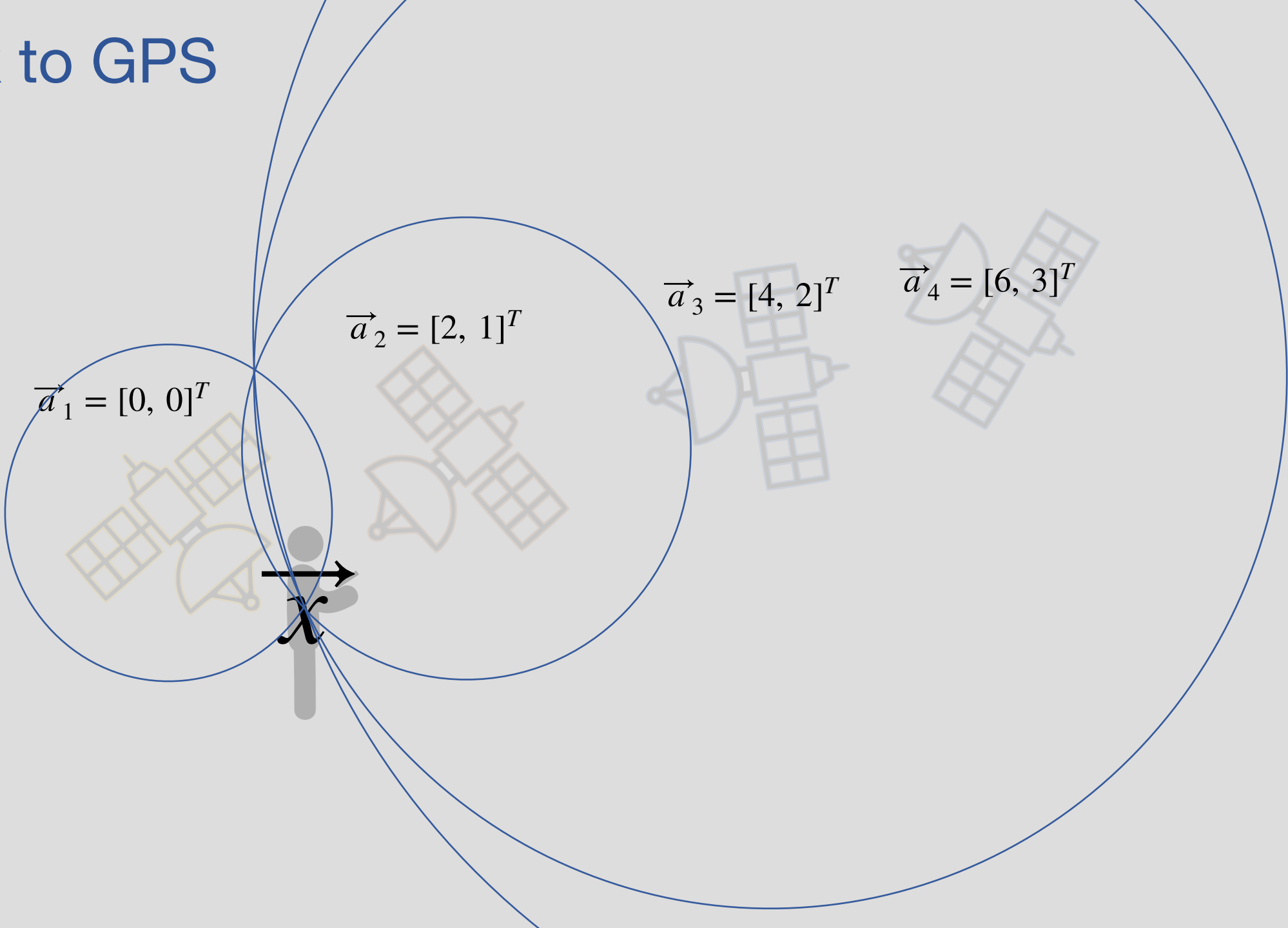
$$\begin{bmatrix} 4 & 2 \\ 8 & 4 \\ 12 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \|\vec{a}_2\|^2 + d_1^2 - d_2^2 \\ \|\vec{a}_3\|^2 + d_1^2 - d_3^2 \\ \|\vec{a}_4\|^2 + d_1^2 - d_4^2 \end{bmatrix}$$

Back to GPS

$$\begin{bmatrix} 4 & 2 \\ 8 & 4 \\ 12 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \|\vec{a}_2\|^2 + d_1^2 - d_2^2 \\ \|\vec{a}_3\|^2 + d_1^2 - d_3^2 \\ \|\vec{a}_4\|^2 + d_1^2 - d_4^2 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 8 & 12 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 8 & 4 \\ 12 & 6 \end{bmatrix} = \begin{bmatrix} 224 & 112 \\ 112 & 56 \end{bmatrix}$$

