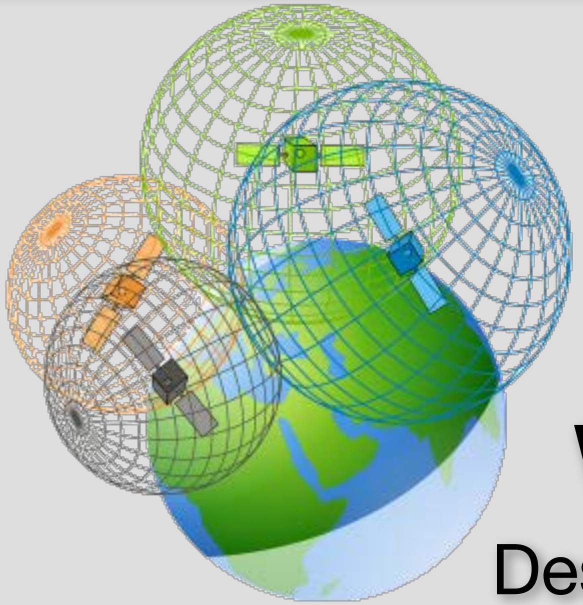


Welcome to EECS 16A!

Designing Information Devices and Systems I

Ana Arias and Miki Lustig

Fall 2021

Lecture ~~2B~~ 1B

Gaussian Elimination

Vectors

Announcements

- Quest: Tuesday 09/14/21 8:30pm
    - Covers everything including today

What is linear algebra?

- The study of linear functions and linear equations, typically using vectors and matrices
- Linearity is not always applicable, but can be a good first-order approximation
- There exist good fast algorithms to solve these problems

Linear Equations

- Definition:

  Consider: $f(x_1, x_2, \cdots, x_N) : \mathbb{R}^n \to \mathbb{R}$

  f is *linear if the following identity holds:*

  (1) Homogeneity:

$$f(ax_1, \cdots, ax_N) = af(x_1, \ldots, x_N)$$

  (2) Super Position: if $x_i = y_i + z_i$, then

$$f(y_1 + z_1, \cdots, y_N + z_N) = f(y_1, \cdots, y_N) + f(z_1, \cdots, z_N)$$

Both: $f(ay_1 + bz_1, \cdots, ay_N + bz_N) = af(y_1, \cdots, y_N) + bf(z_1, \cdots, z_N)$

Claim: linear functions can always be expressed as:

$$f(x_1, x_2, \cdots, x_N) = c_1x_1 + c_2x_2 + \cdots + c_Nx_N$$

Proof for $\mathbb{R}^2$

- $f(x_1, x_2) : \mathbb{R}^2 \Rightarrow \mathbb{R}$ is linear. Need to prove: $f(x_1, x_2) = c_1x_1 + c_2x_2$

Trick:

$$x_1 = 1 \cdot x_1 + 0 \cdot x_2 \qquad \Rightarrow x_1 = x_1y_1 + x_2z_1$$

$$x_2 = 0 \cdot x_1 + 1 \cdot x_2 \qquad \Rightarrow x_2 = x_1y_2 + x_2z_2$$

So,

$$\begin{aligned} f(x_1, x_2) &= f(x_1y_1 + x_2z_1,\ x_1y_2 + x_2z_2) \\ &= x_1f(y_1, y_2) + x_2f(z_1, z_2) \\ &= x_1f(1,0) + x_2f(0,1) \\ &= c_1x_1 + c_2x_2 \end{aligned}$$

Linear Set of Equations

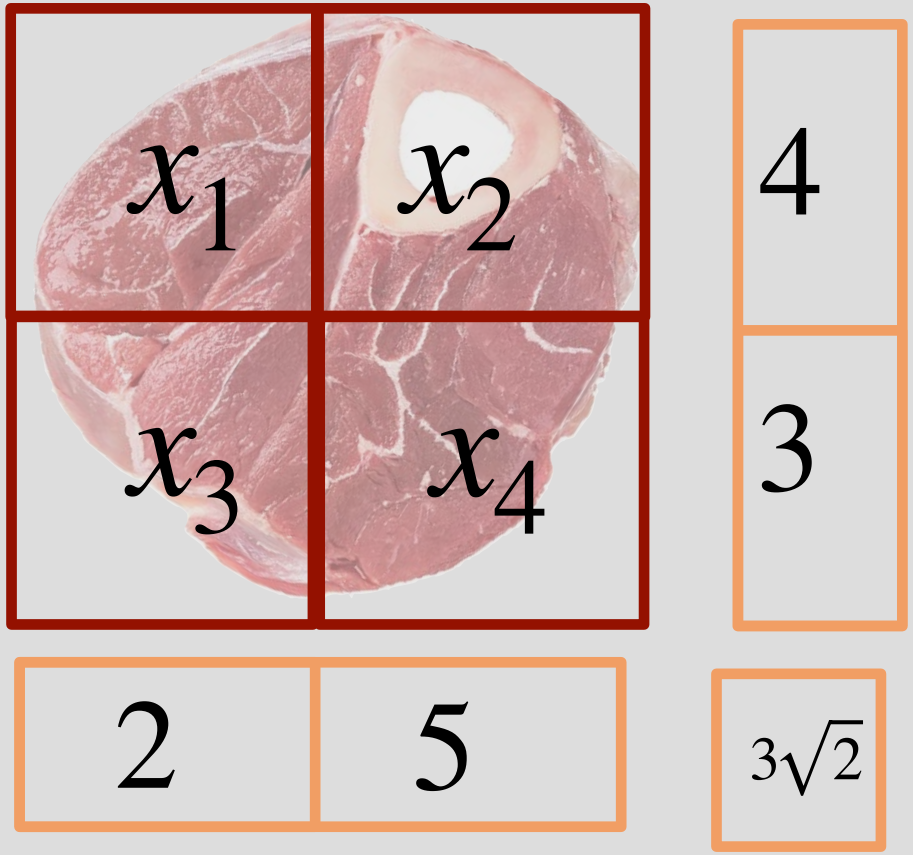
- Consider the set of M linear equations with N variables:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N &= b_2 \\
\vdots \qquad\qquad & \vdots \\
a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N &= b_M
\end{aligned}
$$

- Can be written compactly using augmented matrix:

$$
\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1N} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2N} & b_2 \\
\vdots & & \vdots & & \vdots \\
a_{M1} & a_{M2} & \cdots & a_{MN} & b_M
\end{array}\right]
$$

Back to Tomography



$$1 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 4$$

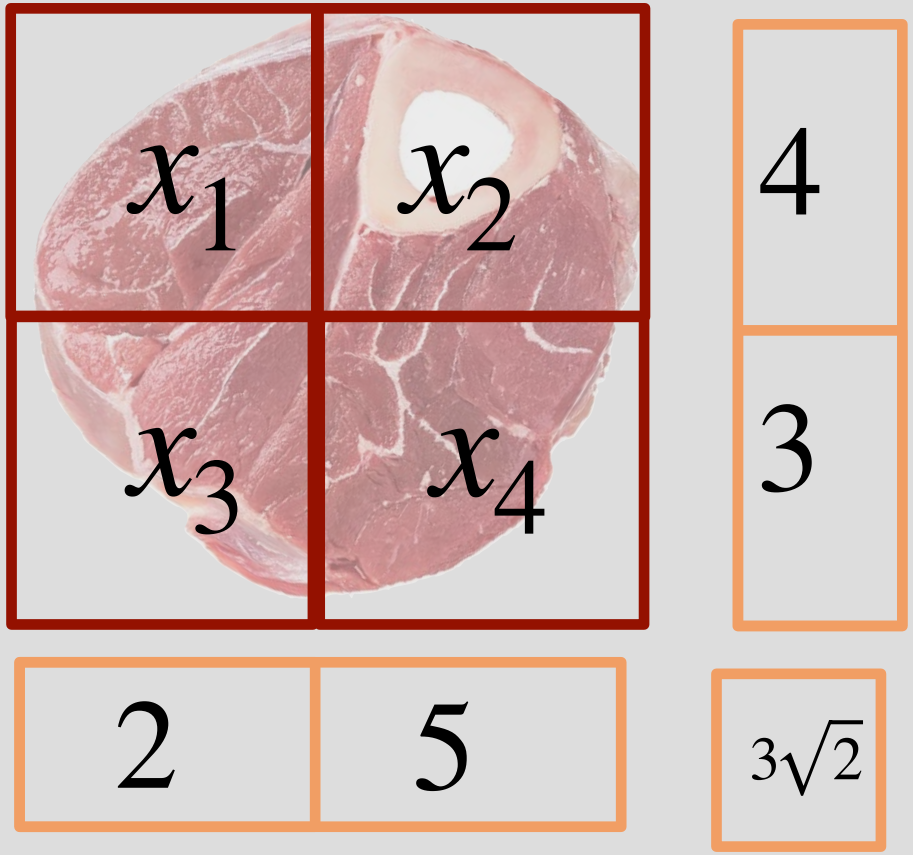
$$0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 1 \cdot x_4 = 3$$

