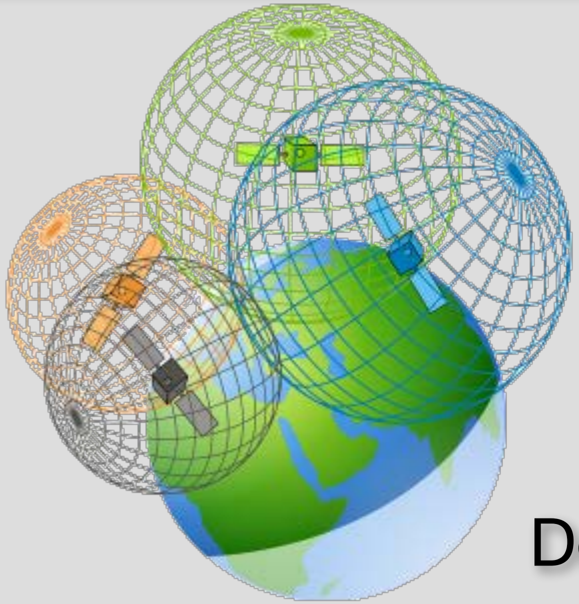


Welcome to EECS 16A!

Designing Information Devices and Systems I

Ana Arias and Miki Lustig

Fall 2021

Lecture ~~3A~~ 2A

Gaussian Elimination

Vectors, Matrices, Multiplications,

And Span

Announcements

- Quest: Tuesday 09/14/21 8:30pm

- Last time:
  - Gaussian Elimination
  - Started vectors
- Today:
  - Continue vectors
  - Matrix-Matrix and Matrix-vector Multiplications
  - Matrix-Vector Multiplications as linear set of equations
  - Intro to span

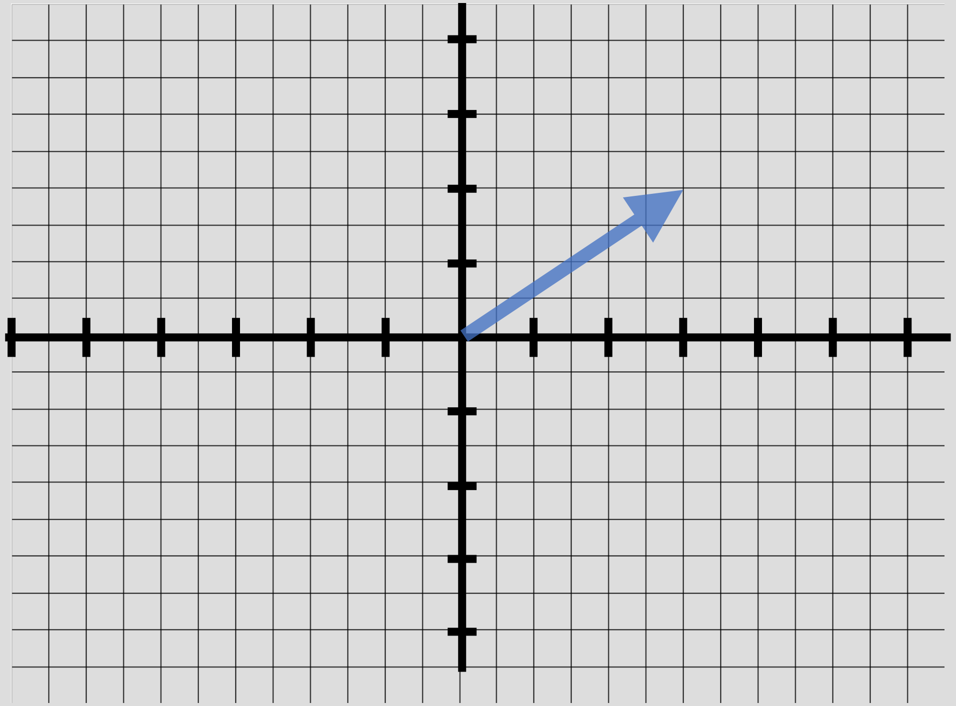
Vectors

- An array of N numbers
  - Represents coordinates in an N-dimensional space

$$\vec{x} = \begin{bmatrix} x_1 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}, \qquad \vec{x} \in \mathbb{R}^N$$

- For example:

$$\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \qquad \vec{x} \in \mathbb{R}^2$$



Matrices

- A collection of numbers in a rectangular form

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1M} \\ x_{21} & x_{22} & \cdots & x_{2M} \\ \vdots & \vdots & \ldots & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{NM} \end{bmatrix}, \quad X \in \mathbb{R}^{\mathbb{N} \times \mathbb{M}}$$

- Or a collection of M, N-length vectors

$$X = \begin{bmatrix} \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_M \\ \\ \end{bmatrix}, \quad X \in \mathbb{R}^{\mathbb{N} \times \mathbb{M}}$$

Matrix Addition

- When matrices are the same size, they can be added element-wise

$$X + Y = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

- Scalar multiplication — by all elements

$$2X = 2\begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Matrix Addition

- When matrices are the same size, they can be added element-wise

$$X + Y = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

diagonal matrix!

- Scalar multiplication — by all elements

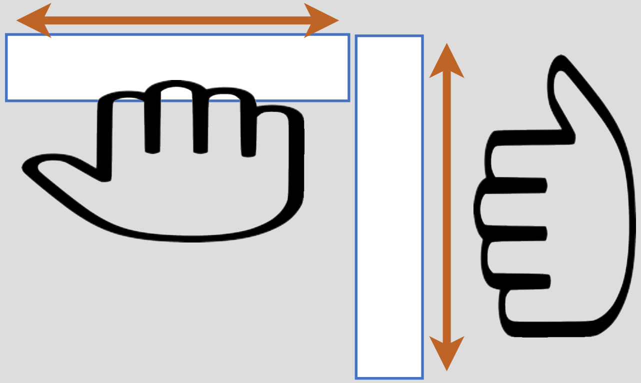
$$2X = 2\begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -2 & -4 \end{bmatrix}$$

Vector-Vector Multiplication

- Multiplication is valid only for specific matching dimensions!
    - Width of the 1st, matches length of the second

Like this....

and like that!

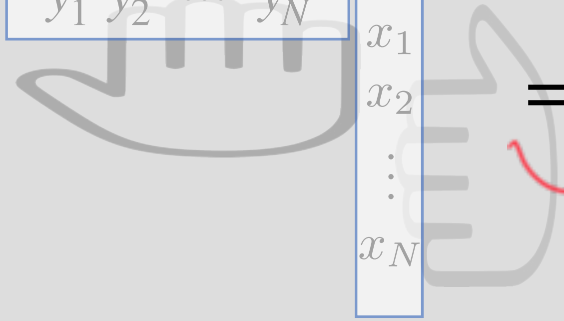


Vector Vector Multiplication

Like this....

and like that!

$$\vec{x}, \vec{y} \in \mathbb{R}^{N\times 1}$$



$$\vec{y}^T\vec{x} = \overset{1\times N}{\begin{bmatrix} y_1 & y_2 & \cdots & y_N \end{bmatrix}} \underset{N\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}} = \underbrace{y_1x_1 + y_2x_2 + y_3x_3 + \cdots + y_Nx_N}_{\text{scalar } 1\times 1} = \underset{1\times 1}{\boxed{}}$$

Also known as “inner product” or “dot product”

Matrix-Vector Multiplication

Like this….

and like that!

$$\vec{x}, \vec{y} \in \mathbb{R}^{N\times 1}$$

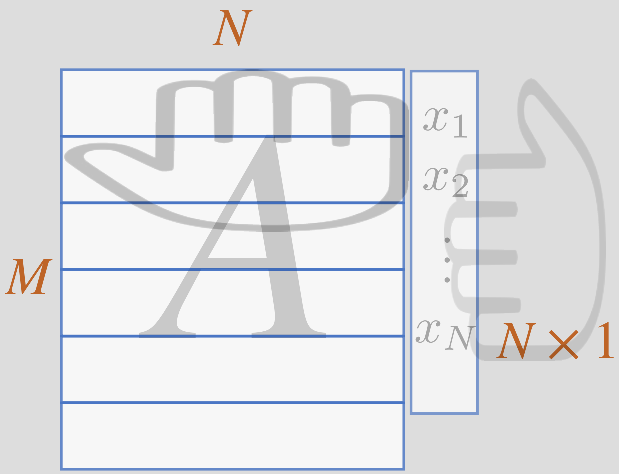
$$\vec{y}^T\vec{x} = \overset{1\times N}{\begin{bmatrix} y_1 & y_2 & \cdots & y_N \end{bmatrix}} \underset{N\times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}} = \underbrace{y_1x_1 + y_2x_2 + y_3x_3 + \cdots + y_Nx_N}_{\text{scalar } 1\times 1} = \underset{1\times 1}{\boxed{}}$$

Also known as "inner product" or "dot product"

What about this case….

$$A \in R^{M\times N}, \vec{x} \in \mathbb{R}^{N\times 1}$$

$$A\vec{x} =$$



$$= ?$$

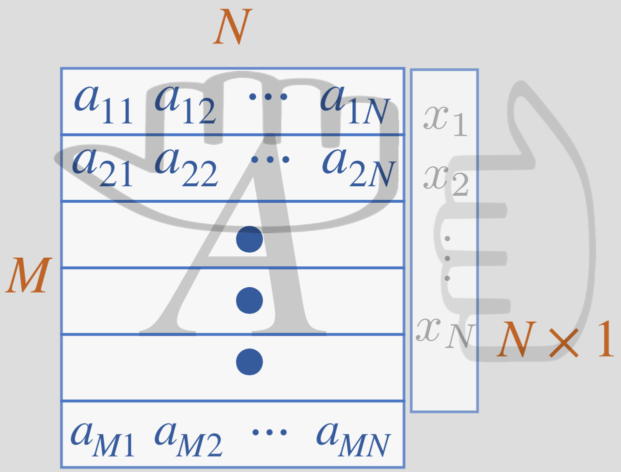
Matrix-Vector Multiplication

Like this....

and like that!