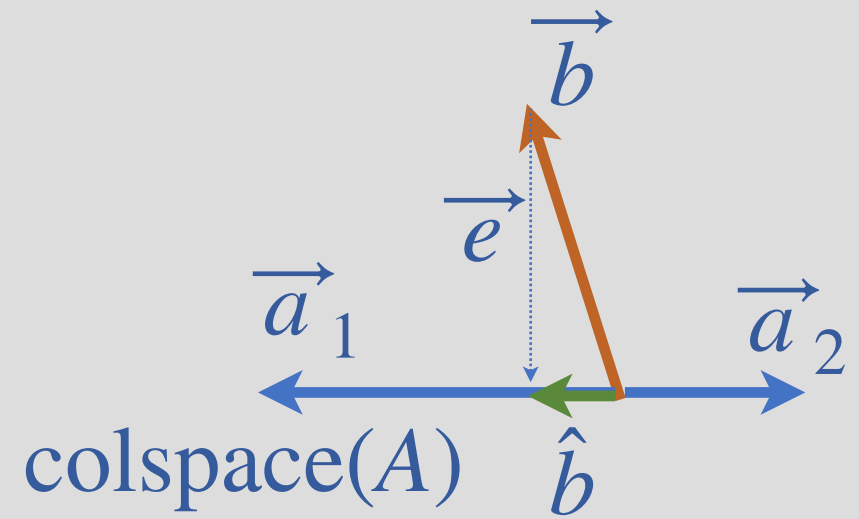
Back to GPS



Invertibility of $A^T A$

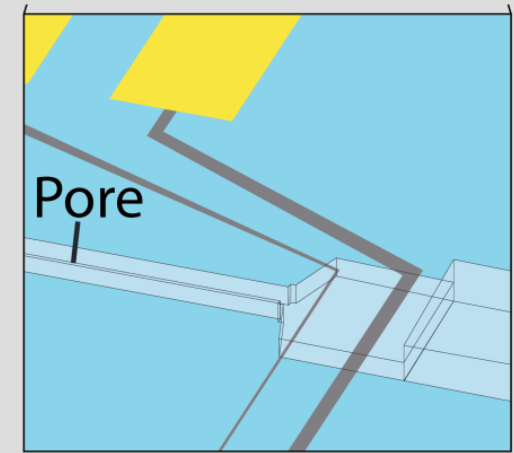
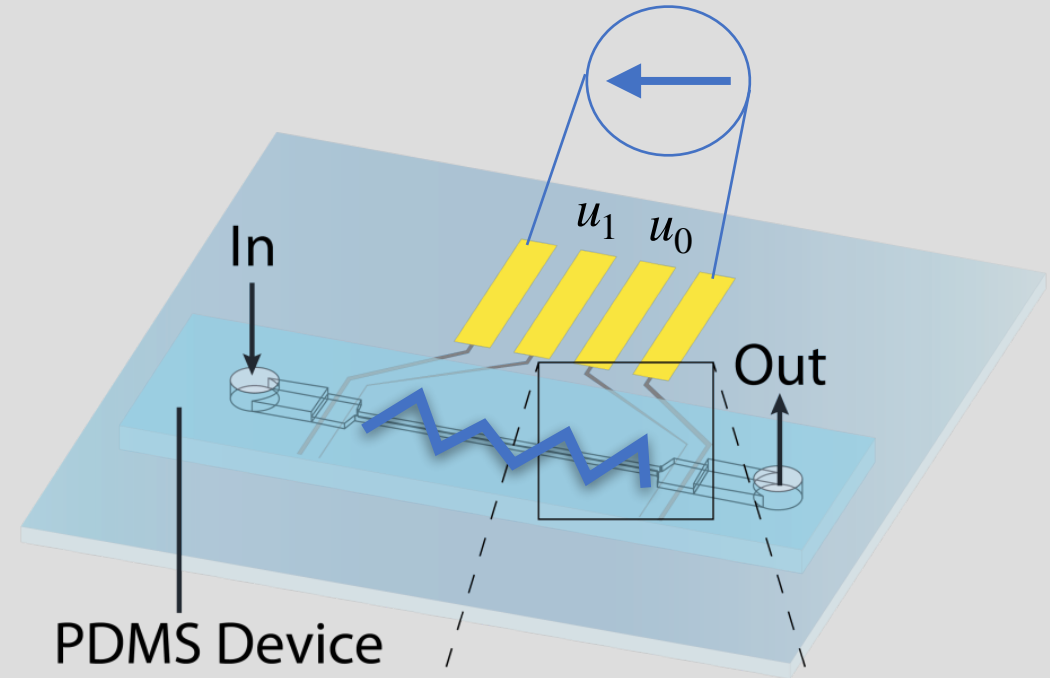
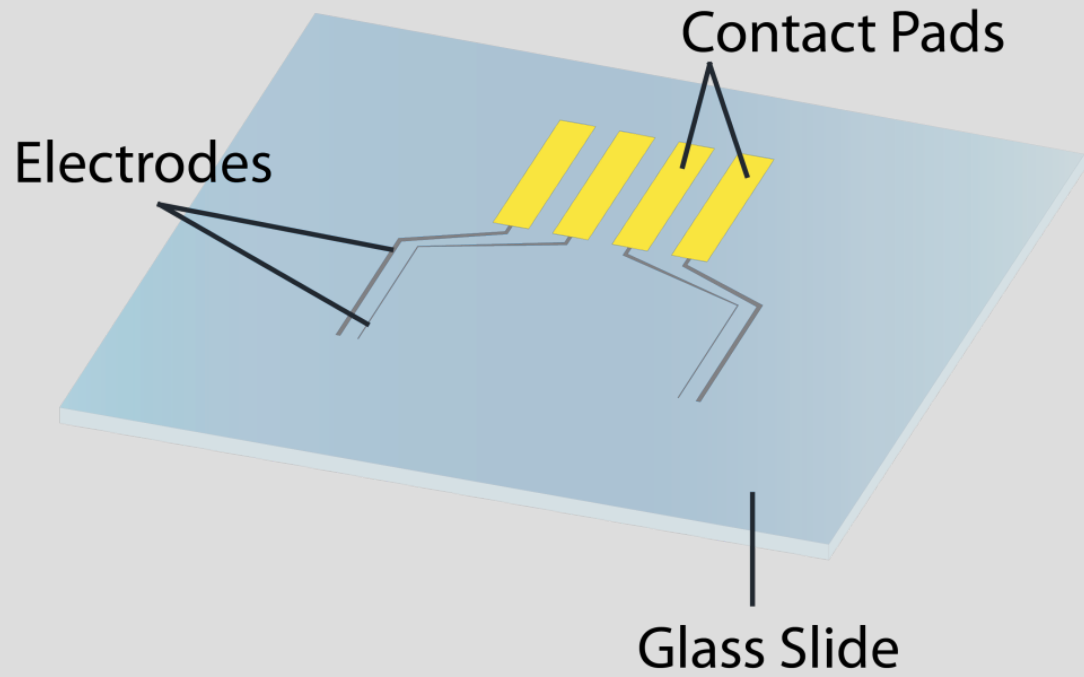
- What if $A^T A$ is not invertible

$$A^T A \hat{x} = A^T \vec{b}$$



A: \hat{x} will have infinite solutions with the same $\vec{e} = A\hat{x} - \vec{b}$

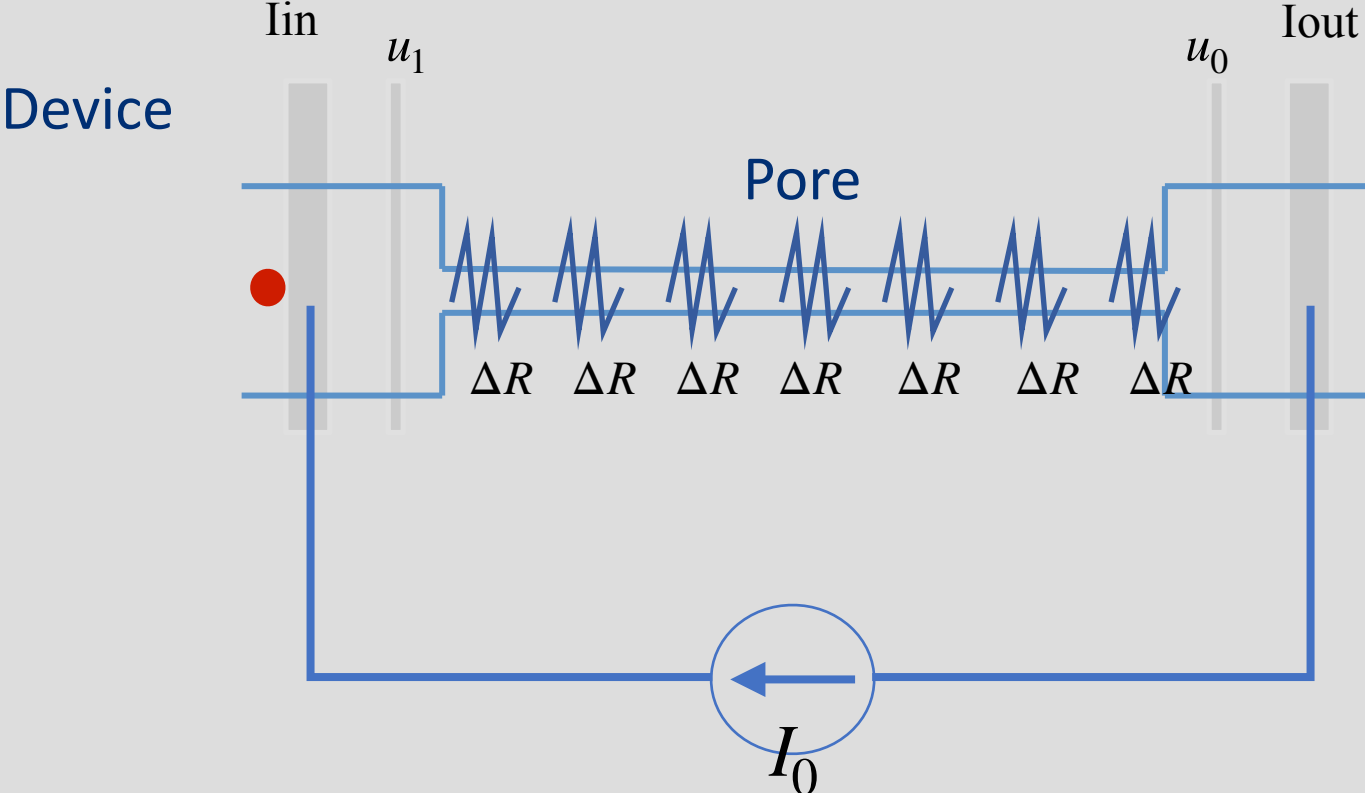
Resistive Pulse Sensing



Prof. Lydia Sohn M.E.