$$1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 = 2$$

$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 5$$

$$\sqrt{2}x_1 + 0 \cdot x_2 + 0 \cdot x_3 + \sqrt{2}x_4 = 3\sqrt{2}$$

Back to Tomography

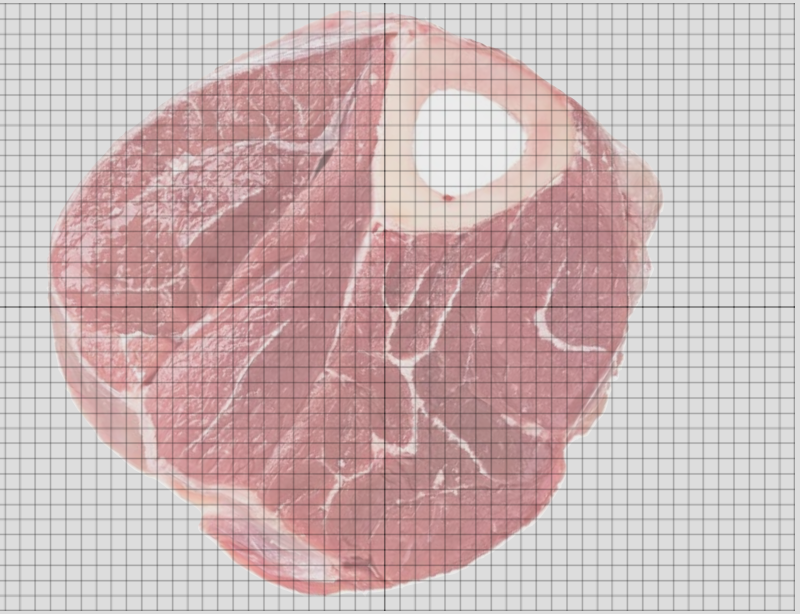


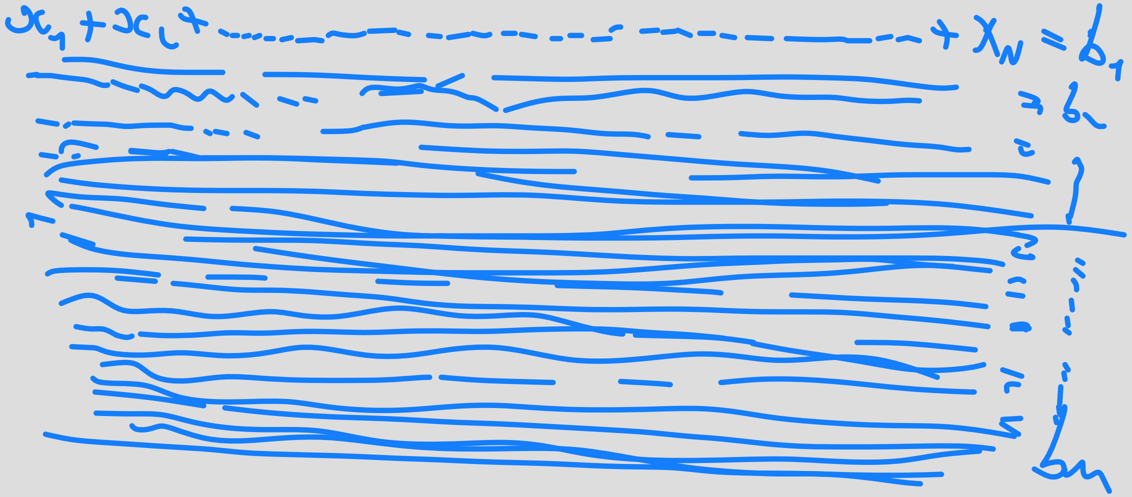
$$1 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 4$$
$$0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 1 \cdot x_4 = 3$$
$$1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 = 2$$
$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 5$$
$$\sqrt{2}x_1 + 0 \cdot x_2 + 0 \cdot x_3 + \sqrt{2}x_4 = 3\sqrt{2}$$

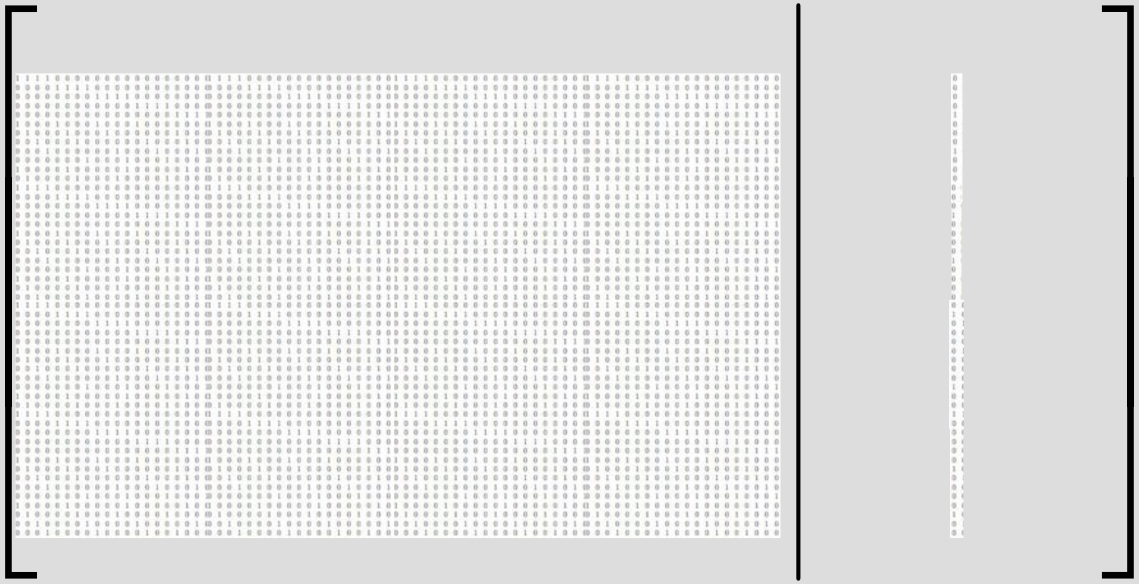
$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \\ \sqrt{2} & 0 & 0 & \sqrt{2} & 3\sqrt{2} \end{array}\right]$$

How do we solve it?

Back to Tomography







How do we systematically solve it?

Algorithm for solving linear equations

- Three basic operations that don’t change a solution:
    1. Multiply an equation with *nonzero* scalar
    2. Adding a scalar constant multiple of one equation to another
    3. Swapping equations

Algorithm for solving linear equations

- Three basic operations that don't change a solution:
  1. Multiply an equation with *nonzero* scalar
  2. Adding a scalar constant multiple of one equation to another
  3. Swapping equations

$$\begin{array}{lrcrcl} (1) & x & + & y & = & 2 \\ (2) & 3x & + & 2y & = & 5 \end{array}$$

and

$$\begin{array}{lrcrcl} (1) & 3x & + & 2y & = & 5 \\ (2) & x & + & y & = & 2 \end{array}$$

Have the same solution

Proof: Pretty obvious!

Algorithm for solving linear equations

- Three basic operations that don't change a solution:
  1. Multiply an equation with *nonzero* scalar

$2x + 3y = 4$ has the same solution as: $4x + 6y = 8$

Proof for N=2:

Let $ax + by = c$, with solution x_0, y_0
$\Rightarrow ax_0 + by_0 = c$

Show that $\beta ax + \beta by = \beta c$,
has the same solution.

Substitute x_0, y_0 for x, y:

$$\beta ax_0 + \beta by_0 = \beta c$$

$$\beta(ax_0 + by_0) = \beta c$$

$$\beta c = \beta c$$

But is it the only solution?

$\beta ax + \beta by = \beta c$, with solution: x_1, y_1
$\Rightarrow \beta ax_1 + \beta by_1 = \beta c$

Show that $ax + by = c$,
has the same solution…..

Since $\beta \neq 0$….

$$\beta ax_1 + \beta by_1 = \beta c \Rightarrow ax_1 + by_1 = c$$

SOLUTION OF ONE, IMPLIES THE OTHER AND VICE-VERSA!

Algorithm for solving linear equations

- Three basic operations that don't change a solution:
    1. Multiply an equation with *nonzero* scalar
    2. Adding a scalar constant multiple of one equation to another

$$\begin{array}{rrcrcr} (1) & x & + & y & = & 2 \\ (2) & 3x & + & 2y & = & 5 \end{array}$$

and

$$\begin{array}{rrcrcr} (1) & x & + & y & = & 2 \\ 3 \times (1) + (2) & 6x & + & 5y & = & 11 \end{array}$$

Have the same solution

Concept of proof: look at explicit solution, show they are the same

Also show the reverse — by applying the reverse operations

Upper Diagonal Systems

- Consider the following equations:

$$
\begin{array}{rcrcrcl}
x & - & y & + & 2z & = & 1 \\
  &   & y & - & z & = & 2 \\
  &   &   &   & z & = & 1
\end{array}
$$

Upper Diagonal Systems

- Consider the following equations:

$$\begin{aligned} x - y + 2z &= 1 \\ y - z &= 2 \\ z &= 1 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

- Why are they easy to solve?