$A \in R^{M\times N}, \vec{x} \in \mathbb{R}^{N\times 1}$

$A\vec{x} =$



$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N \\ \\ \\ \\ \end{bmatrix}$$

$[a_{11}\ a_{12}\ \cdots\ a_{1N}]\vec{x} = \vec{y}_1^{\,T}\vec{x}$

Matrix-Vector Multiplication

Like this....

and like that!

$A \in R^{M\times N}, \vec{x} \in \mathbb{R}^{N\times 1}$

$$A\vec{x} = \begin{array}{cc} & N \\ M & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ & \bullet & & \\ & \bullet & & \\ & \bullet & & \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} N \times 1 = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N \\ \\ \\ \\ \end{bmatrix}$$

$$[a_{11}\ a_{12}\ \cdots\ a_{1N}]\vec{x} = \vec{y}_1^{T}\vec{x}$$

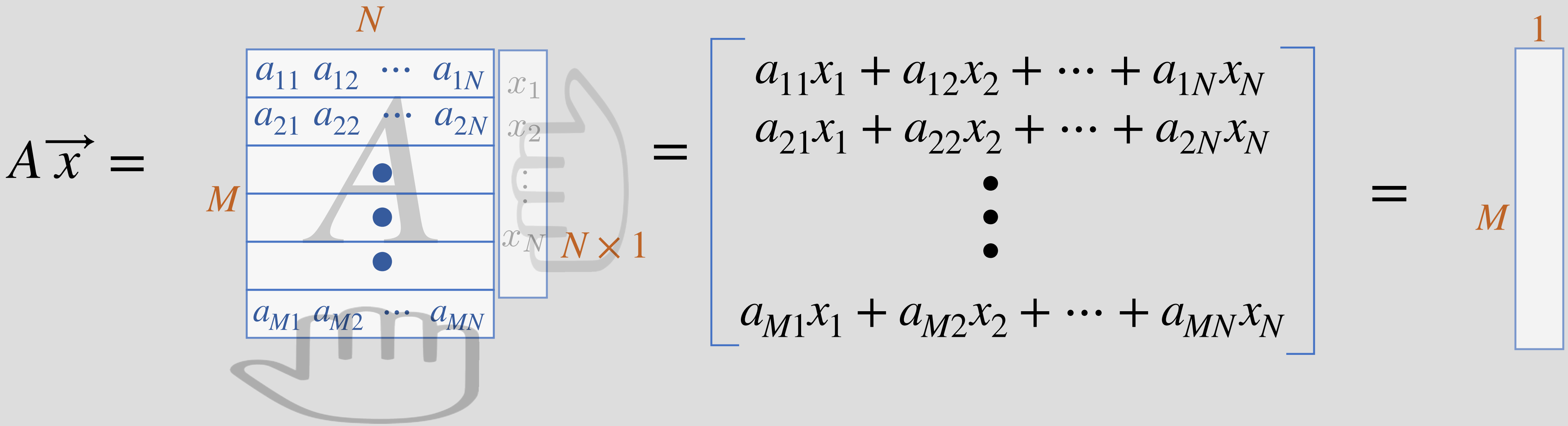
$$[a_{21}\ a_{22}\ \cdots\ a_{2N}]\vec{x} = \vec{y}_2^{T}\vec{x}$$

Matrix-Vector Multiplication

Like this....

and like that!

$A \in R^{M\times N}, \vec{x} \in \mathbb{R}^{N\times 1}$



$$A\vec{x} = \begin{array}{cc} & N \\ M & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ & \vdots & & \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} N\times 1 = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N \\ \vdots \\ a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N \end{bmatrix} = \begin{array}{cc} & 1 \\ M & \begin{bmatrix} \; \\ \; \\ \; \end{bmatrix} \end{array}$$

$$[a_{11}\ a_{12}\ \cdots\ a_{1N}]\vec{x} = \vec{y}_1^T\vec{x}$$

$$[a_{21}\ a_{22}\ \cdots\ a_{2N}]\vec{x} = \vec{y}_2^T\vec{x}$$

$$\vdots$$

$$[a_{M1}\ a_{M2}\ \cdots\ a_{MN}]\vec{x} = \vec{y}_M^T\vec{x}$$

Matrix-Vector Multiplication

Like this....

and like that!

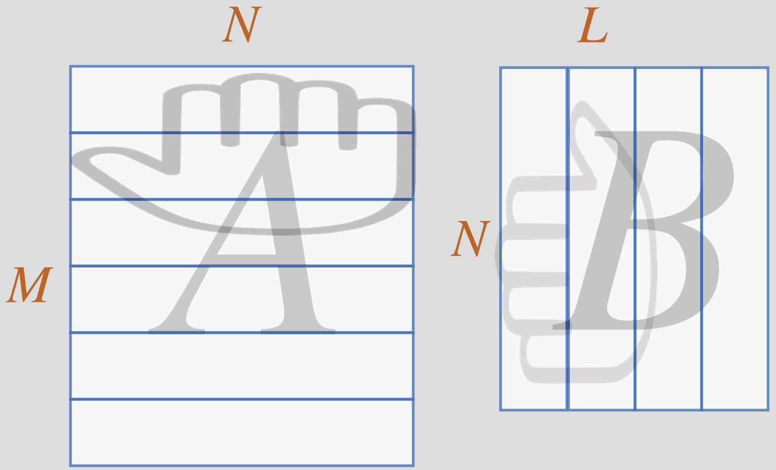
$A \in R^{M\times N}, \vec{x} \in \mathbb{R}^{N\times 1}$

$$A\vec{x} = \begin{array}{cc} & N \\ M & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ & \vdots & & \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} N \times 1 = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N \\ \vdots \\ a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N \end{bmatrix} = \begin{array}{cc} & 1 \\ M & \begin{bmatrix} \\ \\ \\ \end{bmatrix} \end{array}$$

What about this case....

$A \in R^{M\times N}, B \in \mathbb{R}^{N\times L}$

$AB =$



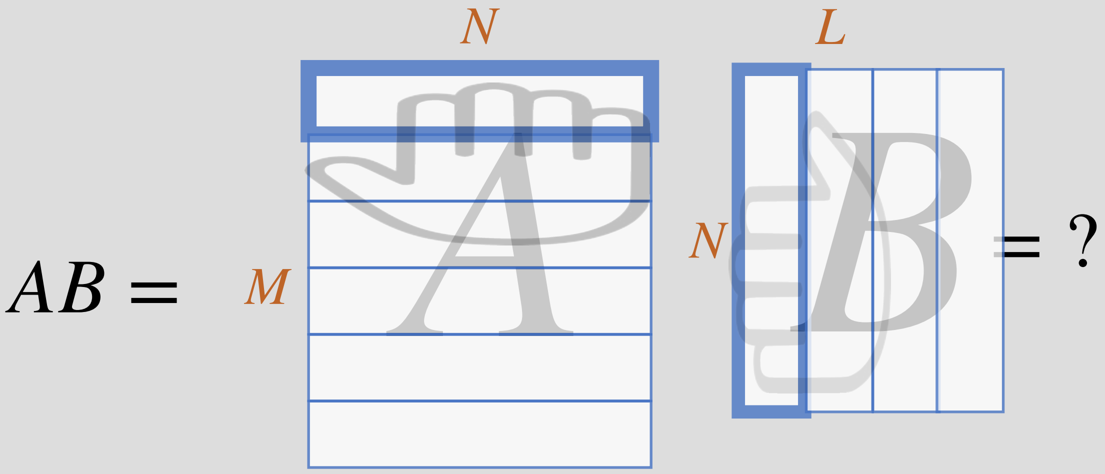
$= \,?$

Matrix-Matrix Multiplication

Like this....

and like that!

$A \in R^{M\times N}, B \in \mathbb{R}^{N\times L}$



$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1N}b_{N1} & & \\ & & \\ & & \end{bmatrix}$$

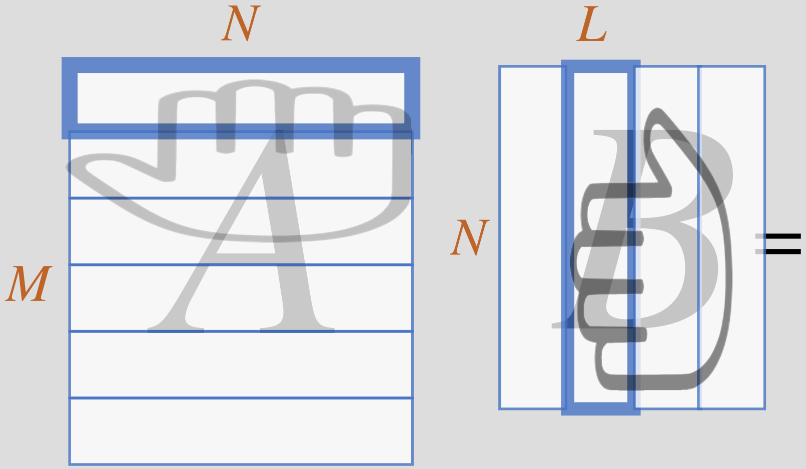
Matrix-Matrix Multiplication

Like this....

and like that!