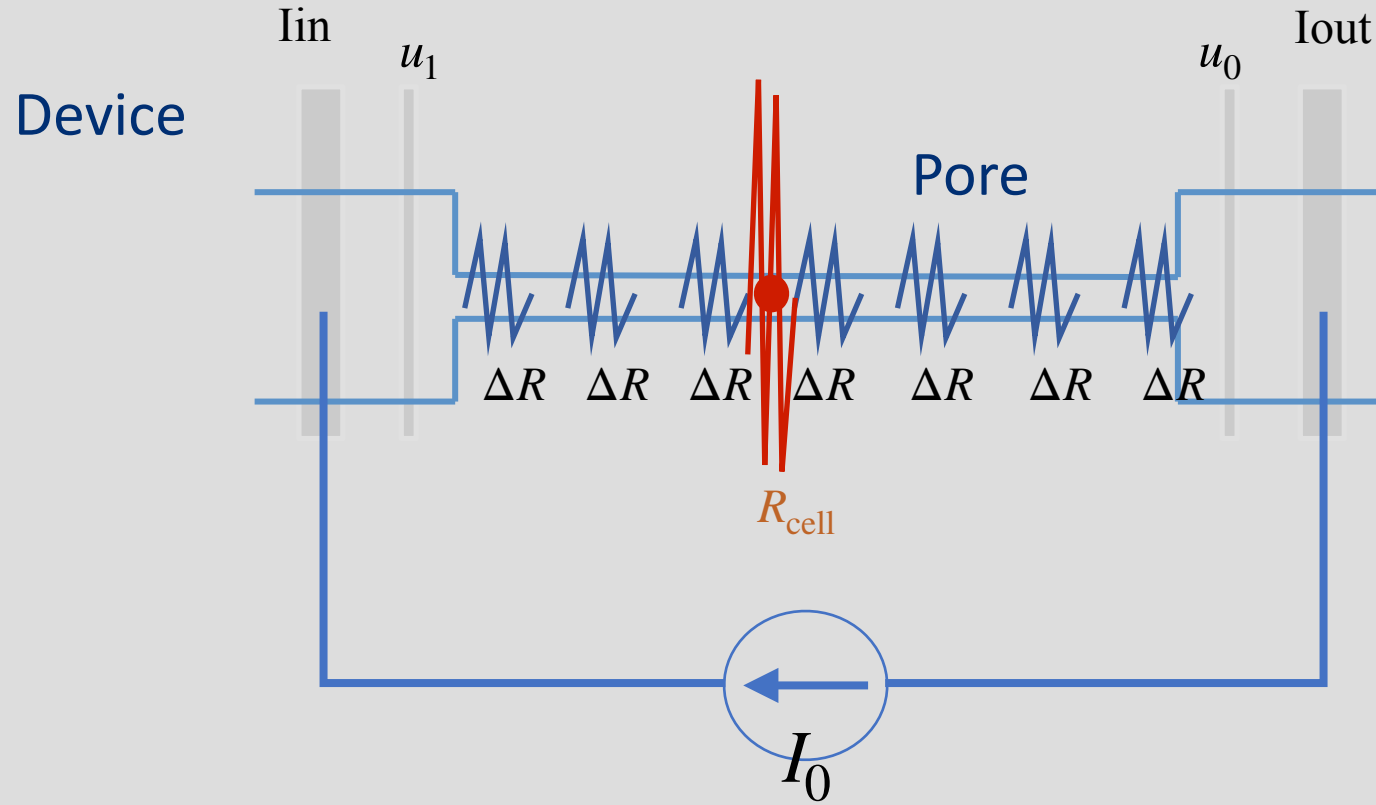


Resistive Pulse Sensing



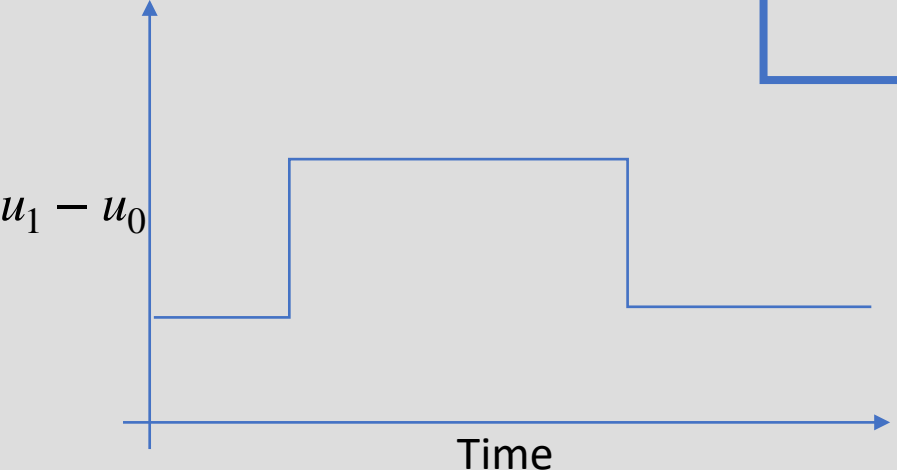
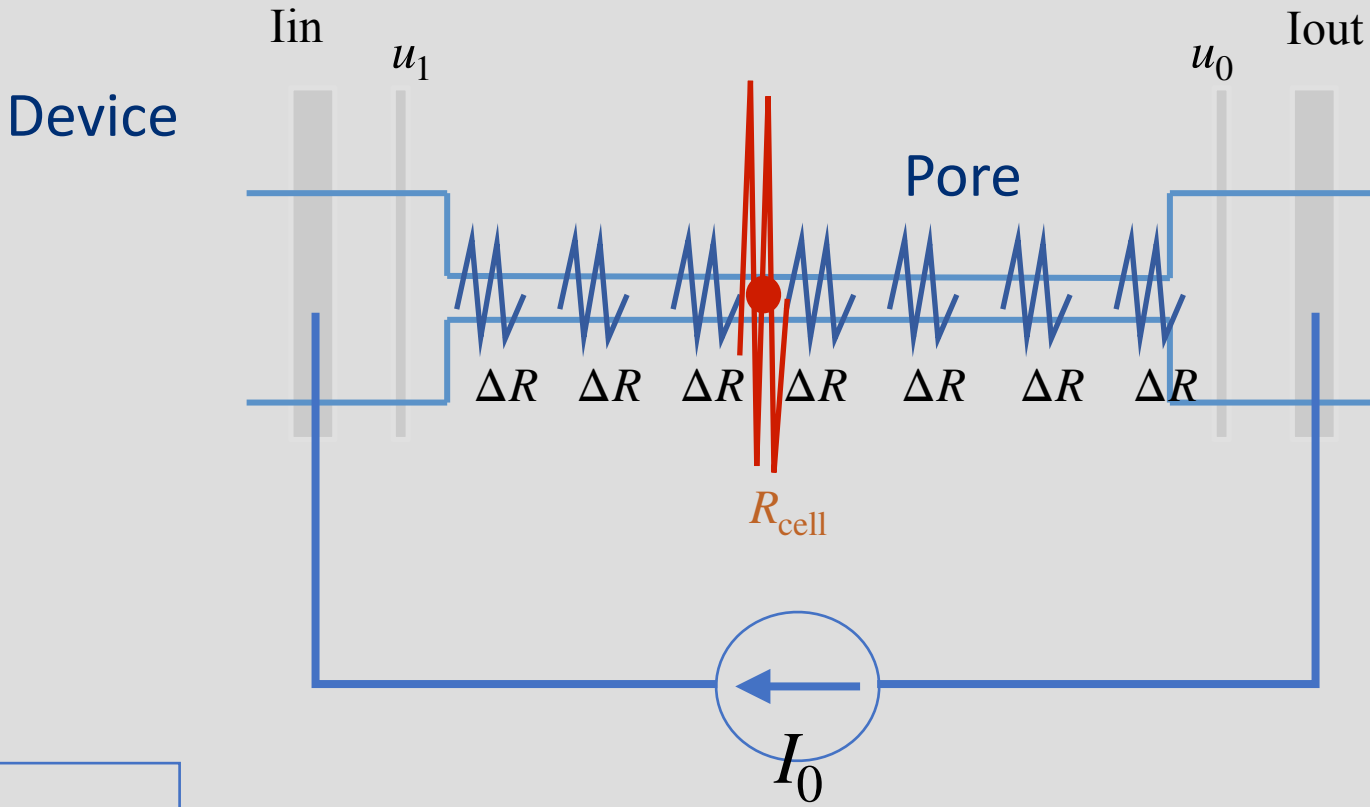
$$u_1 - u_0 = I_0 R$$