Upper Triangular matrix \ Row Echelon

$$\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right]$$

Pivots

More general Row Echelon in the notes!

Gaussian Elimination

$$\begin{array}{rcrcrcl} x & - & y & + & 2z & = & 1 \\ 2x & + & y & + & z & = & 8 \\ -4x & + & 5y & & & = & 7 \end{array}$$

- Row-reduction to upper triang (Row echelon):

Step I

$$\left[\begin{array}{ccc|c} & & & \\ & & & \\ & & & \end{array}\right]$$

(1)
(2)
(3)

Gaussian Elimination

$$\begin{array}{rcrcrcl} x & - & y & + & 2z & = & 1 \\ 2x & + & y & + & z & = & 8 \\ -4x & + & 5y & & & = & 7 \end{array}$$

- Row-reduction to upper triang (Row echelon):

Step I

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 8 \\ -4 & 5 & 0 & 7 \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

Step II

$$\begin{array}{r} \\ (2) - 2 \times (1) \\ (3) + 4 \times (1) \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ & & & \\ & & & \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

Gaussian Elimination

$$\begin{array}{rcrcrcr} x & - & y & + & 2z & = & 1 \\ 2x & + & y & + & z & = & 8 \\ -4x & + & 5y & & & = & 7 \end{array}$$

- Row-reduction to upper triang (Row echelon):

Step I

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 8 \\ -4 & 5 & 0 & 7 \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

Step II

$$\begin{array}{r} \\ (2) - 2 \times (1) \\ (3) + 4 \times (1) \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 3 & -3 & 6 \\ 0 & 1 & 8 & 11 \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

Step III

$$(2)/3 \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ & & & \\ & & & \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

Gaussian Elimination

$$\begin{array}{rcrcrcl} x & - & y & + & 2z & = & 1 \\ 2x & + & y & + & z & = & 8 \\ -4x & + & 5y & & & = & 7 \end{array}$$

- Row-reduction to upper triang (Row echelon):

Step I

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 8 \\ -4 & 5 & 0 & 7 \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

Step II

$$\begin{array}{r} \\ (2) - 2 \times (1) \\ (3) + 4 \times (1) \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 3 & -3 & 6 \\ 0 & 1 & 8 & 11 \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

Step III

$$(2)/3 \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & 8 & 11 \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

Step IV

$$\begin{array}{r} \\ \\ (3) - (2) \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ & & & \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

Gaussian Elimination

$$\begin{array}{rcrcrcl} x & - & y & + & 2z & = & 1 \\ 2x & + & y & + & z & = & 8 \\ -4x & + & 5y & & & = & 7 \end{array}$$

- Row-reduction to upper triang (Row echelon):

Step I

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 8 \\ -4 & 5 & 0 & 7 \end{array}\right] \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

Step II

$$\begin{array}{r} \\ (2)-2\times(1) \\ (3)+4\times(1) \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 3 & -3 & 6 \\ 0 & 1 & 8 & 11 \end{array}\right] \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

Step III

$$\begin{array}{r} \\ (2)/3 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & 8 & 11 \end{array}\right] \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

Step IV

$$\begin{array}{r} \\ \\ (3)-(2) \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 9 & 9 \end{array}\right] \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

Step V

$$\begin{array}{r} \\ \\ (3)/9 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ & & & \end{array}\right] \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

Gaussian Elimination

$$\begin{aligned} x - y + 2z &= 1 \\ 2x + y + z &= 8 \\ -4x + 5y &= 7 \end{aligned}$$

- Row-reduction to upper triang (Row echelon):

Step I

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 8 \\ -4 & 5 & 0 & 7 \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

Step II

$$\begin{array}{r} \\ (2) - 2 \times (1) \\ (3) + 4 \times (1) \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 3 & -3 & 6 \\ 0 & 1 & 8 & 11 \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

Step III

$$\begin{array}{r} \\ (2)/3 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & 8 & 11 \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

Step IV

$$\begin{array}{r} \\ \\ (3) - (2) \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 9 & 9 \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

Step V

$$\begin{array}{r} \\ \\ (3)/9 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

Pivots

Gaussian Elimination Cont.

$$\begin{array}{rcrcrcl} x & - & y & + & 2z & = & 1 \\ 2x & + & y & + & z & = & 8 \\ -4x & + & 5y & & & = & 7 \end{array}$$

- Back substitution:

Step V

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]\begin{array}{c}(1)\\(2)\\(3)\end{array}$$

Step VI

$$\begin{array}{r}(1)-2\times(3)\\(2)+(3)\\ \end{array}\left[\begin{array}{ccc|c} & & & \\ & & & \\ 0 & 0 & 1 & 1 \end{array}\right]\begin{array}{c}(1)\\(2)\\(3)\end{array}$$

Gaussian Elimination Cont.

$$\begin{array}{rcrcrcl} x & - & y & + & 2z & = & 1 \\ 2x & + & y & + & z & = & 8 \\ -4x & + & 5y & & & = & 7 \end{array}$$

- Back substitution:

Step V

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]\begin{array}{l}(1)\\(2)\\(3)\end{array}$$

Step VI

$$\begin{array}{r}(1)-2\times(3)\\(2)+(3)\\ \end{array}\left[\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]\begin{array}{l}(1)\\(2)\\(3)\end{array}$$

Step VII

$$\begin{array}{r}(1)+(2)\\ \\ \end{array}\left[\begin{array}{ccc|c} & & & \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]\begin{array}{l}(1)\\(2)\\(3)\end{array}$$

Gaussian Elimination Cont.

$$\begin{array}{rcrcrcl} x & - & y & + & 2z & = & 1 \\ 2x & + & y & + & z & = & 8 \\ -4x & + & 5y & & & = & 7 \end{array}$$

- Back substitution:

Step V

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]\begin{array}{l}(1)\\(2)\\(3)\end{array}$$

Step VI

$$\begin{array}{r}(1)-2\times(3)\\(2)+(3)\\ \end{array}\left[\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]\begin{array}{l}(1)\\(2)\\(3)\end{array}$$

Step VII

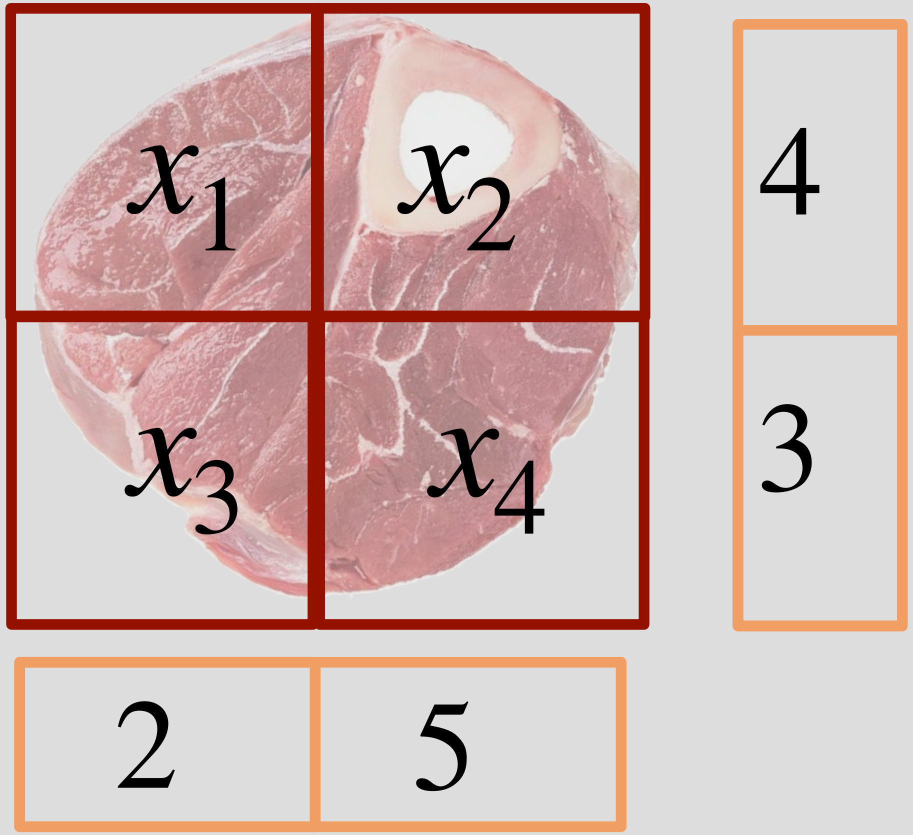
$$\begin{array}{r}(1)+(2)\\ \\ \\ \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]\begin{array}{l}(1)\\(2)\\(3)\end{array}$$

$$\begin{array}{ccc} x & y & z \end{array}$$

Diagonal/identity matrix (reduced Row-Echelon form)

$$\begin{array}{rcl} x & = & 2 \\ y & = & 3 \\ z & = & 1 \end{array}$$

Back to Tomography



$$1 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 4$$
$$0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 1 \cdot x_4 = 3$$
$$1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 = 2$$
$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 5$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \end{array}\right]$$

How do we solve it?