$A \in R^{M\times N}, B \in \mathbb{R}^{N\times L}$

$AB =$



$= ?$

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1N}b_{N1} & a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1N}b_{N2} \end{bmatrix}$$

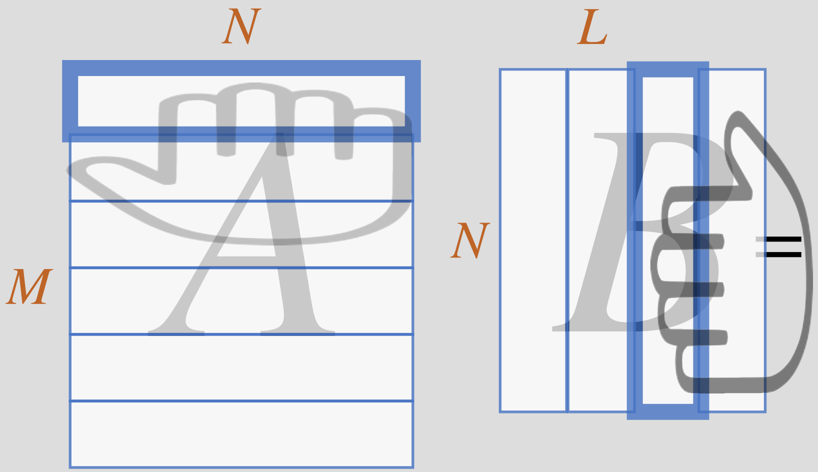
Matrix-Matrix Multiplication

$A \in R^{M\times N}, B \in \mathbb{R}^{N\times L}$

Like this....

and like that!

$AB =$



$= ?$

$$\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+\cdots+a_{1N}b_{N1} & a_{11}b_{12}+a_{12}b_{22}+\cdots+a_{1N}b_{N2} & \cdots \end{bmatrix}$$

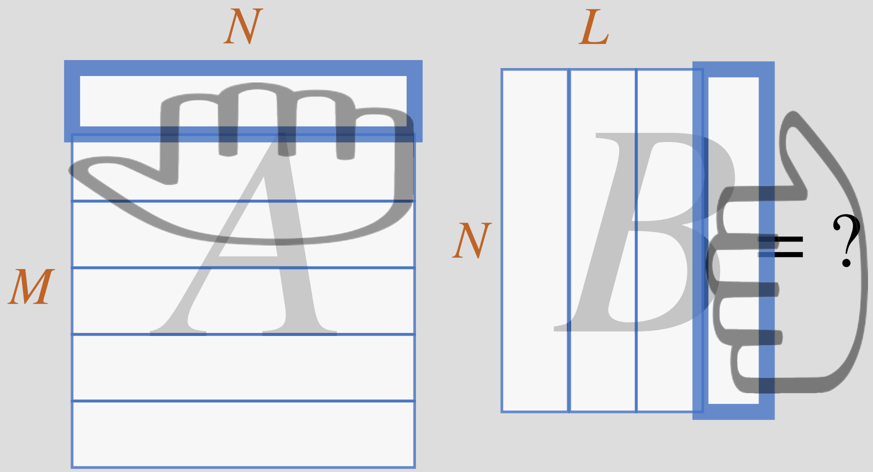
Matrix-Matrix Multiplication

Like this....

and like that!

$A \in R^{M\times N}, B \in \mathbb{R}^{N\times L}$

$AB =$



$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1N}b_{N1} & a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1N}b_{N2} & \cdots & a_{11}b_{1L} + a_{12}b_{2L} + \cdots + a_{1N}b_{NL} \\ & & & \\ & & & \end{bmatrix}$$

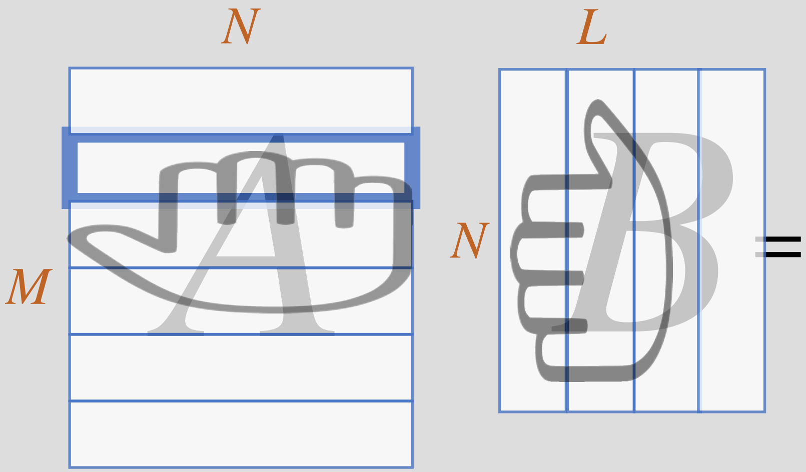
Matrix-Matrix Multiplication

Like this....

and like that!

$A \in R^{M\times N}, B \in \mathbb{R}^{N\times L}$

$AB =$



$= ?$

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1N}b_{N1} & a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1N}b_{N2} & \cdots & a_{11}b_{1L} + a_{12}b_{2L} + \cdots + a_{1N}b_{NL} \\ a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2N}b_{N1} & a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2N}b_{N2} & \cdots & a_{21}b_{1L} + a_{22}b_{2L} + \cdots + a_{2N}b_{NL} \\ & & & \\ & & & \end{bmatrix}$$

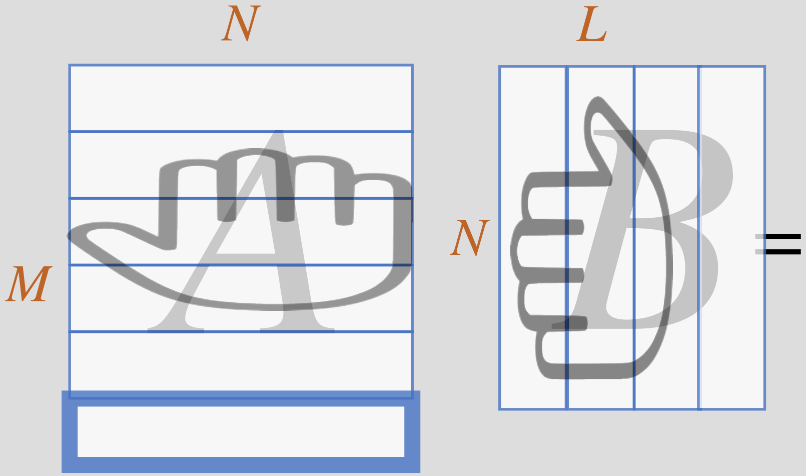
Matrix-Matrix Multiplication

Like this....

and like that!

$A \in R^{M\times N}, B \in \mathbb{R}^{N\times L}$

$$AB =$$



$$= ? \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+\cdots+a_{1N}b_{N1} & a_{11}b_{12}+a_{12}b_{22}+\cdots+a_{1N}b_{N2} & \cdots & a_{11}b_{1L}+a_{12}b_{2L}+\cdots+a_{1N}b_{NL} \\ a_{21}b_{11}+a_{22}b_{21}+\cdots+a_{2N}b_{N1} & a_{21}b_{12}+a_{22}b_{22}+\cdots+a_{2N}b_{N2} & \cdots & a_{21}b_{1L}+a_{22}b_{2L}+\cdots+a_{2N}b_{NL} \\ \vdots & \vdots & & \vdots \\ a_{M1}b_{11}+a_{M2}b_{21}+\cdots+a_{MN}b_{N1} & a_{M1}b_{12}+a_{M2}b_{22}+\cdots+a_{MN}b_{N2} & & a_{M1}b_{1L}+a_{M2}b_{2L}+\cdots+a_{MN}b_{NL} \end{bmatrix}$$

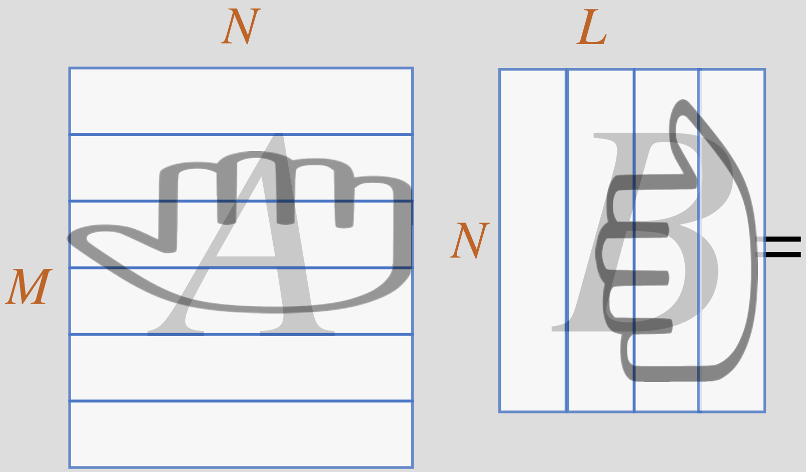
Matrix-Matrix Multiplication

Like this....

and like that!