Resistive Pulse Sensing



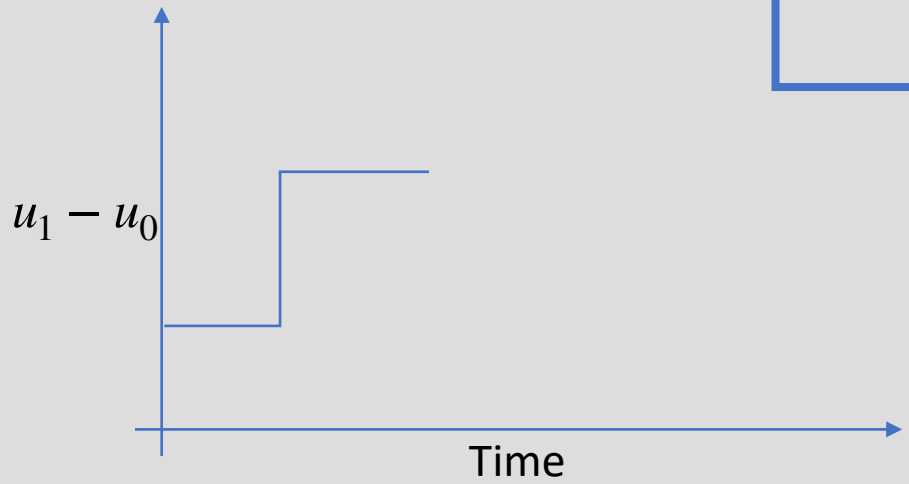
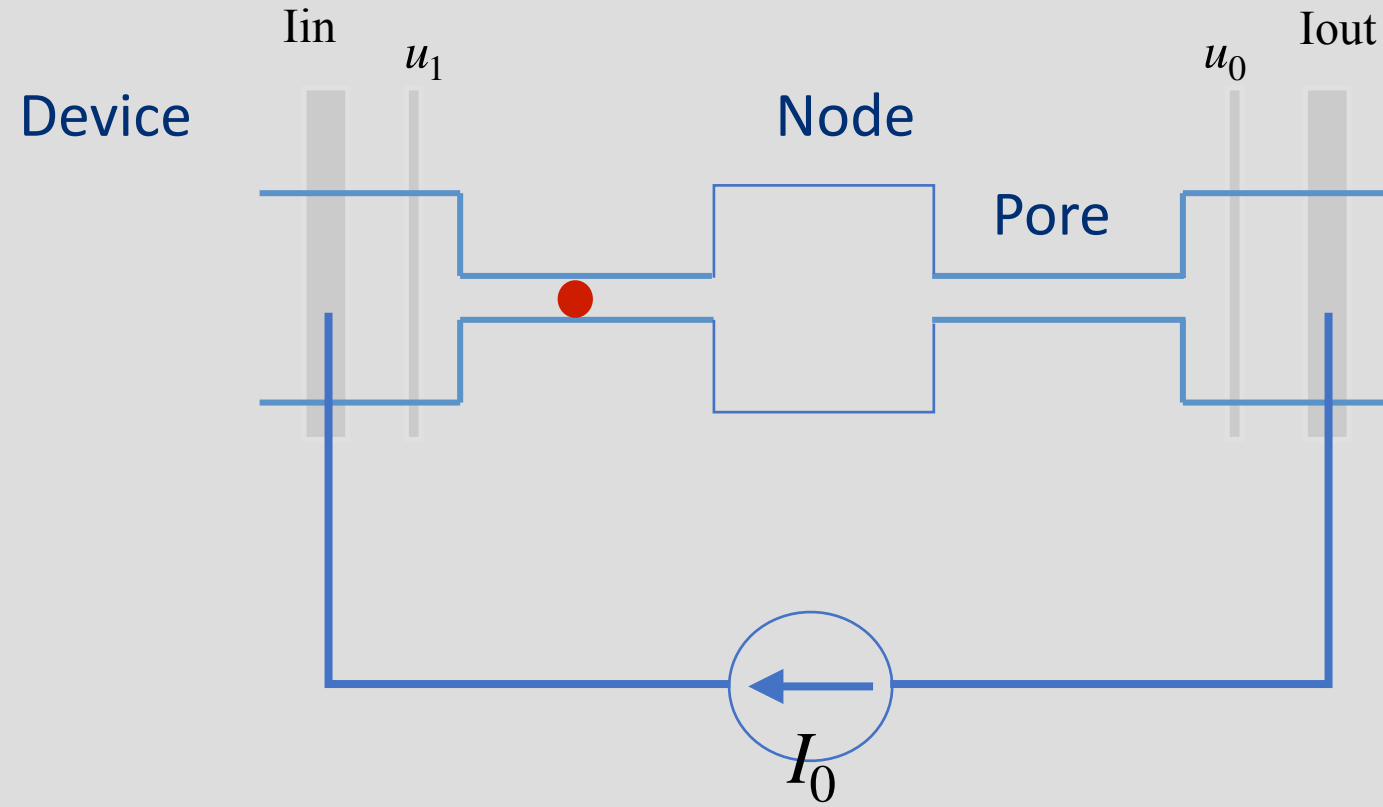
$$u_1 - u_0 = I_0 R$$

