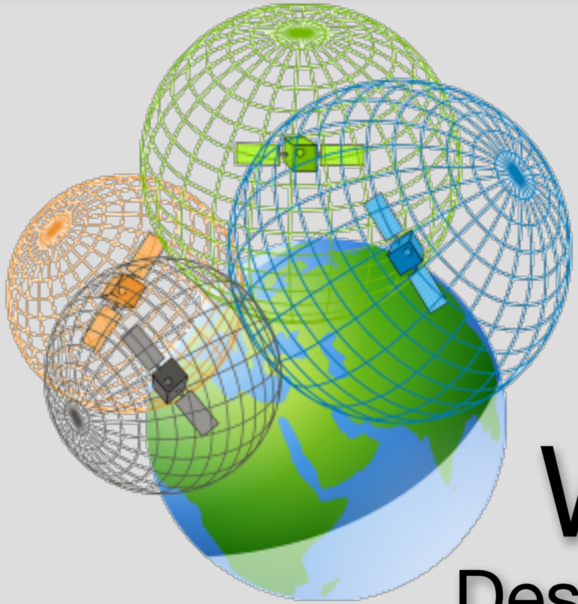
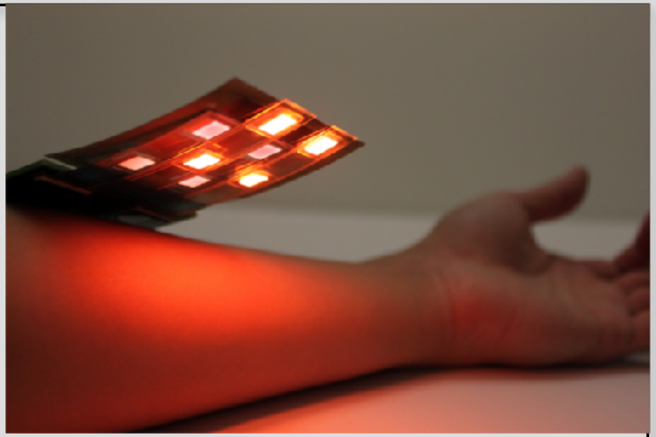

Almost Austin!
DAVID OLNEY
SERGIO WEBB
ONE TOUGH TOUR
2007
Malcolm Holcombe
Corona
Corona Extra





Welcome to EECS 16A!

Designing Information Devices and Systems I

Ana Arias and Miki Lustig

Fall 2021

Lecture 2B
Proofs
Linear (in)dependance
Matrix Transformations

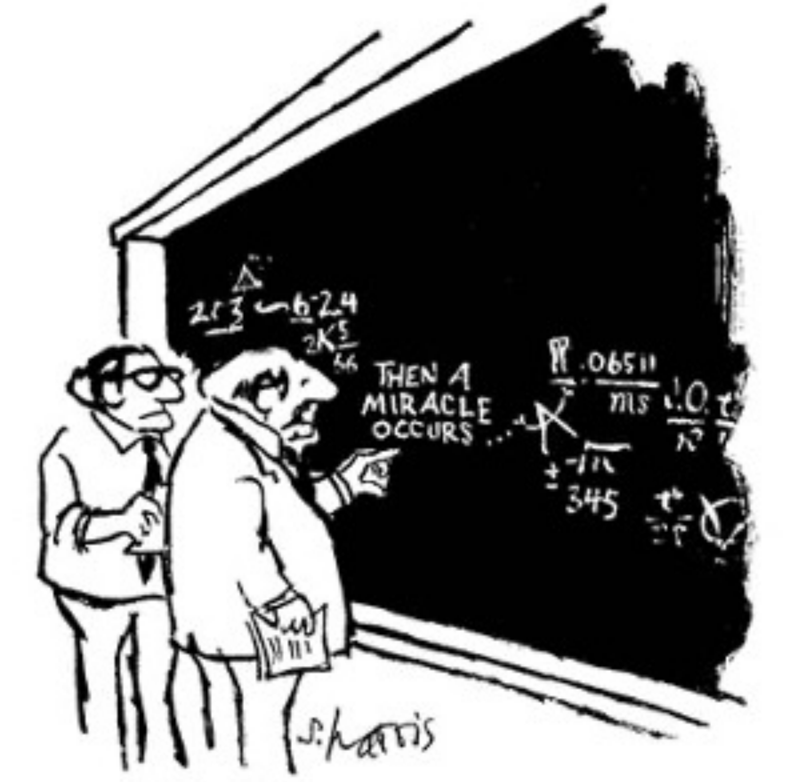
Announcements

- Quest: Tuesday 09/14/21 8:30pm

- Last time:
  - Continue vectors
  - Matrix-Matrix and Matrix-vector Multiplications
  - Matrix-Vector Multiplications as linear set of equations
  - Intro to span
- Today:
  - Proofs
  - Linear (in)dependance
  - Matrix Transformations

Steps for a proof

- Write out the statement, note direction ("if" → "then")
- Try a simple example (to see a pattern)
  - Use what is known, definitions and other theorems
- Manipulate both sides of the arguments
  - Must justify each step
- Know the different styles of proofs to try
  - Constructive
  - Proof by contradiction



"I think you should be more explicit here in step two."