$A \in R^{M\times N}, B \in \mathbb{R}^{N\times L}$

$AB =$



$= ?$

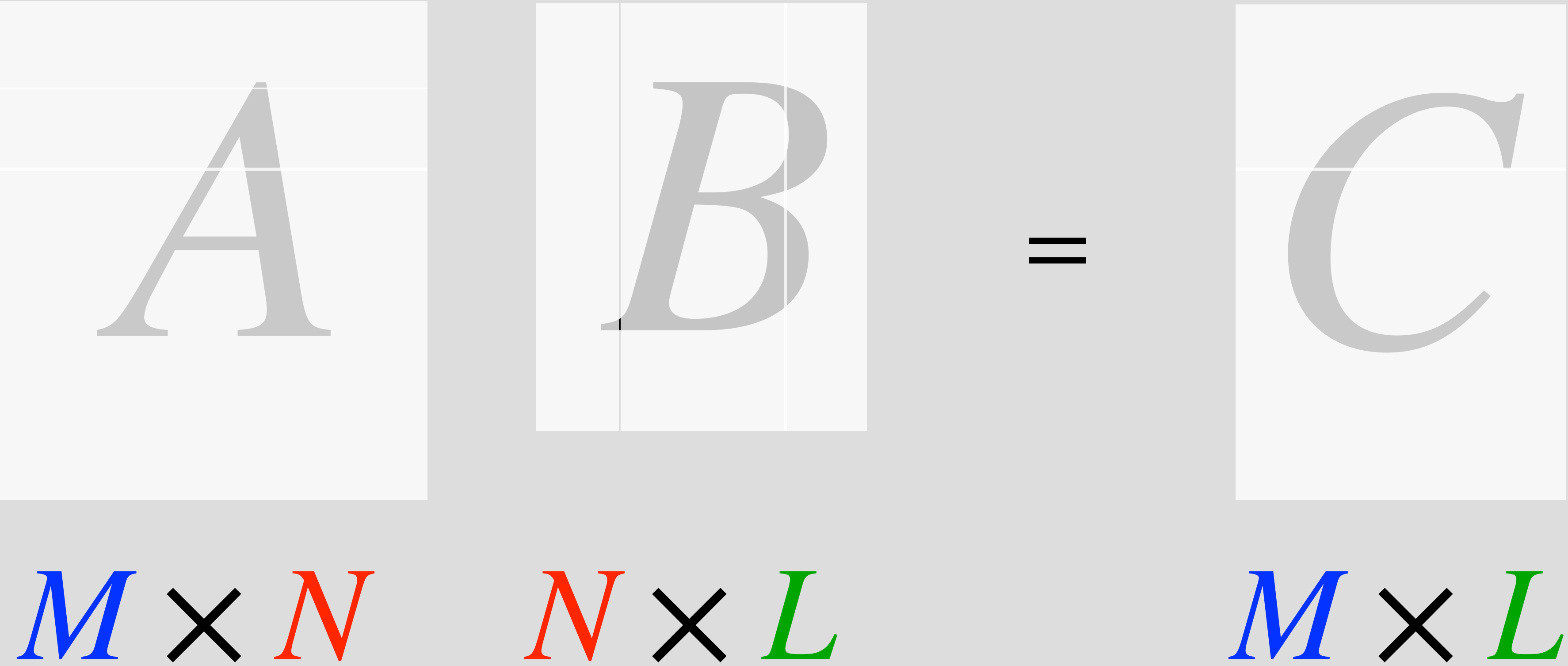
$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1N}b_{N1} & a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1N}b_{N2} & \cdots & a_{11}b_{1L} + a_{12}b_{2L} + \cdots + a_{1N}b_{NL} \\ a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2N}b_{N1} & \boxed{a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2N}b_{N2}} & \cdots & a_{21}b_{1L} + a_{22}b_{2L} + \cdots + a_{2N}b_{NL} \\ \vdots & \vdots & & \vdots \\ a_{M1}b_{11} + a_{M2}b_{21} + \cdots + a_{MN}b_{N1} & a_{M1}b_{12} + a_{M2}b_{22} + \cdots + a_{MN}b_{N2} & & a_{M1}b_{1L} + a_{M2}b_{2L} + \cdots + a_{MN}b_{NL} \end{bmatrix}$$

Result at location 2x2 = $a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2N}b_{N2}$

Matrix-Vector Multiplication

Like this….

and like that!



$$A \quad B = C$$

$M \times N$ $\quad$ $N \times L$ $\quad$ $M \times L$

Vector Vector Multiplication

Like this....

and like that!

$\vec{x}, \vec{y} \in \mathbb{R}^{N\times1}$

$$\vec{x}\,\vec{y}^{T} = \begin{array}{cc} & \overset{1\times N}{\boxed{y_1\ y_2\ \cdots\ y_N}} \\ \underset{N\times 1}{\boxed{\begin{matrix}x_1\\x_2\\\vdots\\x_N\end{matrix}}} & \end{array} = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_N \\ x_2y_1 & x_2y_2 & \cdots & x_2y_N \\ \vdots & \vdots & \cdots & \vdots \\ x_Ny_1 & x_Ny_2 & \cdots & x_Ny_N \end{bmatrix} = \underset{N\times N}{\boxed{\phantom{\begin{matrix}X & X\\X & X\end{matrix}}}}$$

Vector Vector Multiplication

Like this....

and like that!

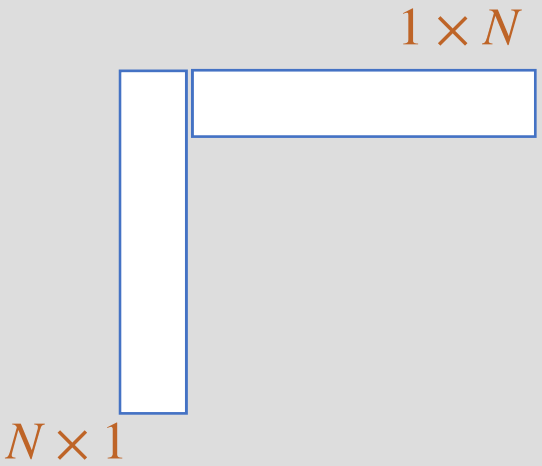
$\vec{x}, \vec{y} \in \mathbb{R}^{N\times 1}$

$\vec{y}^{\,T}\vec{x} =$

$$= \underbrace{y_1x_1 + y_2x_2 + y_3x_3 + \cdots + y_Nx_N}_{\text{scalar } 1\times 1}$$

Also known as “inner product”
or “dot product”

$\vec{x}\,\vec{y}^{\,T} =$

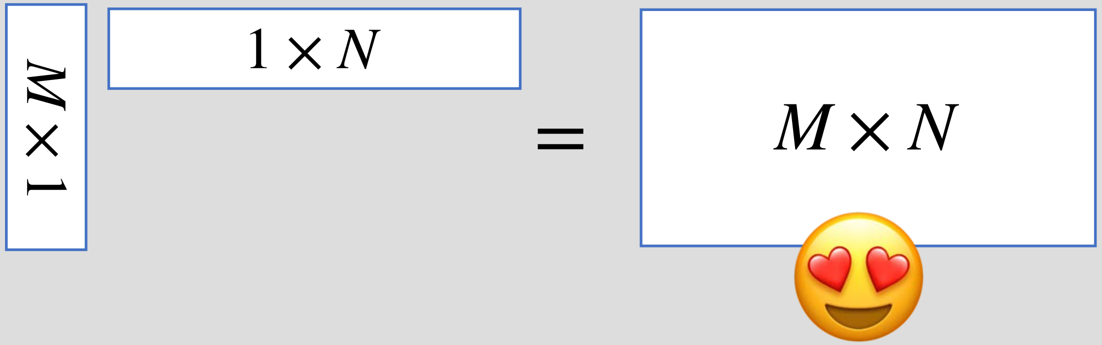


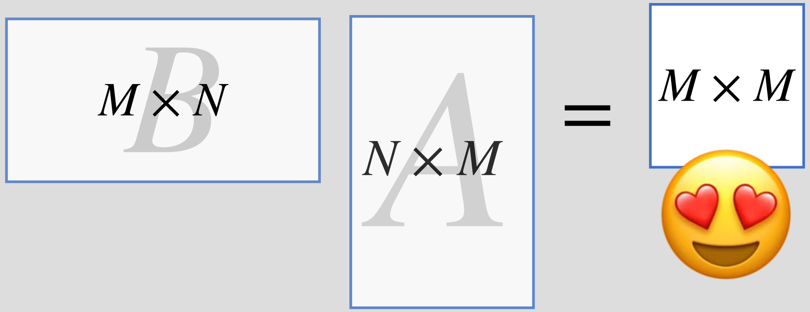
$$= \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_N \\ x_2y_1 & x_2y_2 & \cdots & x_2y_N \\ \vdots & \vdots & \ldots & \vdots \\ x_Ny_1 & x_Ny_2 & \cdots & x_Ny_N \end{bmatrix}$$

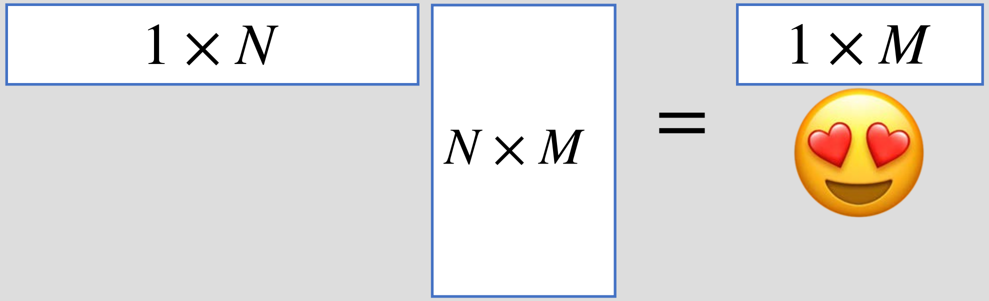
$N \times N$

Do not commute!
Also known as “outer product”

Matrix Matrix Multiplication



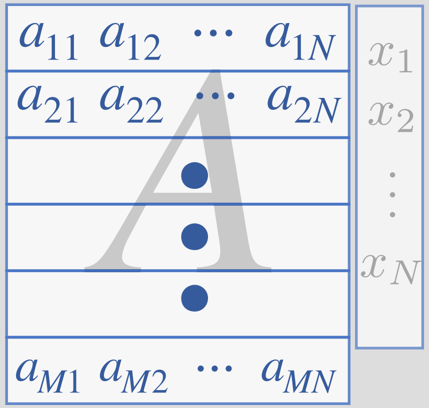




Matrix multiplication does not commute!

Matrix-Vector Form of Systems of Linear Equations

- Consider the matrix equation: $A\vec{x} = \vec{b}$



$$A\vec{x} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ & \bullet & & \\ & \bullet & & \\ & \bullet & & \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix}}_{M \times N} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}}_{N \times 1} = \underbrace{\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N \\ \bullet \\ \bullet \\ \bullet \\ a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N \end{bmatrix}}_{M \times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{bmatrix}$$

Same as the Augmented Matrix!