Resistive Pulse Sensing



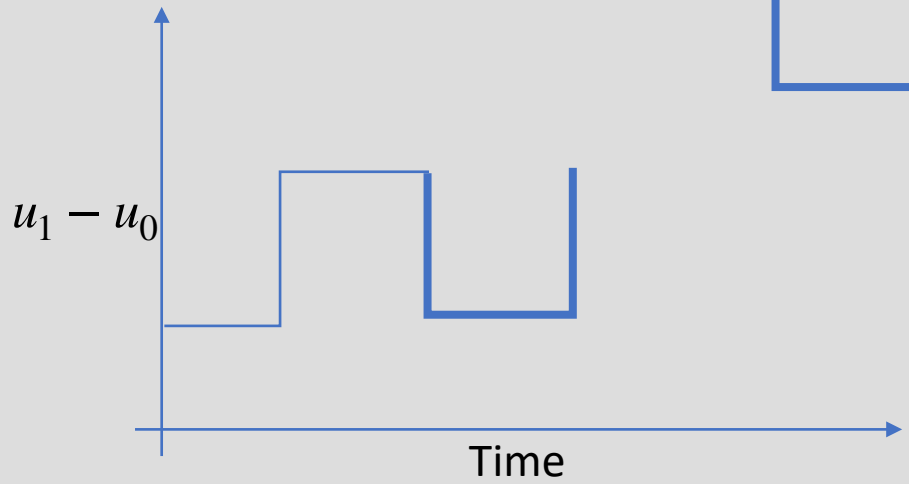
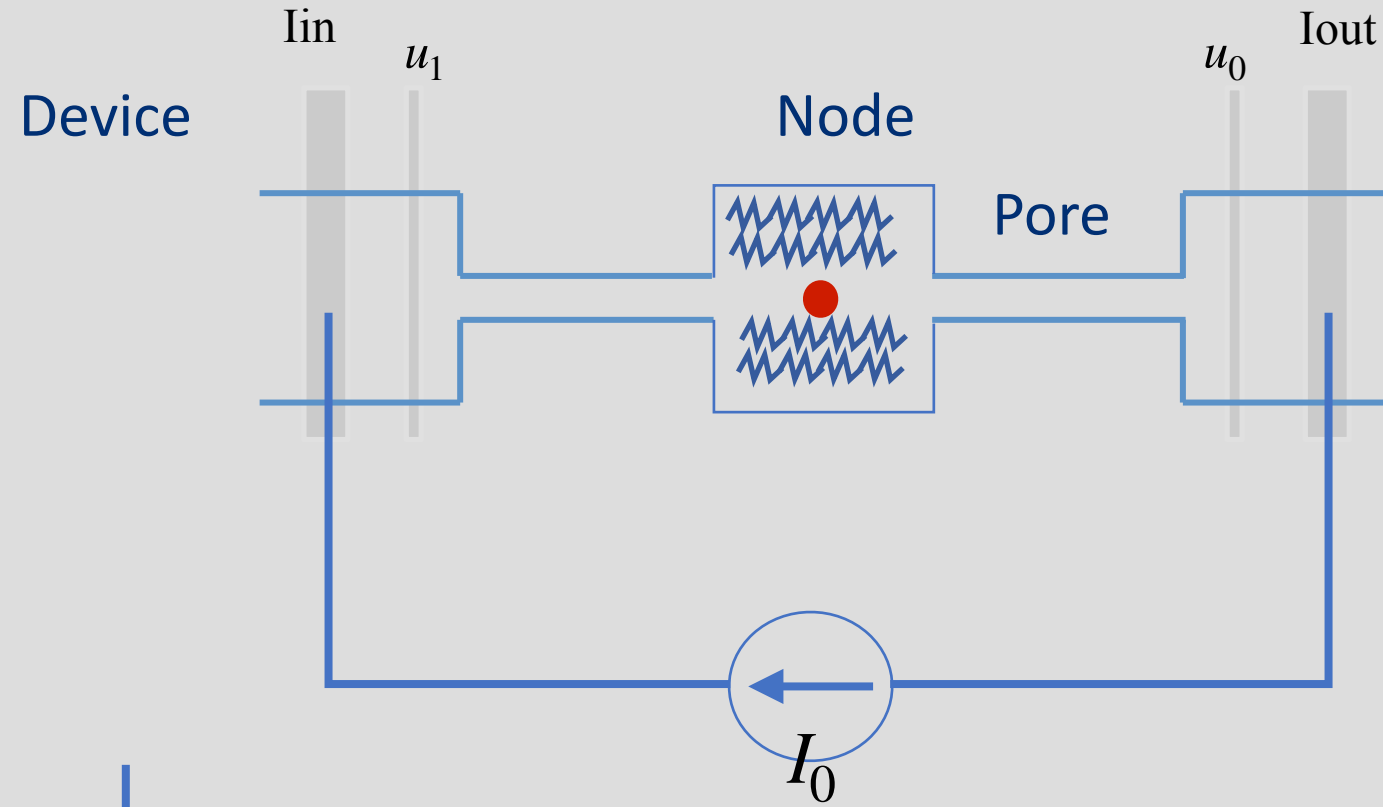
$$u_1 - u_0 = I_0(R + R_{\text{cell}})$$

Node-Pore Sensing



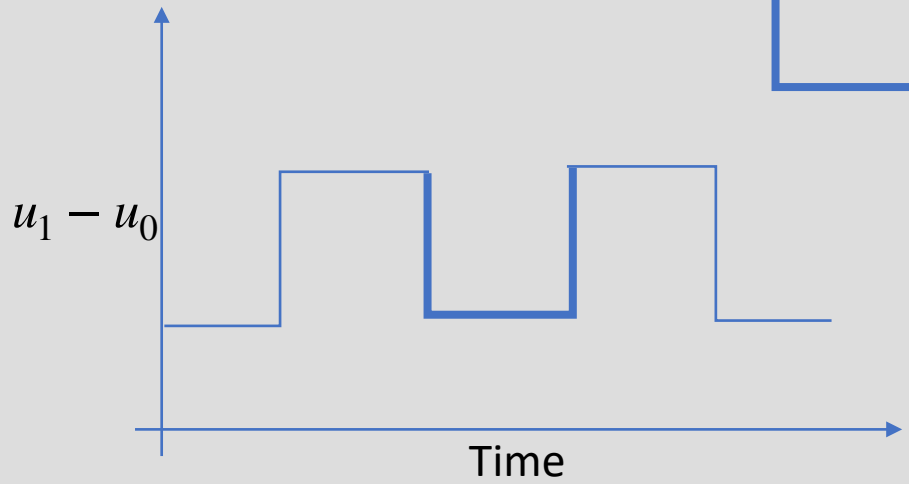
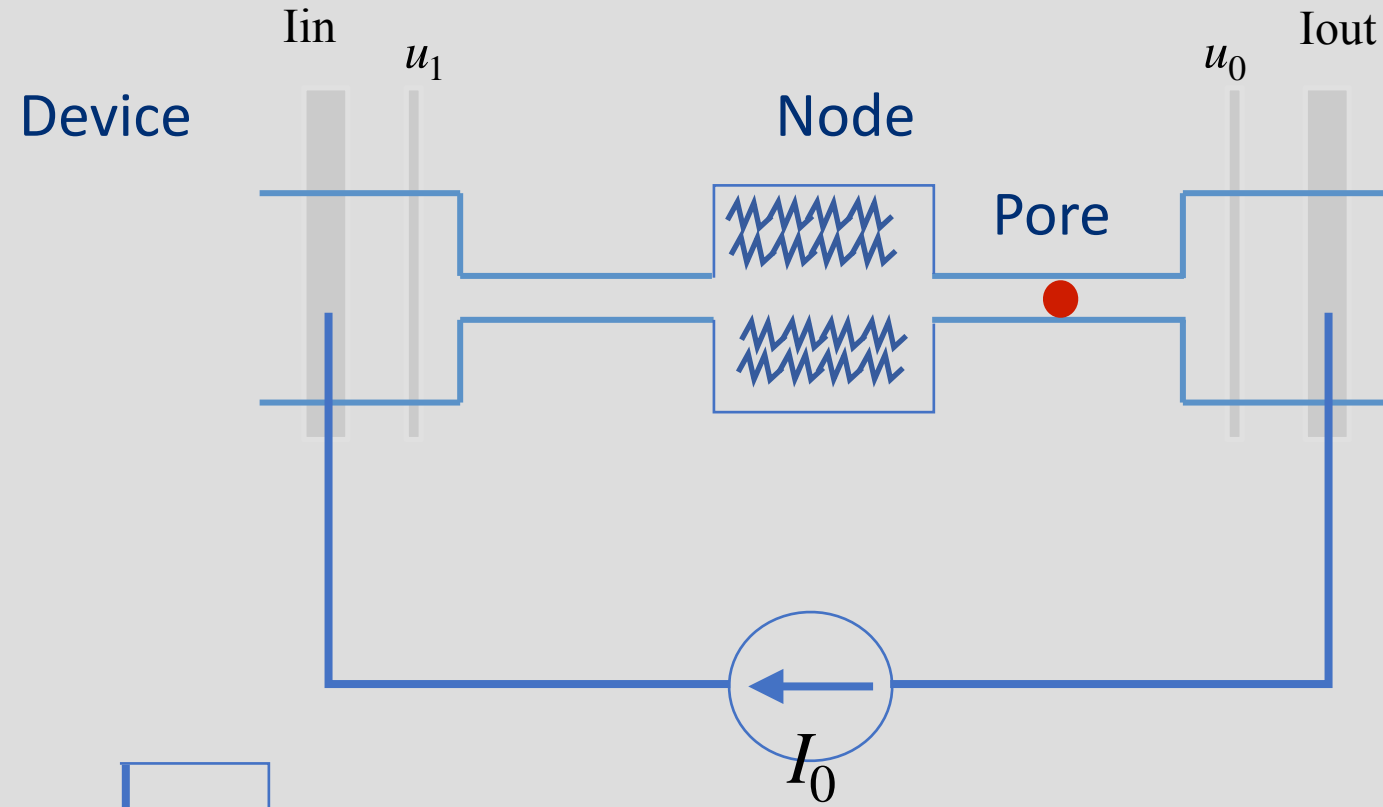
$$u_1 - u_0 = I_0(R + R_{cell} + R_{node})$$