Back to Tomography

Step I

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \\ (4) \end{array}$$

Step II

$$\begin{array}{c} \\ \\ (3)-(1) \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ & & & & \\ & & & & \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \\ (4) \end{array}$$

Step III

$$\begin{array}{c} \\ (4) \\ \\ (2) \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ & & & & \\ & & & & \\ & & & & \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \\ (4) \end{array}$$

Step IV

$$\begin{array}{c} \\ \\ (3)+(2) \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 5 \\ & & & & \\ & & & & \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \\ (4) \end{array}$$

Step V

$$\begin{array}{c} \\ \\ \\ (4)-(3) \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 \\ & & & & \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \\ (4) \end{array}$$

Back to Tomography

Step I

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \end{array}\right]\begin{array}{c}(1)\\(2)\\(3)\\(4)\end{array}$$

Step II

$$\begin{array}{c}\\ \\(3)-(1)\\ \\ \end{array}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 1 & 0 & 1 & 5 \end{array}\right]\begin{array}{c}(1)\\(2)\\(3)\\(4)\end{array}$$

Step III

$$\begin{array}{c}\\(4)\\ \\(2)\end{array}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right]\begin{array}{c}(1)\\(2)\\(3)\\(4)\end{array}$$

Step IV

$$\begin{array}{c}\\ \\(3)+(2)\\ \\ \end{array}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right]\begin{array}{c}(1)\\(2)\\(3)\\(4)\end{array}$$

Step V

$$\begin{array}{c}\\ \\ \\(4)-(3)\end{array}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]\begin{array}{c}(1)\\(2)\\(3)\\(4)\end{array}$$

Infinite solutions!

Back to Tomography

Back substitution:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(1)-(2)

$$\left[\begin{array}{cccc|c} & & & & \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Row Echelon

Pivots

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Reduced Row Echelon

Pivots

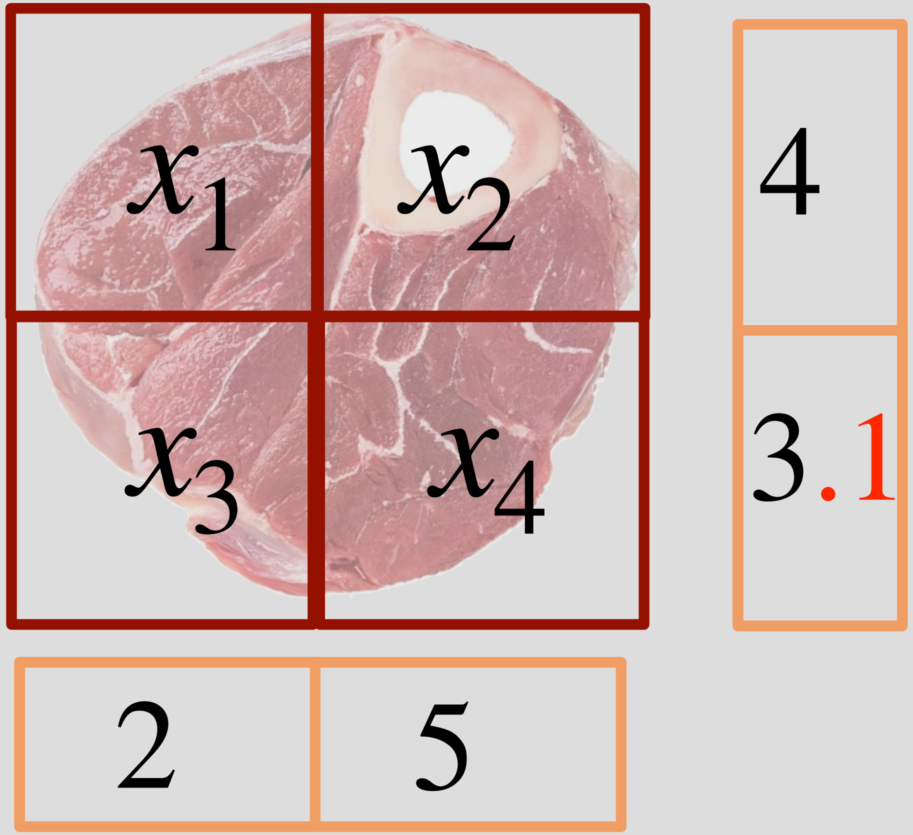
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & * & * \\ 0 & 1 & 0 & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Basic variables

Free variables

Back to Tomography

Perturbations in the measurements



$$1 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 4$$
$$0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 1 \cdot x_4 = 3.1$$
$$1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 = 2$$
$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 5$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3.1 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \end{array}\right]$$

How do we solve it?

Back to Tomography

Step I

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3.1 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \end{array}\right]$$

Step II

(3) − (1)

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3.1 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 1 & 0 & 1 & 5 \end{array}\right]$$

Step III

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 3.1 \end{array}\right]$$

Step IV

(3) + (2)

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 3.1 \end{array}\right]$$

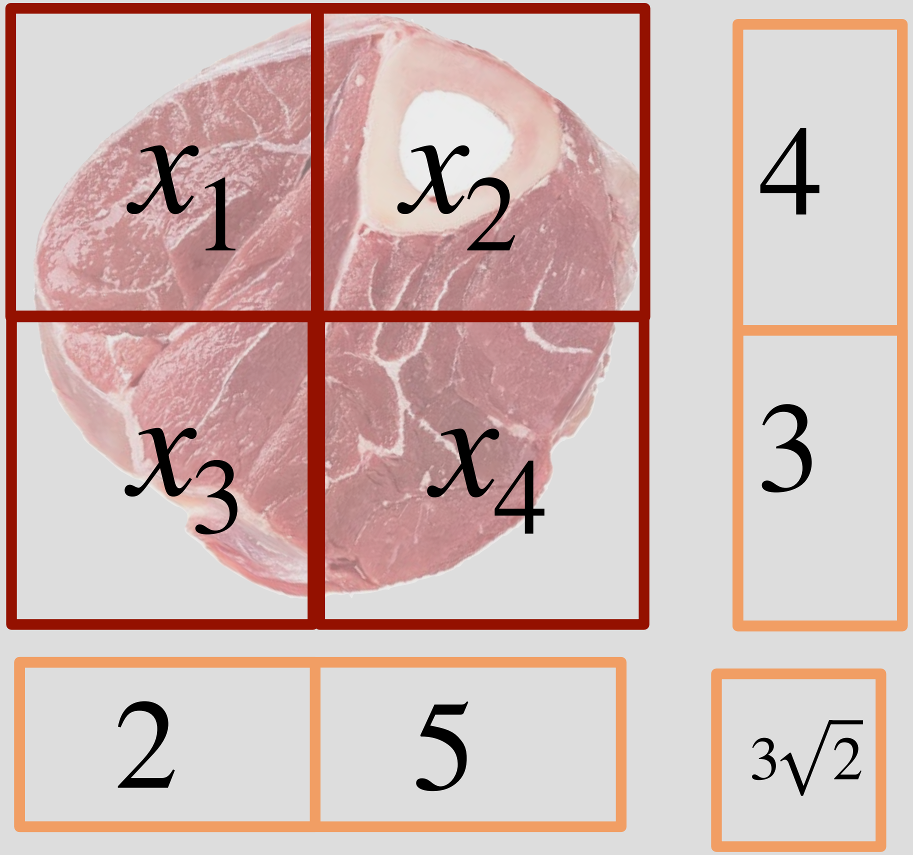
Step V

(4) − (3)

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 \\ & & & & \end{array}\right]$$

No Solution!

Back to Tomography



$$1 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 4$$

$$0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 1 \cdot x_4 = 3$$

$$1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 = 2$$

$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 5$$

$$\sqrt{2}x_1 + 0 \cdot x_2 + 0 \cdot x_3 + \sqrt{2}x_4 = 3\sqrt{2}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \\ \sqrt{2} & 0 & 0 & \sqrt{2} & 3\sqrt{2} \end{array}\right]$$

How do we solve it?