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1N} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2N} & b_2 \\ \vdots & & \vdots & & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} & b_M \end{array}\right]$$

$A\vec{x} = \vec{b}$ is another way to write
A linear set of equations!

Row vs Column Perspective

- Row / Measurement Perspective of $A\overrightarrow{x} = \overrightarrow{b}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

Row vs Column Perspective

- Row / Measurement Perspective of $A\overrightarrow{x} = \overrightarrow{b}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Q: What does a row mean?

A: How each variable affect a particular measurement

Row vs Column Perspective

- Column Perspective of $A\overrightarrow{x} = \overrightarrow{b}$

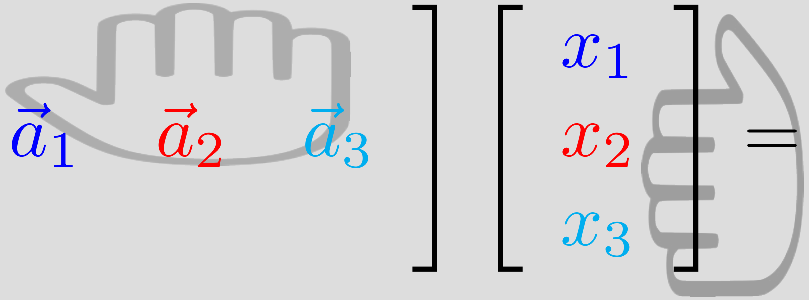
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} \quad & \quad & \quad \\ \\ \\ \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

Row vs Column Perspective

- Column Perspective of $A\overrightarrow{x} = \overrightarrow{b}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

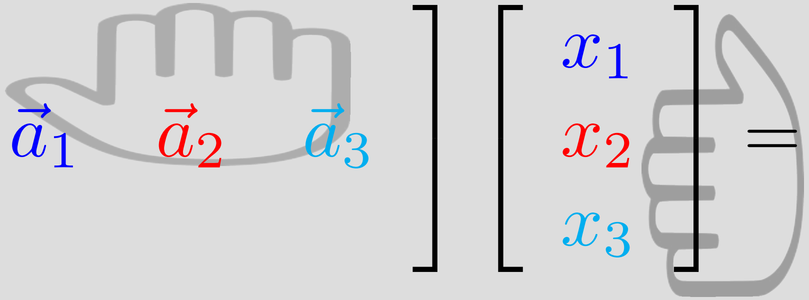


$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

Row vs Column Perspective

- Column Perspective of $A\overrightarrow{x} = \overrightarrow{b}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$



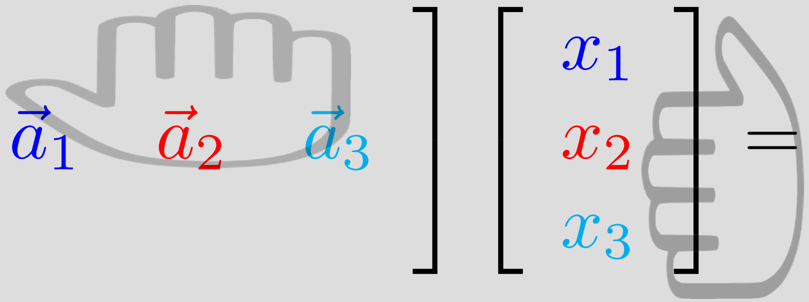
$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

$$x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 =$$

Row vs Column Perspective

- Column Perspective of $A\overrightarrow{x} = \overrightarrow{b}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$



$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 =$$

$$= \begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ a_{22}x_2 \end{bmatrix} + \begin{bmatrix} a_{13}x_3 \\ a_{23}x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Q: What does a column mean?

A: How a particular variable affects all measurements.

Linear combination of vectors

- Given set of vectors $\{\overrightarrow{a}_1, \overrightarrow{a}_2, \cdots, \overrightarrow{a}_M\} \in \mathbb{R}^N$, and coefficients $\{\alpha_1, \alpha_2, \cdots, \alpha_M\} \in \mathbb{R}$
- A linear combination of vectors is defined as: $\overrightarrow{b} \triangleq \alpha_1 \overrightarrow{a}_1 + \alpha_2 \overrightarrow{a}_2 + \cdots + \alpha_M \overrightarrow{a}_M$

Recall: $A\overrightarrow{x}$:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

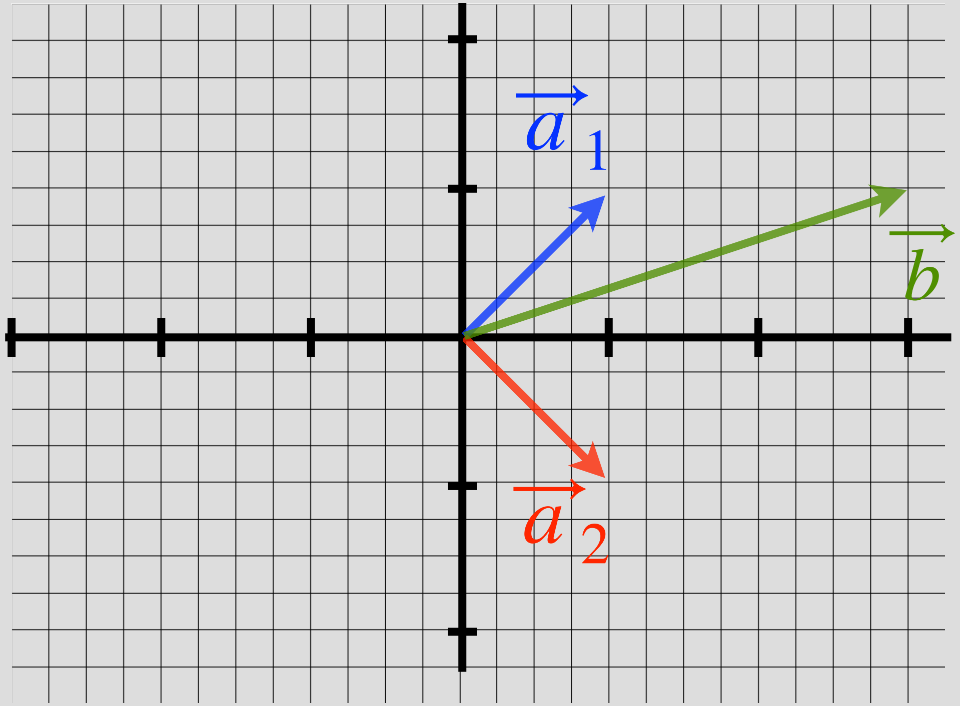
$$= x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3$$

Matrix-vector multiplication is a linear combination of the columns of A!

Linear Set of Equations as a Linear Combination

- Consider the problem: $A\overrightarrow{x} = \overrightarrow{b}$:

$$\begin{array}{cccc} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} & = & \begin{bmatrix} 3 \\ 1 \end{bmatrix} \\ \swarrow \quad \swarrow & & & \swarrow \\ \overrightarrow{a}_1 \quad \overrightarrow{a}_2 & & & \overrightarrow{b} \end{array}$$



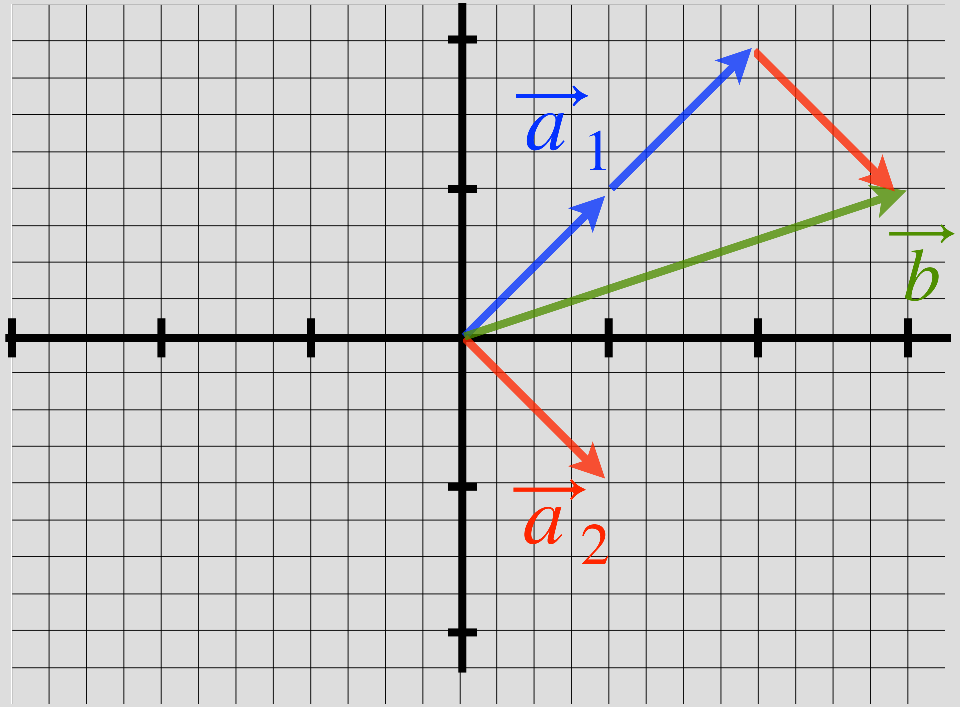
Linear Set of Equations as a Linear Combination

- Consider the problem: $A\overrightarrow{x} = \overrightarrow{b}$:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\begin{array}{ccc} \swarrow & \swarrow & \swarrow \\ \overrightarrow{a}_1 & \overrightarrow{a}_2 & \overrightarrow{b} \end{array}$$

Q: What linear combination of $\overrightarrow{a}_1, \overrightarrow{a}_2$ will give $\overrightarrow{b}$?