Node-Pore Sensing



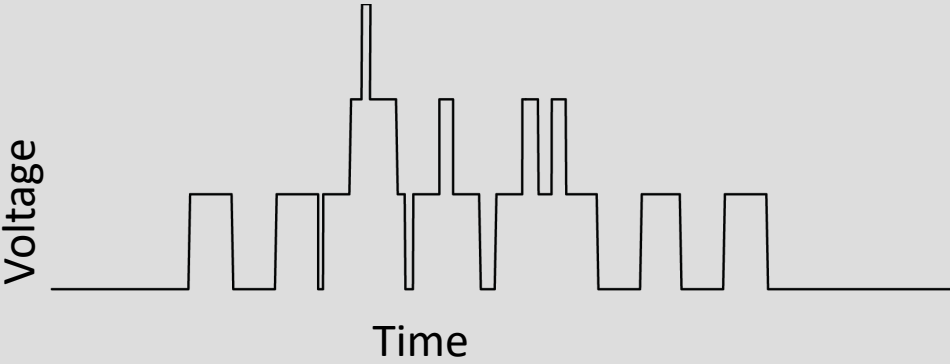
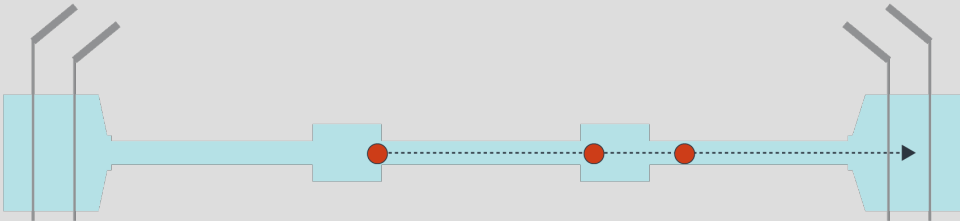
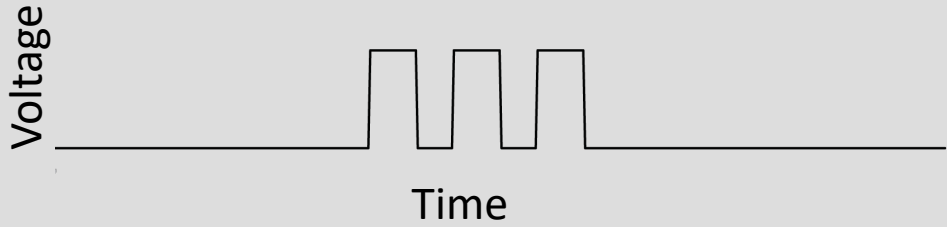
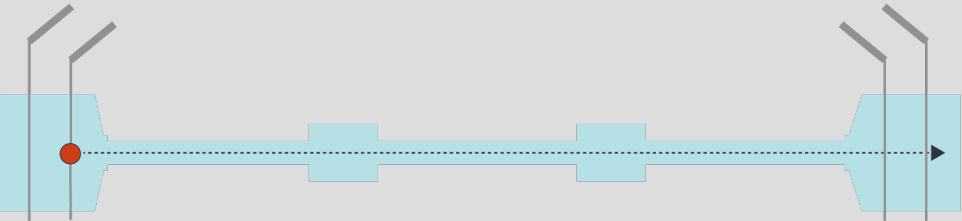
$$u_1 - u_0 = I_0 (R + R_{cell} \parallel R_{node})$$

Node-Pore Sensing



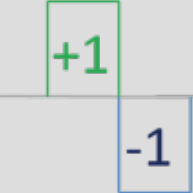

$$u_1 - u_0 = I_0(R + R_{cell} + R_{node})$$

Sensing Complexities



Barker Codes

- 9 unique sequences

Barker 2 : +1 -1 or +1 +1 ->  or 

Barker 3 : +1 +1 -1

Barker 4 : +1 +1 -1 +1 or +1 +1 +1 +1 -1

Barker 5 : +1 +1 +1 -1 +1

Barker 7 : +1 +1 +1 -1 -1 +1 -1

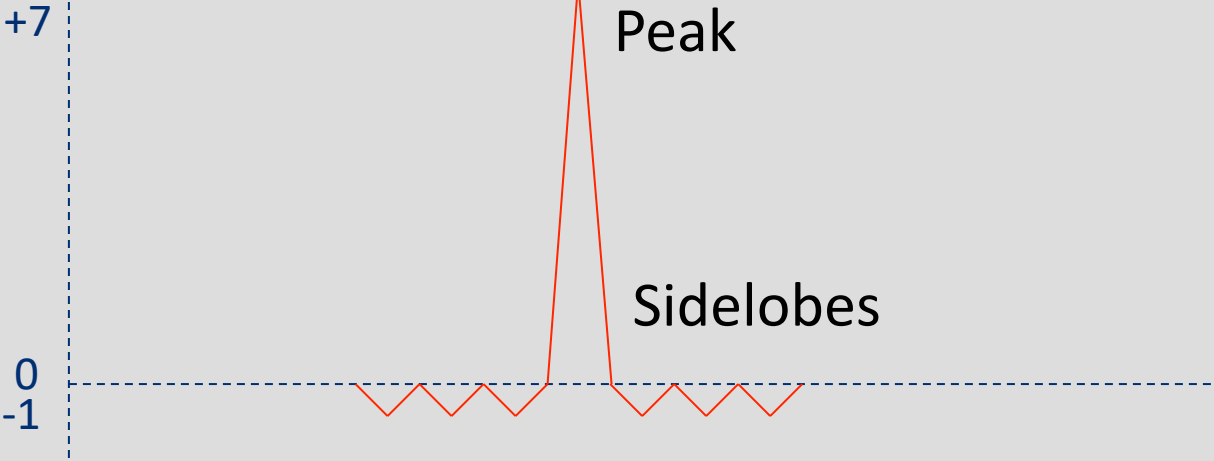
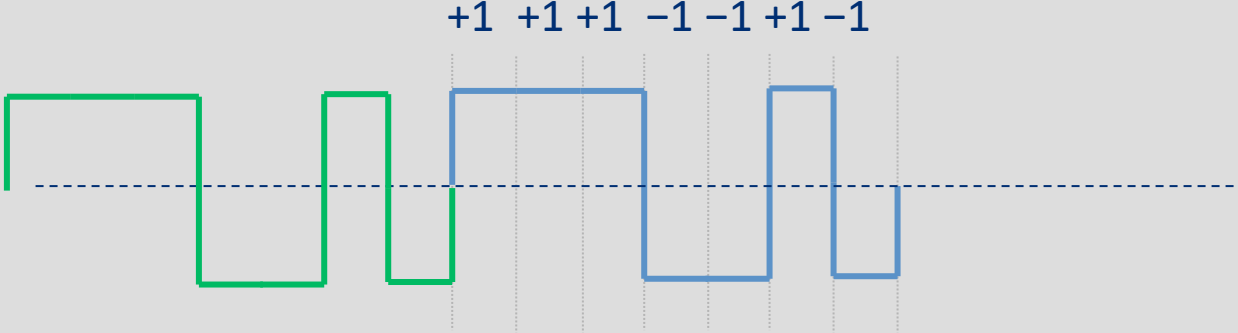
Barker 11 : +1 +1 +1 -1 -1 -1 +1 -1 -1 +1 -1