Gaussian Elimination

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \\ \sqrt{2} & 0 & 0 & \sqrt{2} & 3\sqrt{2} \end{array}\right]$$

$(3) - (1)$

$(5) - \sqrt{2} \times (1)$

$-(5)/\sqrt(2)$

(2)

$(3) + (2)$

$(4) - (2)$

$(5) - (3)$

$(4)/2$

$(5) - (4)$

$(2) + (4)$

$(3) + (4)$

$(1) - (2)$

Gaussian Elimination

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \\ \sqrt{2} & 0 & 0 & \sqrt{2} & 3\sqrt{2} \end{array}\right]$$

$(3)-(1)$

$(5)-\sqrt{2}\times(1)$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & -\sqrt{2} & 0 & \sqrt{2} & -\sqrt{2} \end{array}\right]$$

$-(5)/\sqrt(2)$

(2)

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right]$$

$(3)+(2)$

$(4)-(2)$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right]$$

$(5)-(3)$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 2 & 4 \end{array}\right]$$

$(4)/2$

$(5)-(4)$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$(2)+(4)$

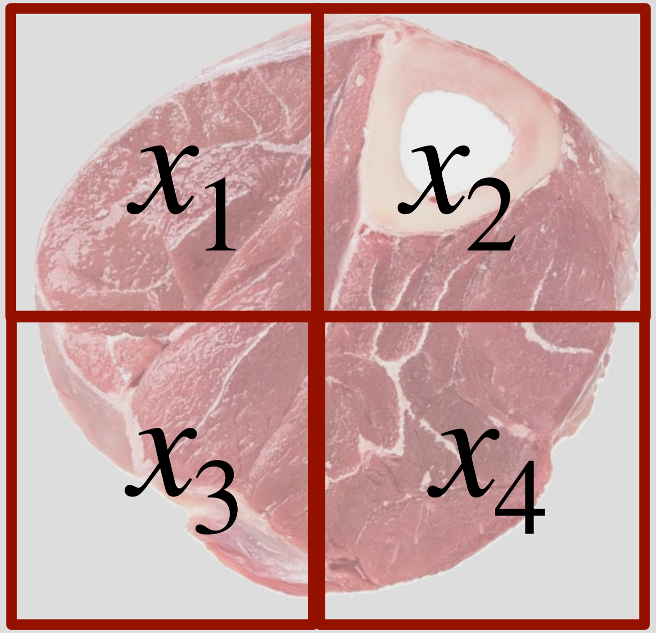
$(3)+(4)$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

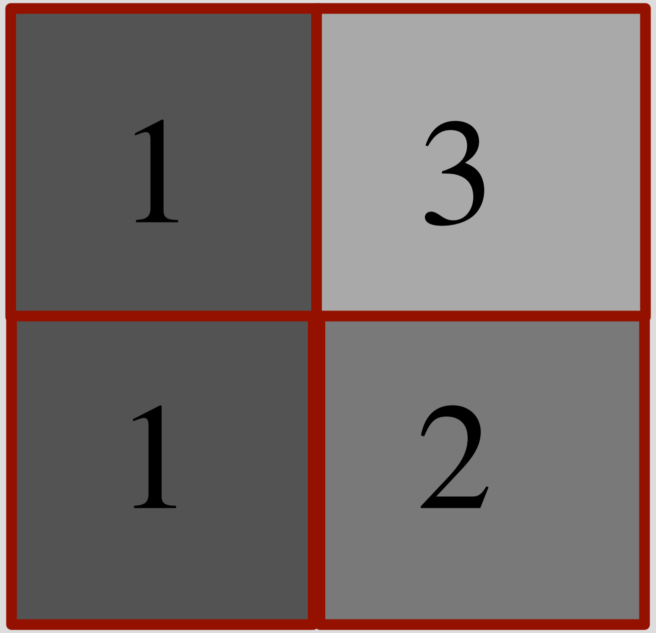
$(1)-(2)$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Tomography Solved!



Possible reconstruction



$$
\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}
$$

$$
\left[\begin{array}{cccc|c}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 3 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

Blurred version of :

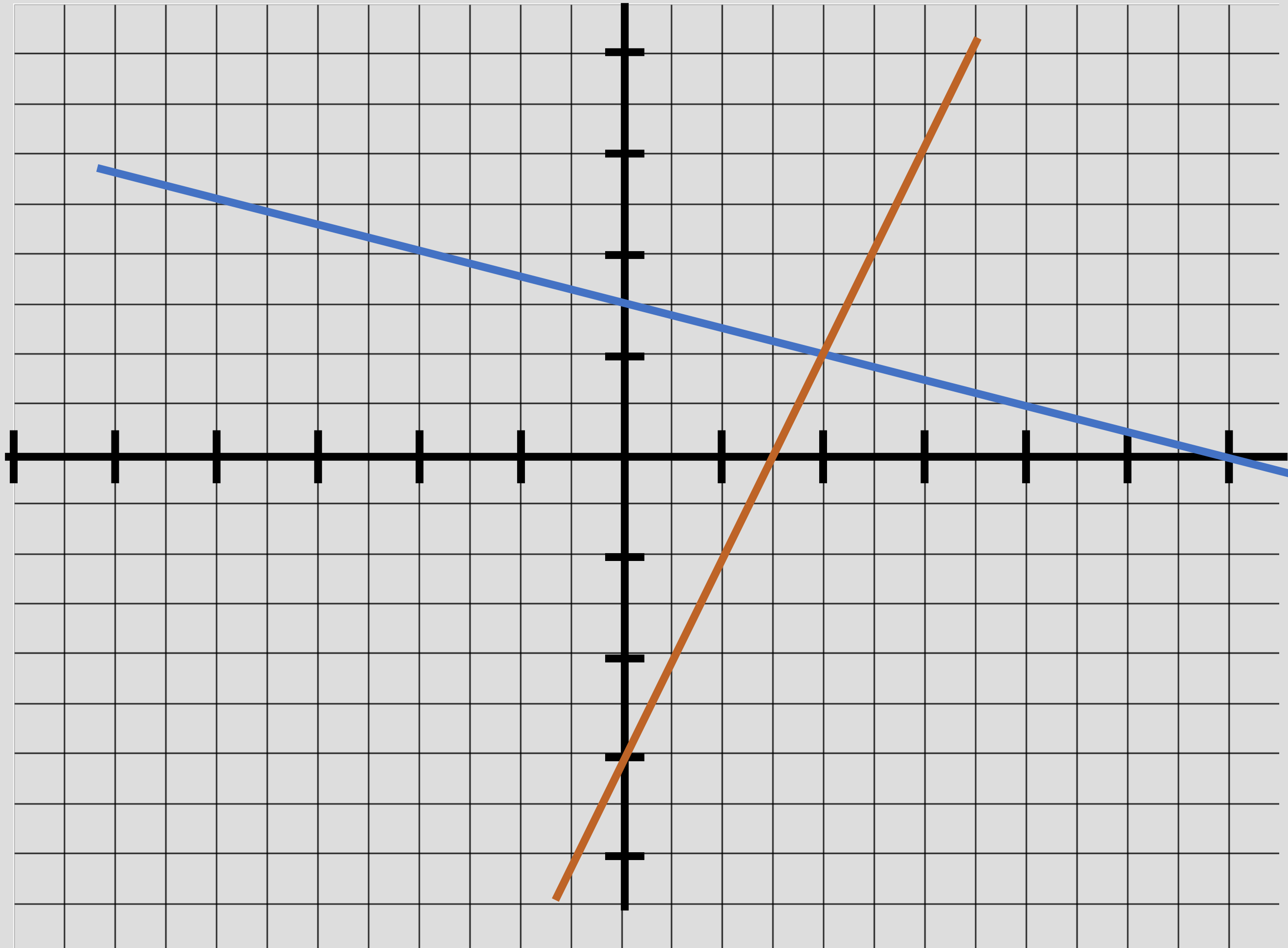
Geometric Interpretation

(1) $x + 4y = 6$ $\quad x = 0 \Rightarrow y = 1.5 \quad y = 0 \Rightarrow x = 6$

(2) $2x - y = 3$ $\quad x = 0 \Rightarrow y = -3 \quad y = 0 \Rightarrow x = 1.5$

Single Solution!