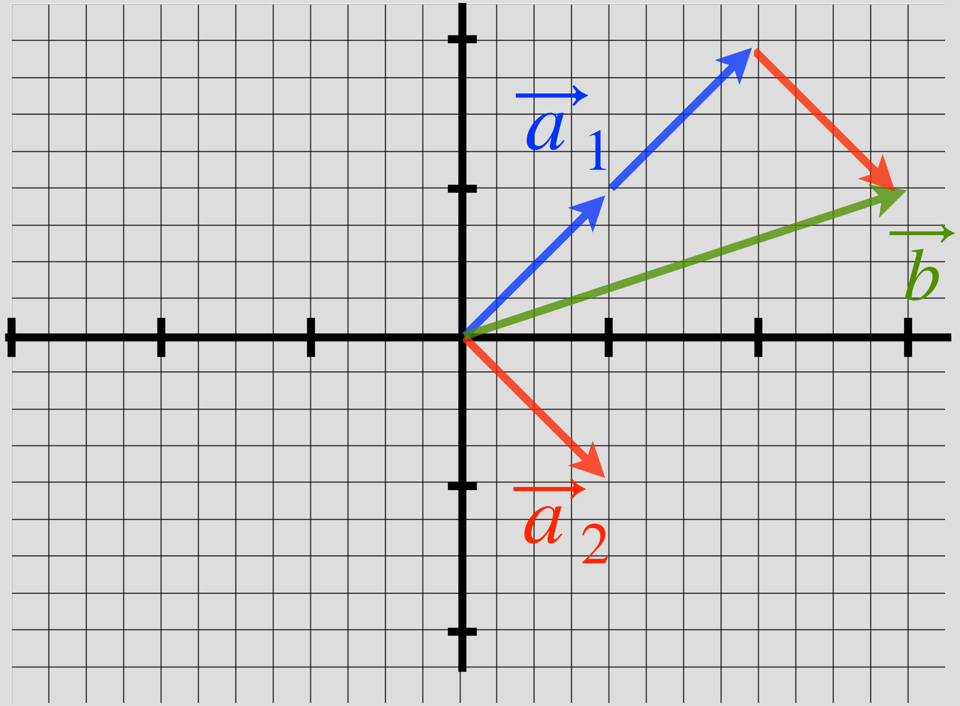
Linear Set of Equations as a Linear Combination

- Consider the problem: $A\overrightarrow{x} = \overrightarrow{b}$:

$$\begin{array}{cccc} \left[\begin{array}{cc}1 & 1\\ 1 & -1\end{array}\right] & \left[\begin{array}{c}x_1\\ x_2\end{array}\right] & = & \left[\begin{array}{c}3\\ 1\end{array}\right] \\ \swarrow \qquad \swarrow & & & \swarrow \\ \overrightarrow{a}_1 \quad \overrightarrow{a}_2 & & & \overrightarrow{b} \end{array}$$

Q: What linear combination of $\overrightarrow{a}_1, \overrightarrow{a}_2$ will give $\overrightarrow{b}$?

A: $2\overrightarrow{a}_1 + 1\overrightarrow{a}_2$



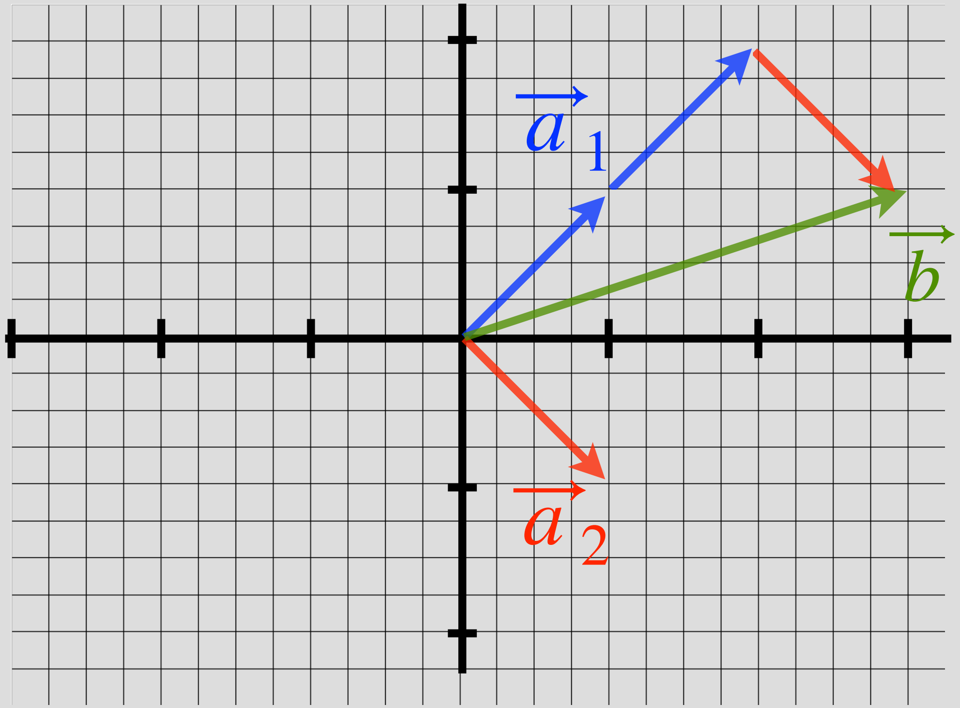
Linear Set of Equations as a Linear Combination

- Consider the problem: $A\vec{x} = \vec{b}$:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\downarrow \qquad \downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\vec{a}_1 \qquad \vec{a}_2 \qquad\qquad\qquad\qquad \vec{b}$$



Q: What linear combination of $\vec{a}_1, \vec{a}_2$ will give $\vec{b}$?

A: $2\vec{a}_1 + 1\vec{a}_2$

Gaussian Elimination:

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & 1 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -2 & -2 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right] \longrightarrow$$

$x_1 = 2$

$x_2 = 1$

same as

$$\Rightarrow \vec{b} = \underline{\underline{2}}\,\vec{a}_1 + \underline{\underline{1}}\cdot\vec{a}_2 \quad !$$

Linear Set of Equations as a Linear Combination

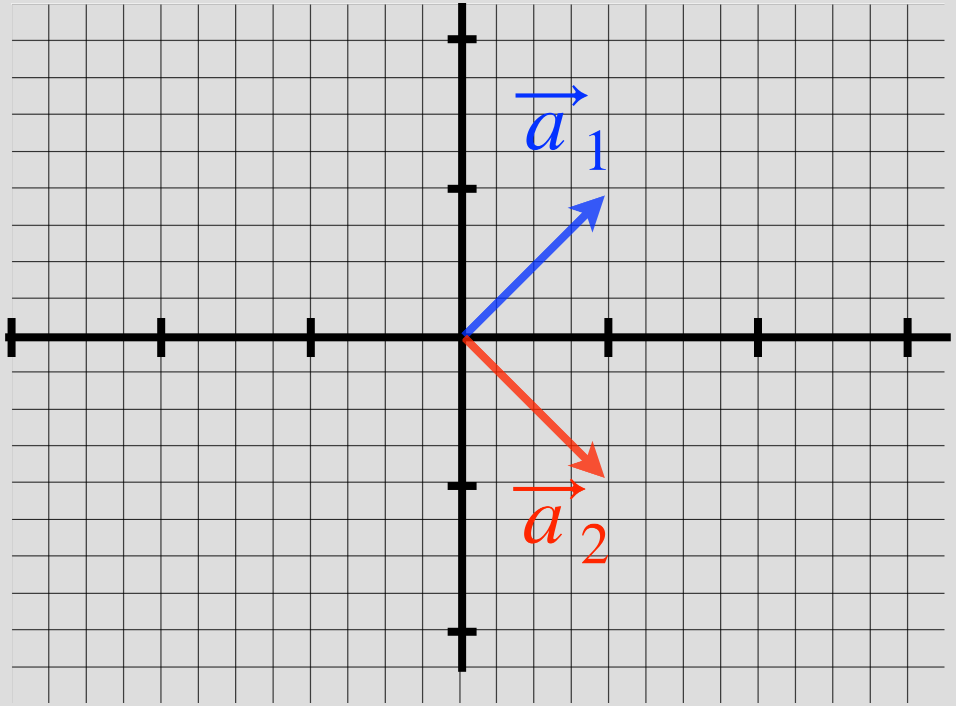
- Consider the problem: $A\overrightarrow{x} = \overrightarrow{b}$:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{b}$$

$$\begin{array}{cc} \swarrow & \swarrow \\ \overrightarrow{a}_1 & \overrightarrow{a}_2 \end{array}$$

Q: Can linear combination of $\overrightarrow{a}_1, \overrightarrow{a}_2$ give any $\overrightarrow{b}$?

A: Hmmm….I think so….