Barker 13 : +1 +1 +1 +1 +1 -1 -1 +1 +1 -1 +1 -1 +1

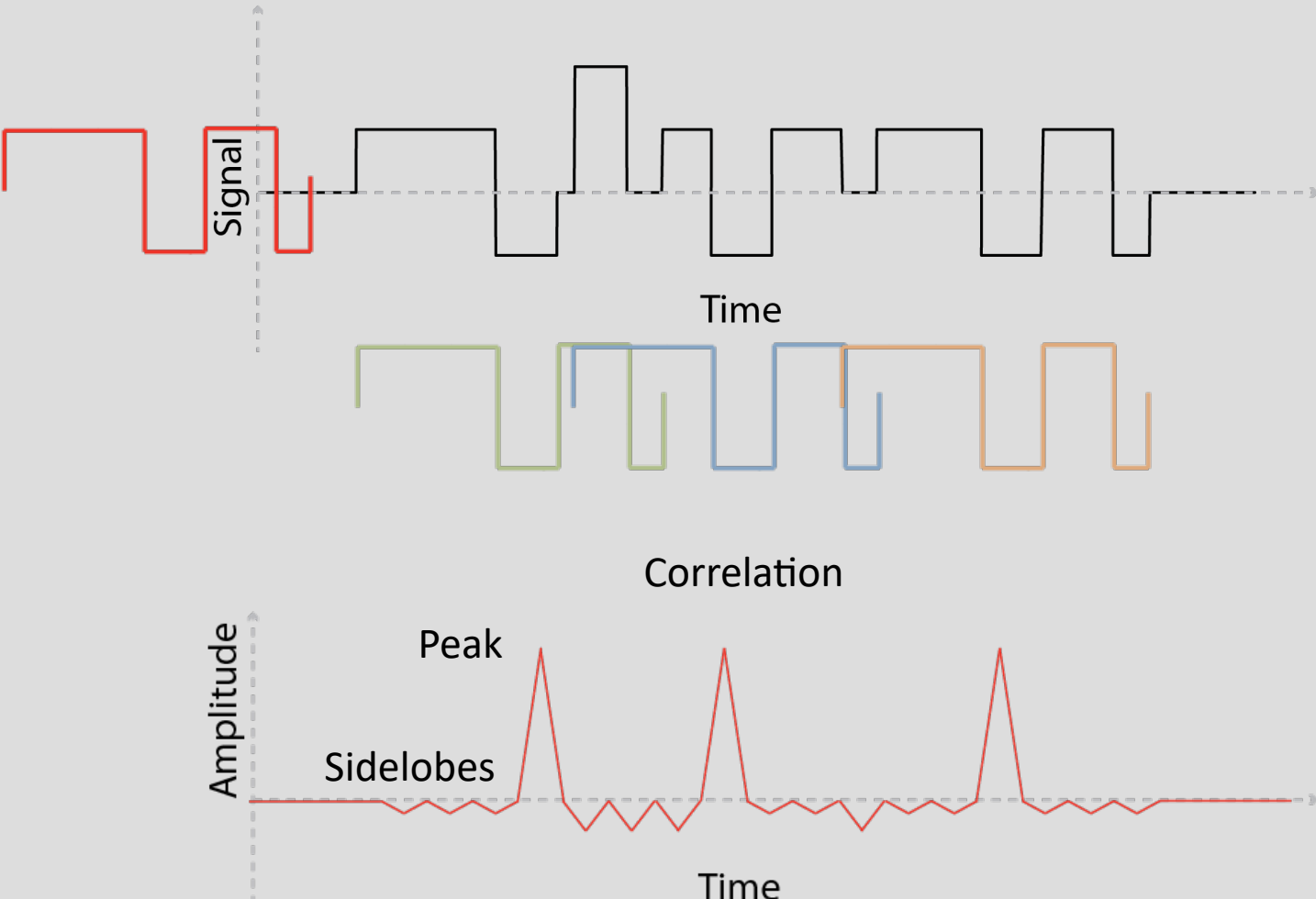


Auto-Correlation of Barker Codes

Barker 7

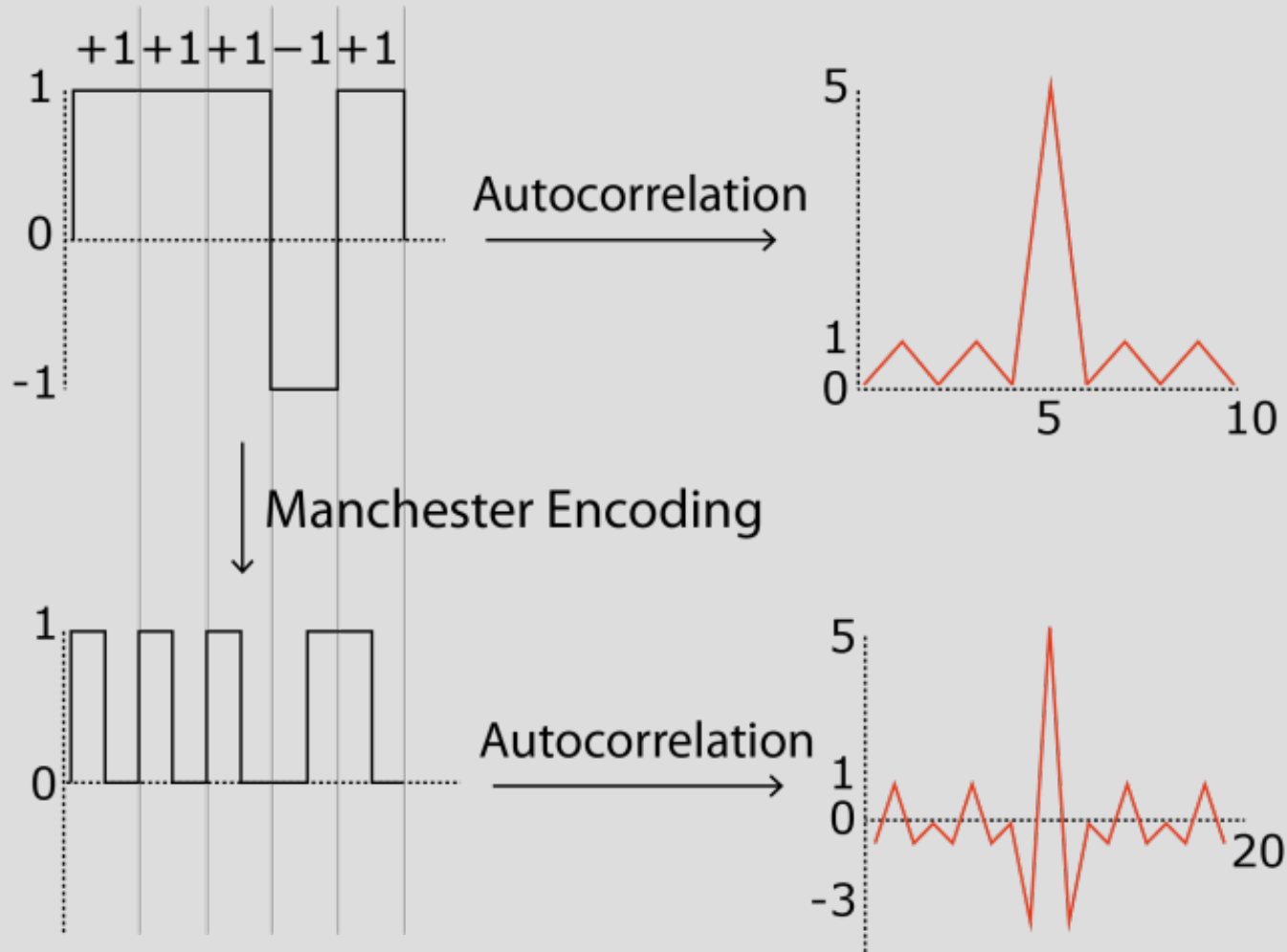


Cross Correlation with Barker Codes



Implementing Barker Codes in NPS

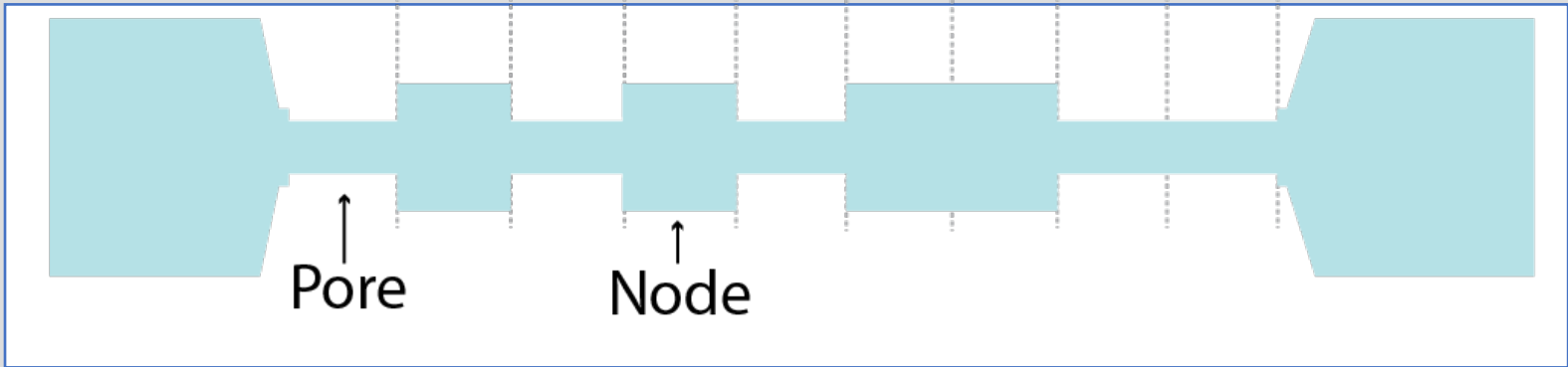