$x = 2, \ y = 1$



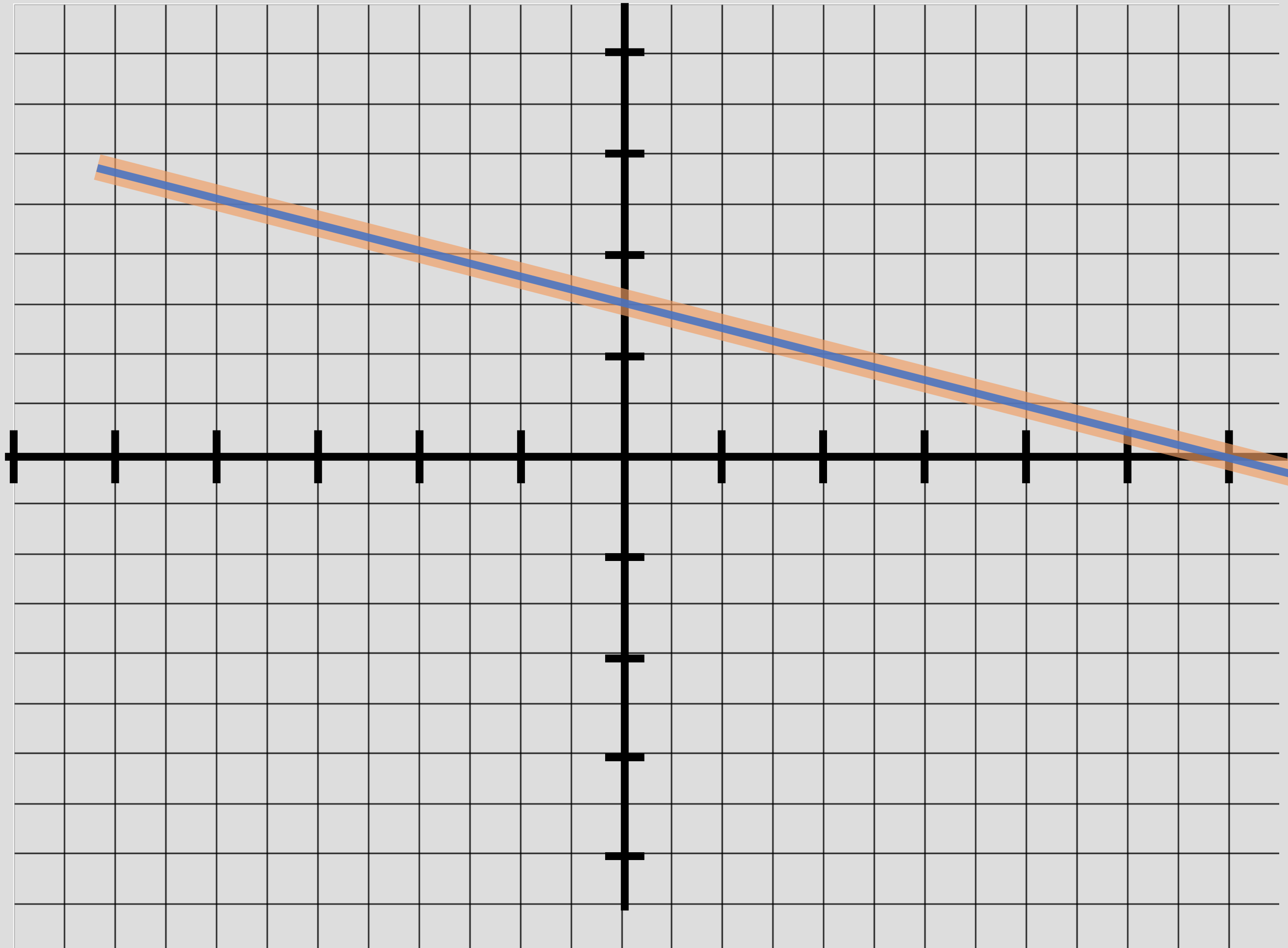
Geometric Interpretation

(1) $x + 4y = 6$ $\quad x = 0 \Rightarrow y = 1.5 \quad y = 0 \Rightarrow x = 6$

(2) $2x + 8y = 12$ $\quad x = 0 \Rightarrow y = 1.5 \quad y = 0 \Rightarrow x = 6$

Infinite Solutions!
anything that satisfies:

$$x = 6 - 4y_0$$



Geometric Interpretation

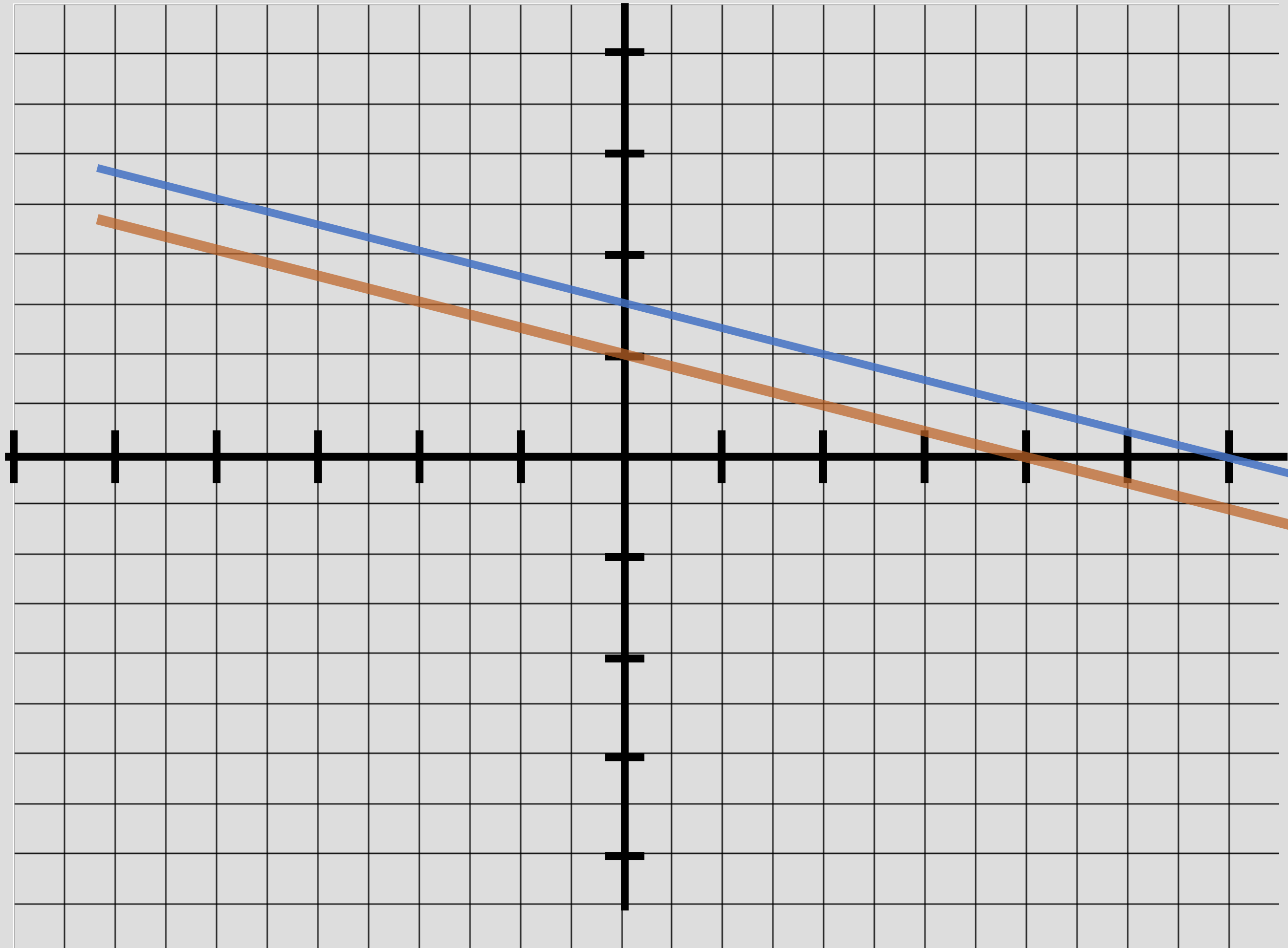
(1) $x + 4y = 6$ $\quad x = 0 \Rightarrow y = 1.5 \quad y = 0 \Rightarrow x = 6$

(2) $2x + 8y = \underline{8}$ $\quad x = 0 \Rightarrow y = 1 \quad y = 0 \Rightarrow x = 4$

No Solutions!

Parallel lines do not intersect!

$\varnothing$



Gaussian Elimination Summary

- Reduce to row-echelon form, from left-to-right by using:
  1. Multiply an equation with *nonzero* scalar
  2. Adding a scalar constant multiple of one equation to another
  3. Swapping equations

Single solution

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \end{array}\right]$$

Infinite solutions

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

No solution

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & * \end{array}\right]$$

- Back substitute to reduced row-echelon form, from right-to-left

Single solution

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \end{array}\right]$$

Pivots

Infinite solutions

$$\left[\begin{array}{cccc|c} 1 & 0 & * & 0 & * \\ 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Basic variables