Linear Set of Equations as a Linear Combination

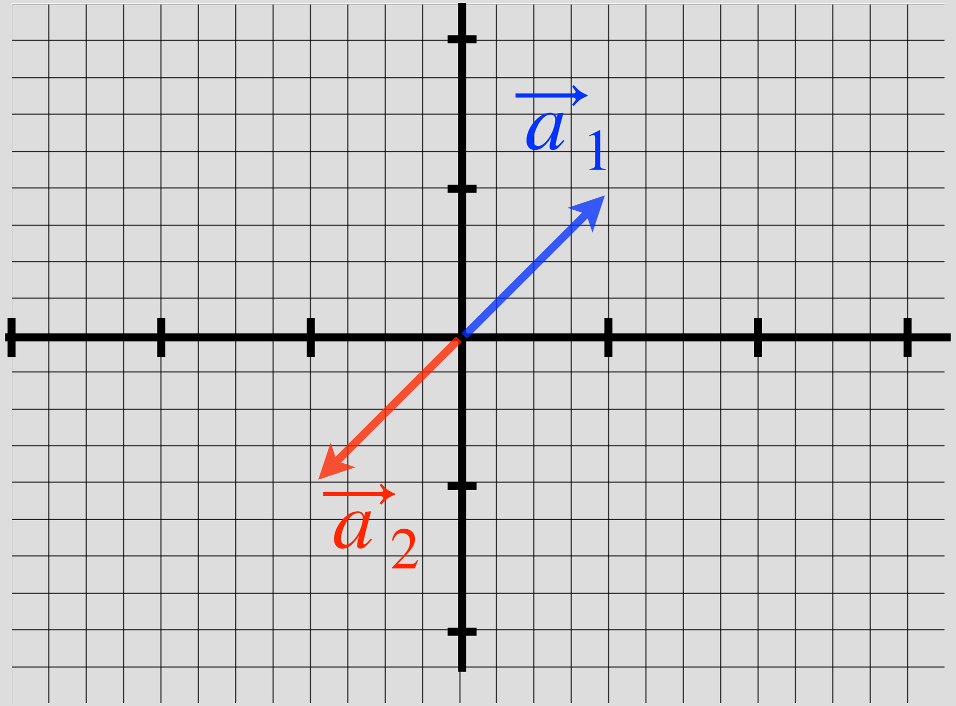
- Consider the problem: $A\overrightarrow{x} = \overrightarrow{b}$:

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{b}$$

$$\underset{\overrightarrow{a}_1}{\downarrow} \qquad \underset{\overrightarrow{a}_2}{\downarrow}$$

Q: Can linear combination of $\overrightarrow{a}_1, \overrightarrow{a}_2$ give any $\overrightarrow{b}$?

A: Hmmm….I don't think so…. Unless its along the line $\overrightarrow{a}_1$



Linear Set of Equations as a Linear Combination

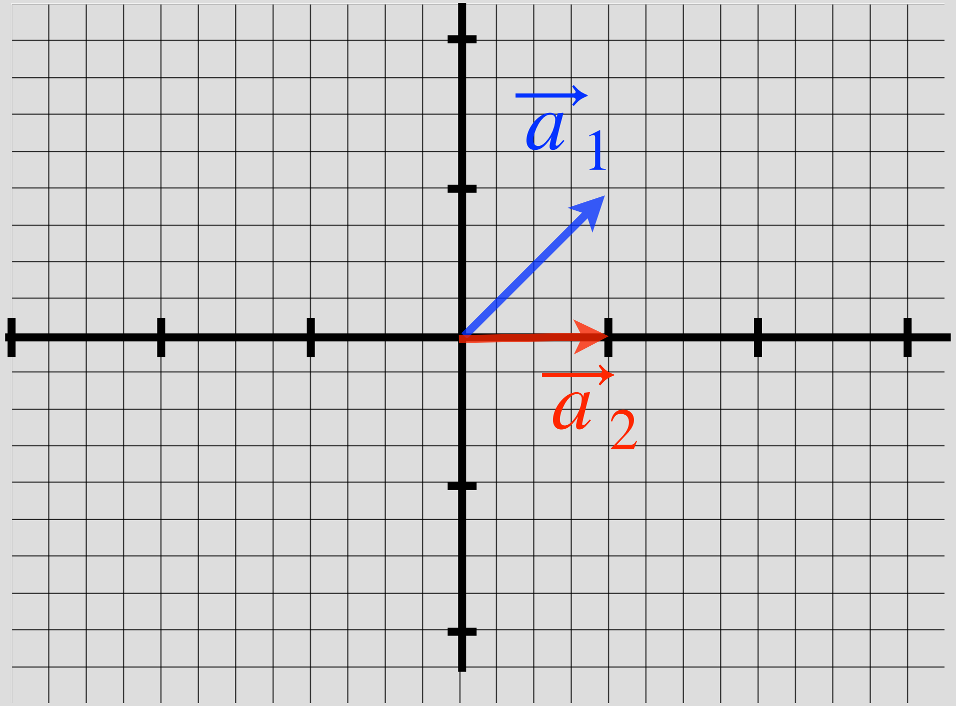
- Consider the problem: $A\overrightarrow{x} = \overrightarrow{b}$:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{b}$$

$$\underset{\overrightarrow{a}_1}{\downarrow} \quad \underset{\overrightarrow{a}_2}{\downarrow}$$

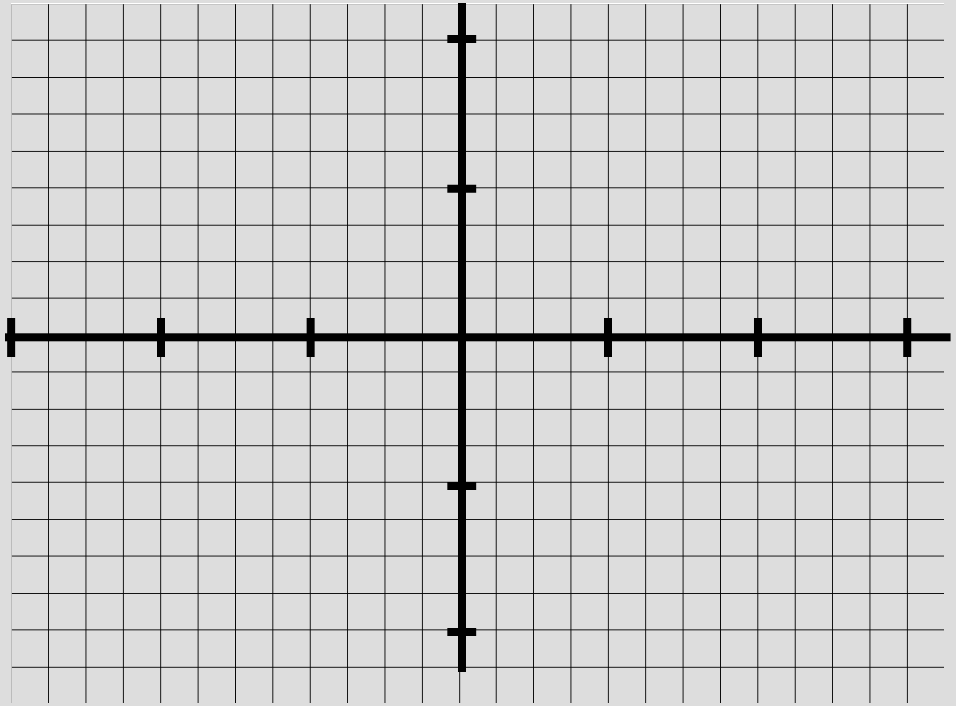
Q: Can linear combination of $\overrightarrow{a}_1, \overrightarrow{a}_2$ give any $\overrightarrow{b}$?

A: Hmmm….yes!



Span / Column Space / Range

- Span of the columns of A is the set of all vectors $\overrightarrow{b}$ such that $A\overrightarrow{x} = \overrightarrow{b}$ has a solution
  - i.e. the set of all vectors that can be reached by all possible linear combinations of the columns of A



Span / Column Space / Range

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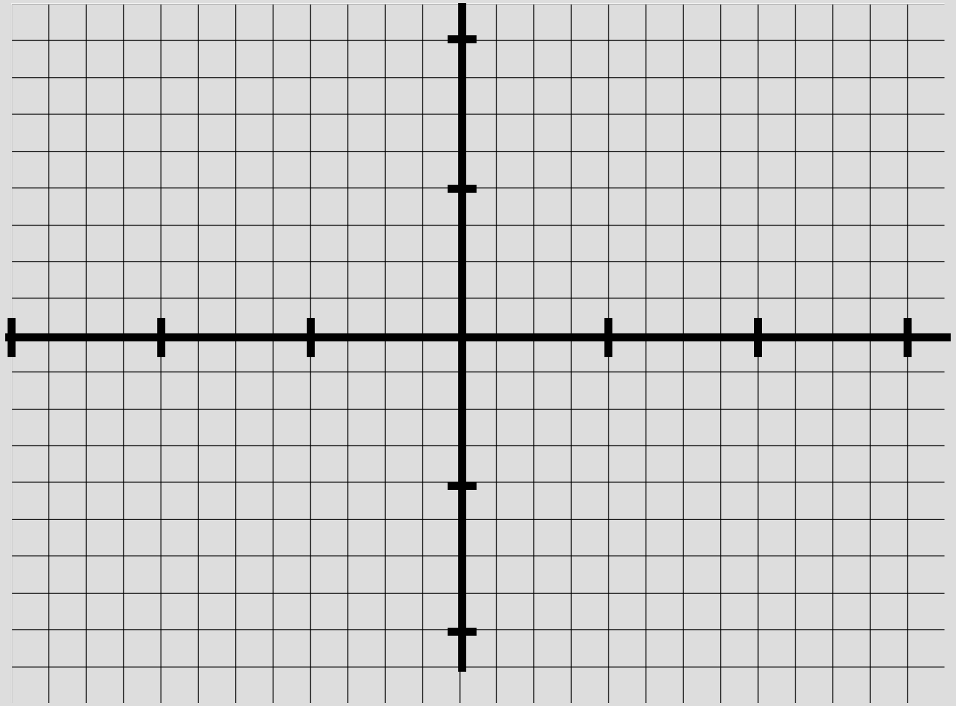
  - i.e. the set of all vectors that can be reached by all possible linear combinations of the columns of A

Example: What is the span of the cols of A?