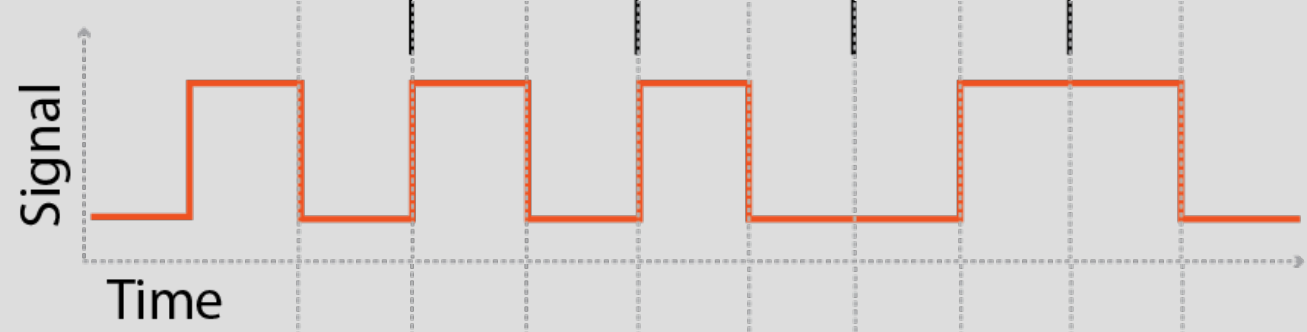
Barker 5: +1,+1,+1,-1,+1



Encoding a Channel


Barker 5 : +1 +1 +1 -1 +1

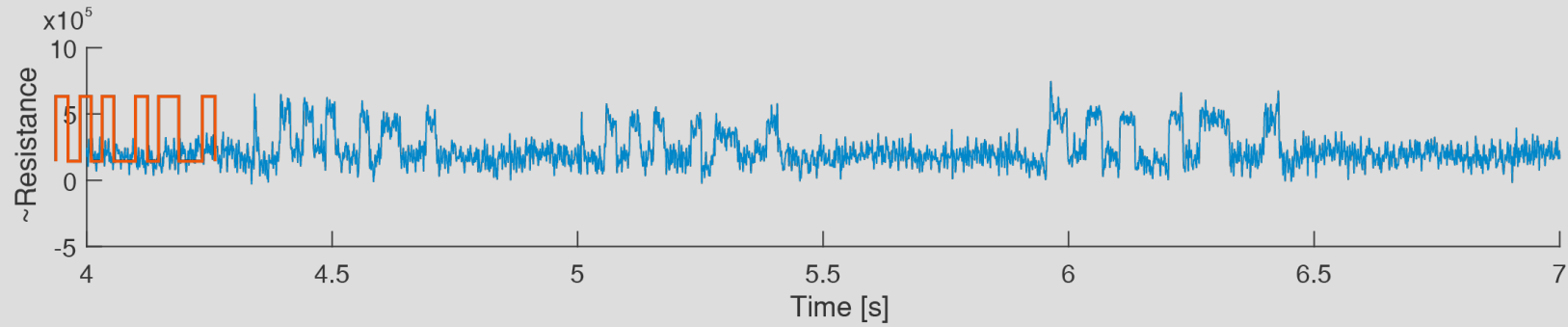
Encoded Signal : 1 0 1 0 1 0 0 1 1 0



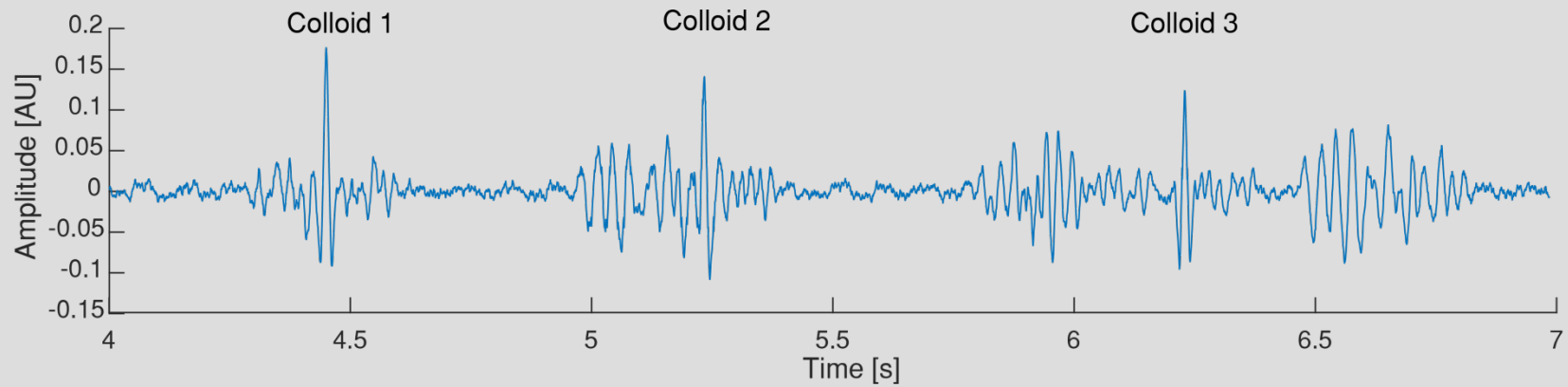
(Rivest et al., 2015)

Real Data

 26 mm/s



Correlation plot



Speed and Time