Free variables

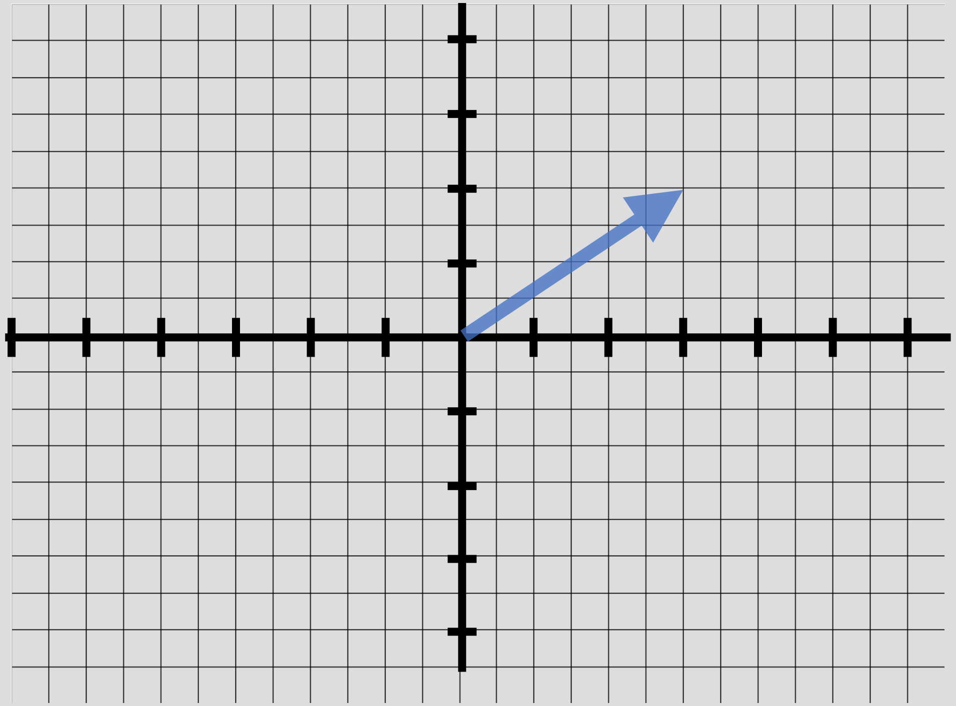
```
Data: Augmented matrix A ∈ ℝ^{m×(n+1)}, for a system of m equations with n variables
Result: Reduced form of augmented matrix
# Forward elimination procedure:
for each variable index i from 1 to n do
    if entry in row i, column i of A is 0 then
        if all entries in column i and row > i of A are 0 then
            proceed to next variable index;
        else
            find j, the smallest row index > i of A for which entry in column i ≠ 0 ;
            # The following rows implement the "swap" operation:
            old_row_j ⟵ row j of A;
            row j of A ⟵ row i of A;
            row i of A ⟵ old_row_j;
        end
    end
    divide row i of A by entry in row i, column i of A;
    for each row index k from i+1 to m do
        scaled_row_i ⟵ row i of A times entry in row k, column i of A;
        row k of A ⟵ row k of A − scaled_row_i;
    end
end
# Back substitution procedure:
for each variable index u from n−1 to 1 do
    if entry in row u, column u of A ≠ 0 then
        for each row v from u−1 to 1 do
            scaled_row_u ⟵ row u of A times entry in row v, column u of A;
            row v of A ⟵ row v of A − scaled_row_u;
        end
    end
end
```

Algorithm 1: The Gaussian elimination algorithm.

Vectors

- An array of N numbers
  - Represents coordinates in an N-dimensional space

$$\vec{x} = \begin{bmatrix} x_1 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}, \qquad \vec{x} \in \mathbb{R}^N$$



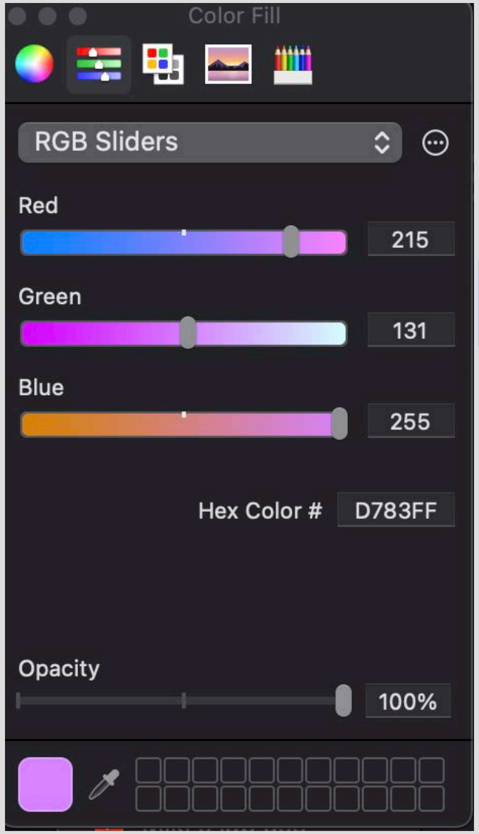
- For example:

$$\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \qquad \vec{x} \in \mathbb{R}^2$$

Vectors

- Since it’s an array of numbers, it can represent other things….

pixel color



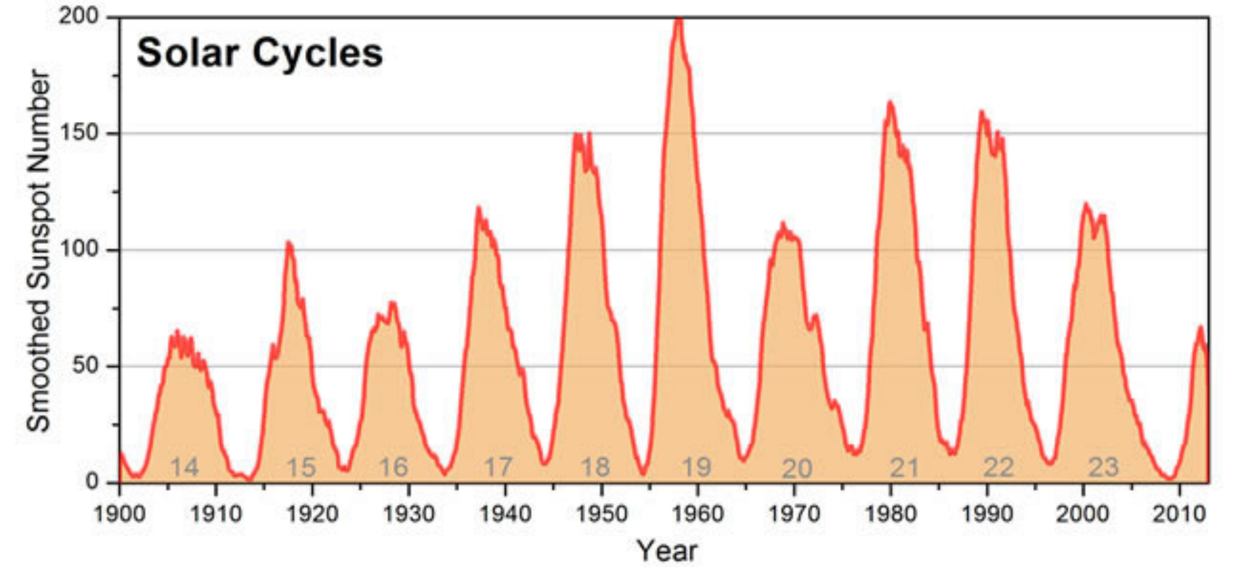
$$\vec{x} = \begin{bmatrix} 215 \\ 131 \\ 25 \end{bmatrix}$$

pixel values in an image

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Vectors

Data



$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{120} \end{bmatrix}$$

Special Vectors

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e}_N = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Matrices

- A collection of numbers in a rectangular form

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1M} \\ x_{21} & x_{22} & \cdots & x_{2M} \\ \vdots & \vdots & \ldots & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{NM} \end{bmatrix}, \quad X \in \mathbb{R}^{\mathbb{N} \times \mathbb{M}}$$

- Or a collection of M, N-length vectors

$$X = \begin{bmatrix} \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_M \\ \\ \end{bmatrix}, \quad X \in \mathbb{R}^{\mathbb{N} \times \mathbb{M}}$$

Vectors as Matrices

- A vector is a degenerate matrix

$$\vec{x} = \begin{bmatrix} x_1 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad \vec{x} \in \mathbb{R}^{N \times 1}$$

- A scalar is a degenerate vector or matrix

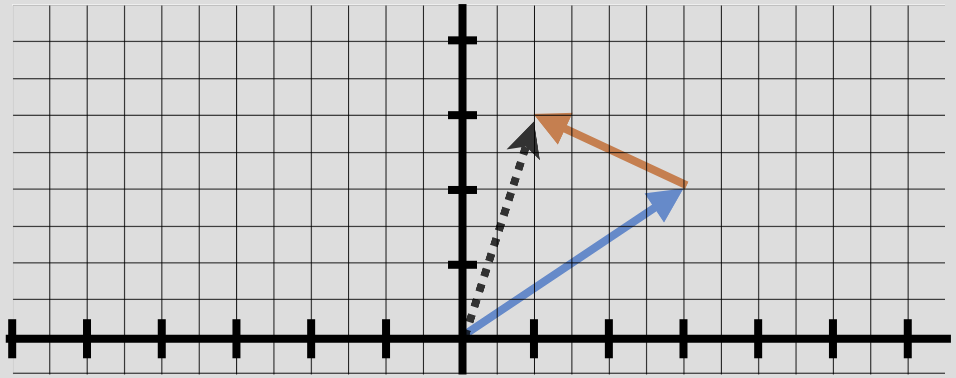
$$a \in \mathbb{R}^{1 \times 1}$$

Vector addition

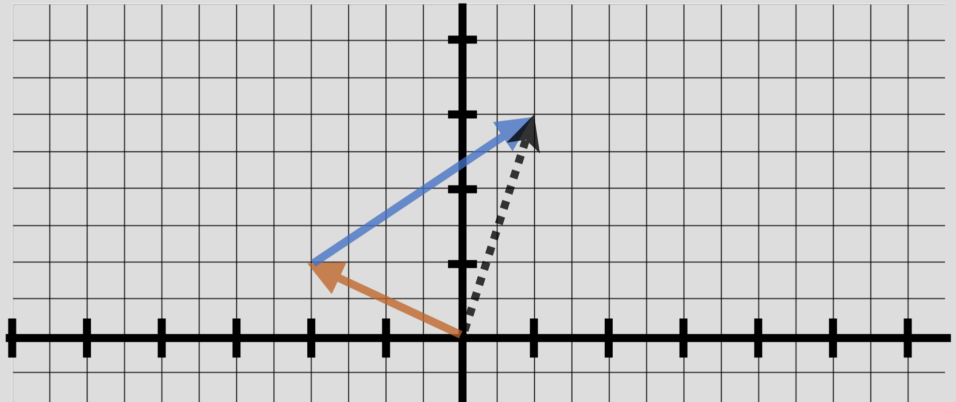
- Two vectors of the same length can be added
  - Addition is element-wise

$$\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \; \vec{y} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$\vec{x} + \vec{y} =$