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$



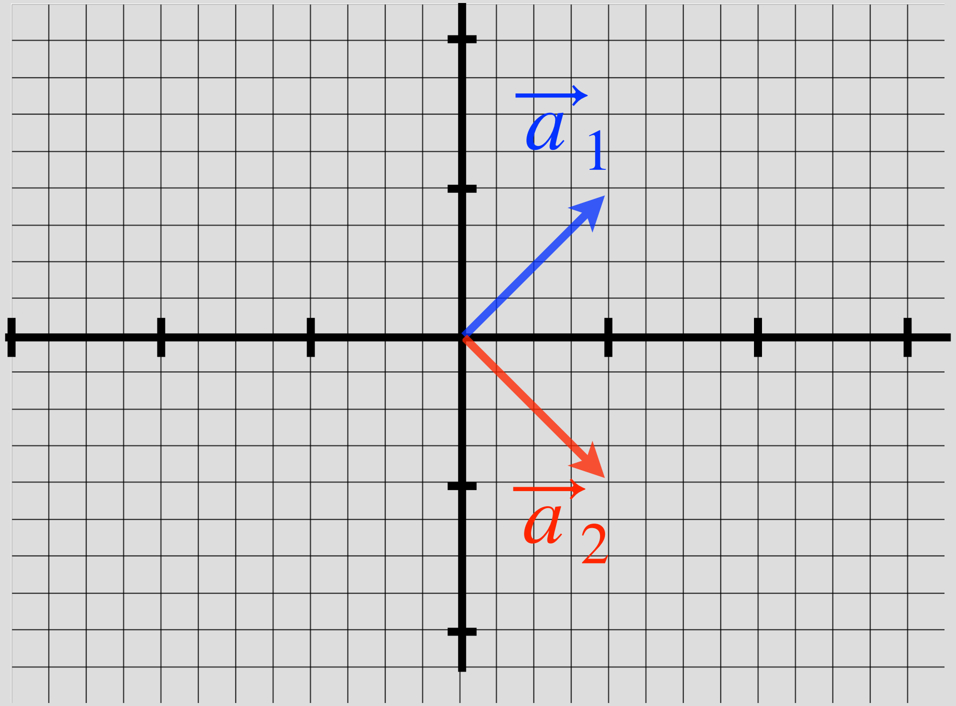
Span / Column Space / Range

- Span of the columns of A is the set of all vectors $\overrightarrow{b}$ such that $A\overrightarrow{x} = \overrightarrow{b}$ has a solution
    - i.e. the set of all vectors that can be reached by all possible linear combinations of the columns of A

Example: What is the span of the cols of A?

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

A: $\mathbb{R}^2$!



Span / Column Space / Range

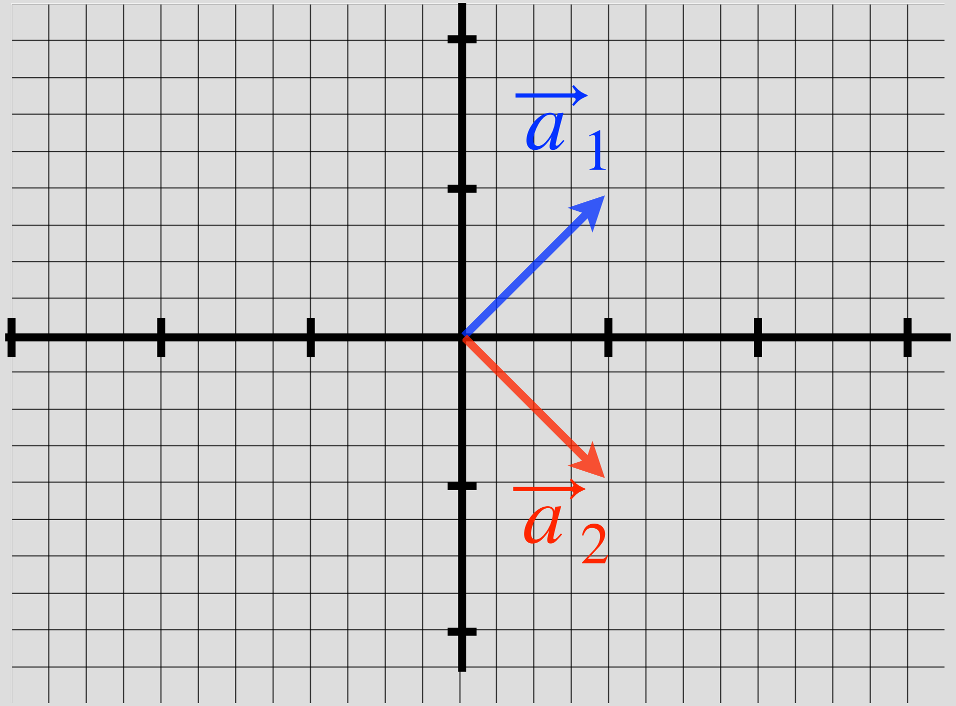
- Span of the columns of A is the set of all vectors $\overrightarrow{b}$ such that $A\overrightarrow{x} = \overrightarrow{b}$ has a solution
  - i.e. the set of all vectors that can be reached by all possible linear combinations of the columns of A

Example: What is the span of the cols of A?

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

A: $\mathbb{R}^2$!

$$\text{span}(A) = \left\{ \vec{v} \middle| \; \vec{v} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \alpha, \beta \in \mathbb{R} \right\}$$



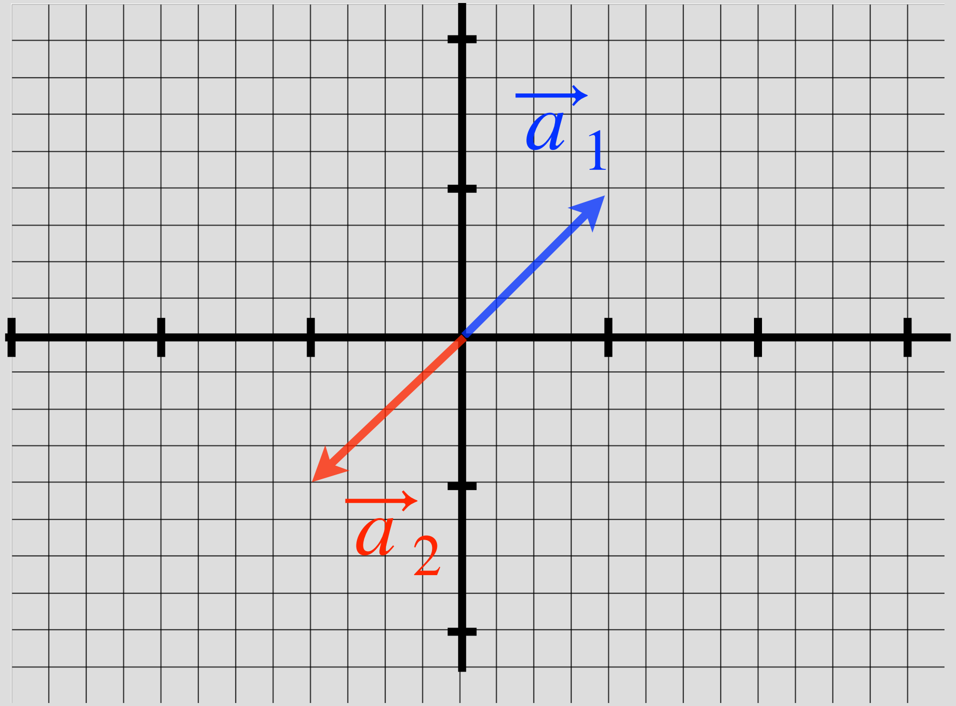
Span / Column Space / Range

Example 2: What is the span of the cols of A?

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

A: The line $x_1 = x_2$

$$\text{span}(A) = \left\{ \vec{v} \middle| \ \vec{v} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \alpha \in \mathbb{R} \right\}$$



Span / Column Space / Range

- Definition:

If $\exists \overrightarrow{x}$ s.t. $A\overrightarrow{x} = \overrightarrow{b}$ then $\overrightarrow{b} \in \text{span}\{A\}$

Converse: $\overrightarrow{b} \in \text{span}\{A\}$, there is a solution for $A\overrightarrow{x} = \overrightarrow{b}$

Q: What if $\overrightarrow{b} \notin \text{span}\{A\}$?

A: There is no solution for $A\overrightarrow{x} = \overrightarrow{b}$

Algorithm for solving linear equations

- Three basic operations that don't change a solution:
  1. Multiply an equation with *nonzero* scalar

$2x + 3y = 4$ has the same solution as: $4x + 6y = 8$

Proof for N=2:

Let $ax + by = c$, with solution x_0, y_0
$\Rightarrow ax_0 + by_0 = c$

Show that $\beta ax + \beta by = \beta c$,
has the same solution.

Substitute x_0, y_0 for x, y:

$$\beta ax_0 + \beta by_0 = \beta c$$

$$\beta(ax_0 + by_0) = \beta c$$

$$\beta c = \beta c$$

But is it the only solution?

$\beta ax + \beta by = \beta c$, with solution: x_1, y_1
$\Rightarrow \beta ax_1 + \beta by_1 = \beta c$

Show that $ax + by = c$,
has the same solution…..

Since $\beta \neq 0$….

$$\beta ax_1 + \beta by_1 = \beta c \Rightarrow ax_1 + by_1 = c$$

SOLUTION OF ONE, IMPLIES THE OTHER
AND VICE-VERSA!

Algorithm for solving linear equations

- Three basic operations that don't change a solution:
  1. Multiply an equation with *nonzero* scalar
  2. Adding a scalar constant multiple of one equation to another

$$\begin{array}{rrcrcr} (1) & x & + & y & = & 2 \\ (2) & 3x & + & 2y & = & 5 \end{array}$$

and

$$\begin{array}{rrcrcr} (1) & x & + & y & = & 2 \\ 3 \times (1) + (2) & 6x & + & 5y & = & 11 \end{array}$$

Have the same solution

Concept of proof: look at explicit solution, show they are the same

Also show the reverse — by applying the reverse operations