$\vec{y} + \vec{x} =$



Properties of vector addition

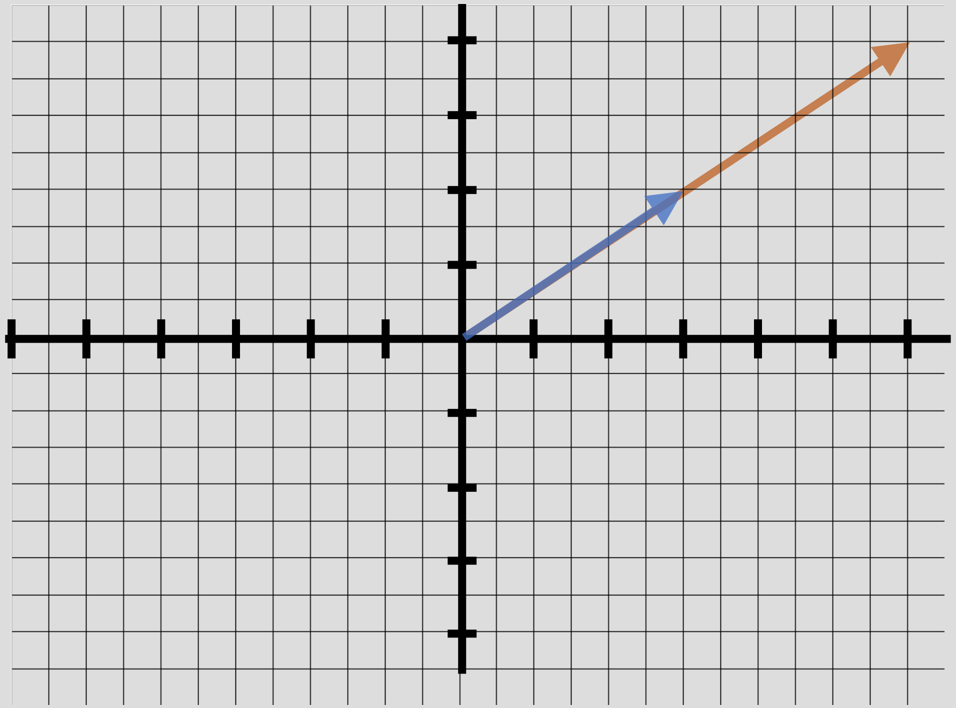
- Commutativity: $\overrightarrow{x} + \overrightarrow{y} = \overrightarrow{y} + \overrightarrow{x}$
- Associativity: $(\overrightarrow{x} + \overrightarrow{y}) + \vec{z} = \overrightarrow{x} + (\overrightarrow{y} + \vec{z})$
- Additive negative: $\overrightarrow{x} + (-\overrightarrow{x}) = \overrightarrow{0}$
- Additive identity: $\overrightarrow{x} + \overrightarrow{0} = \overrightarrow{x}$

Scalar Vector Multiplication

- Multiplying with a scalar result in multiplying each element.

$$a\vec{x} = \begin{bmatrix} ax_1 \\ ax_1 \\ \vdots \\ ax_{N \times 1} \end{bmatrix}$$

$$2 \cdot \vec{x} = 2 \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$



Vector Transpose

- $\overrightarrow{x}^T$ is the transpose of $\overrightarrow{x}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad \vec{x} \in \mathbb{R}^{N \times 1} \qquad \vec{x}^T = \begin{bmatrix} x_1 & x_1 & \cdots & x_N \end{bmatrix}, \quad \vec{x}^T \in \mathbb{R}^{1 \times N}$$

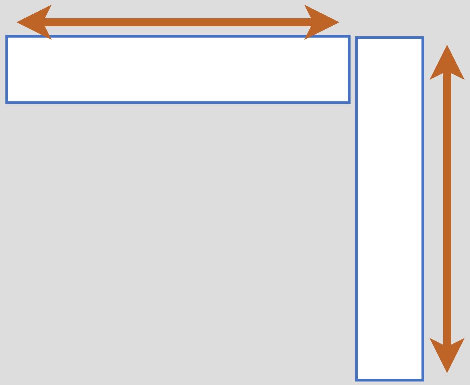
- $\overrightarrow{x}$ is always a column vector
- To represent a row vector, write: $\overrightarrow{x}^T$

Vector Matrix Multiplication

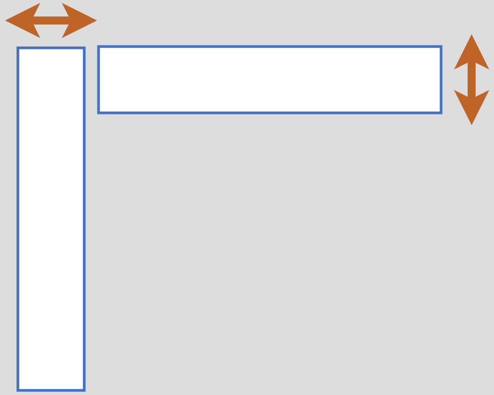
- Let $\overrightarrow{x} \in \mathbb{R}^{N\times 1}$, $\overrightarrow{y} \in \mathbb{R}^{N\times 1}$, $\vec{z} \in \mathbb{R}^{M\times 1}$
- Multiplication is valid only for specific matching dimensions!

Like this....

and like that!



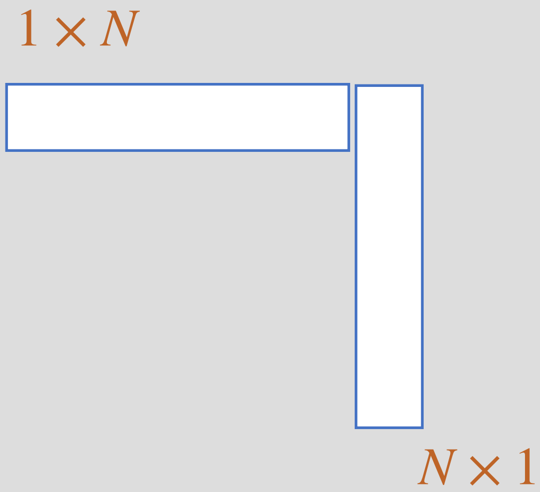
Or



Vector Vector Multiplication

$$\vec{x}, \vec{y} \in \mathbb{R}^{N \times 1}$$

$$\vec{y}^T \vec{x} =$$



Like this....

and like that!

Vector Vector Multiplication

Like this....

and like that!

$\overrightarrow{x}, \overrightarrow{y} \in \mathbb{R}^{N\times1}$

$1 \times N$

$\overrightarrow{y}^T\overrightarrow{x} =$

$N \times 1$

$= y_1x_1 + y_2x_2 + y_3x_3 + \cdots + y_Nx_N$

scalar 1×1

Also known as “inner product” or “dot product”

Vector Vector Multiplication

Like this....

and like that!

$\vec{x}, \vec{y} \in \mathbb{R}^{N \times 1}$

$\vec{y}^T \vec{x} =$  $= y_1 x_1 + y_2 x_2 + y_3 x_3 + \cdots + y_N x_N$

scalar 1×1

Also known as "inner product" or "dot product"

$\vec{x}\,\vec{y}^T =$  $=$ 

$N \times 1$ $1 \times N$ $N \times N$

Vector Vector Multiplication

Like this….

and like that!

$\vec{x}, \vec{y} \in \mathbb{R}^{N\times 1}$

$\vec{y}^T\vec{x} =$

$= y_1x_1 + y_2x_2 + y_3x_3 + \cdots + y_Nx_N$

scalar 1×1

Also known as "inner product" or "dot product"

$1 \times N$

$\vec{x}\,\vec{y}^T =$

$N \times 1$

$$= \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_N \\ x_2y_1 & x_2y_2 & \cdots & x_2y_N \\ \vdots & \vdots & \cdots & \vdots \\ x_Ny_1 & x_Ny_2 & \cdots & x_Ny_N \end{bmatrix}$$

$N \times N$

Do not commute!

Also known as "outer product"

Vector Matrix Multiplication

- Let $\overrightarrow{x} \in \mathbb{R}^{N\times1}$, $\overrightarrow{y} \in \mathbb{R}^{N\times1}$, $\vec{z} \in \mathbb{R}^{M\times1}$
- $N \times 1$ with $1 \times M$ with output: $N \times M$
- $M \times 1$ with $1 \times N$ with output: $M \times N$
- $1 \times N$ with $N \times 1$ with output: 1×1