Algorithm for solving linear equations

- Three basic operations that don't change a solution:
  1. Multiply an equation with *nonzero* scalar

$2x + 3y = 4$ has the same solution as: $4x + 6y = 8$

Proof for N=2:

Let $ax + by = c$, with solution x_0, y_0
$\Rightarrow ax_0 + by_0 = c$

Show that $\beta ax + \beta by = \beta c$,
has the same solution.

Substitute x_0, y_0 for x, y:

$$\beta ax_0 + \beta by_0 = \beta c$$

$$\beta(ax_0 + by_0) = \beta c$$

$$\beta c = \beta c$$

But is it the only solution?

$\beta ax + \beta by = \beta c$, with solution: x_1, y_1
$\Rightarrow \beta ax_1 + \beta by_1 = \beta c$

Show that $ax + by = c$,
has the same solution.....

Since $\beta \neq 0$....

$$\beta ax_1 + \beta by_1 = \beta c \Rightarrow ax_1 + by_1 = c$$

SOLUTION OF ONE, IMPLIES THE OTHER
AND VICE-VERSA!

Algorithm for solving linear equations

- Three basic operations that don’t change a solution:
  1. Multiply an equation with *nonzero* scalar
  2. Adding a scalar constant multiple of one equation to another

$$\begin{array}{rrcrcr} (1) & x & + & y & = & 2 \\ (2) & 3x & + & 2y & = & 5 \end{array}$$

and

$$\begin{array}{rrcrcr} (1) & x & + & y & = & 2 \\ 3\times(1)+(2) & 6x & + & 5y & = & 11 \end{array}$$

Have the same solution

Concept of proof: look at explicit solution, show they are the same

Also show the reverse — by applying the reverse operations

Span / Column Space / Range

- Span of the columns of A is the set of all vectors $\overrightarrow{b}$ such that $A\overrightarrow{x} = \overrightarrow{b}$ has a solution
  - i.e. the set of all vectors that can be reached by all possible linear combinations of the columns of A

- Definition:

  If $\exists\,\overrightarrow{x}$ s.t. $A\overrightarrow{x} = \overrightarrow{b}$ then $\overrightarrow{b} \in \mathrm{span}\{A\}$

Proof: Span

Theorem: $\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\}=\mathbb{R}^2$

Know:

$$\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\}\Rightarrow\left\{\vec{v}\,\middle|\,\vec{v}=\alpha\begin{bmatrix}1\\1\end{bmatrix}+\beta\begin{bmatrix}1\\-1\end{bmatrix}\quad,\alpha,\beta\in\mathbb{R}\right\}=\mathbb{S}$$

Need to show:

$$\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\}=\mathbb{R}^2$$

Concept: pick some specific $\overrightarrow{b}=\begin{bmatrix}b_1\\b_2\end{bmatrix}\in R^2$, and show that it belongs to $\mathbb{S}$

Need to solve:

Proof: Span

Theorem: $\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\} = \mathbb{R}^2$

Know:

$$\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\} \Rightarrow \left\{\vec{v} \,\middle|\, \vec{v} = \alpha\begin{bmatrix}1\\1\end{bmatrix} + \beta\begin{bmatrix}1\\-1\end{bmatrix} \quad, \alpha, \beta \in \mathbb{R}\right\} = \mathbb{S}$$

Need to show:

$$\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\} = \mathbb{R}^2$$

Concept: pick some specific $\vec{b} = \begin{bmatrix}b_1\\b_2\end{bmatrix} \in R^2$, and show that it belongs to $\mathbb{S}$

Need to solve:

$$\alpha\begin{bmatrix}1\\1\end{bmatrix} + \beta\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix} \qquad \Rightarrow \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$$

unknown

Known
and $\in \mathbb{R}^2$

Proof: Span

Need to solve:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Gaussian Elimination:

Proof: Span

Need to solve:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\frac{b_1+b_2}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{b_1-b_2}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Constructive proof

Gaussian Elimination:

$$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 1 & -1 & b_2 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & -2 & b_2-b_1 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & \frac{b_1-b_2}{2} \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{b_1+b_2}{2} \\ 0 & 1 & \frac{b_1-b_2}{2} \end{array}\right] \Rightarrow \alpha = \frac{b_1+b_2}{2}, \beta = \frac{b_1-b_2}{2},$$

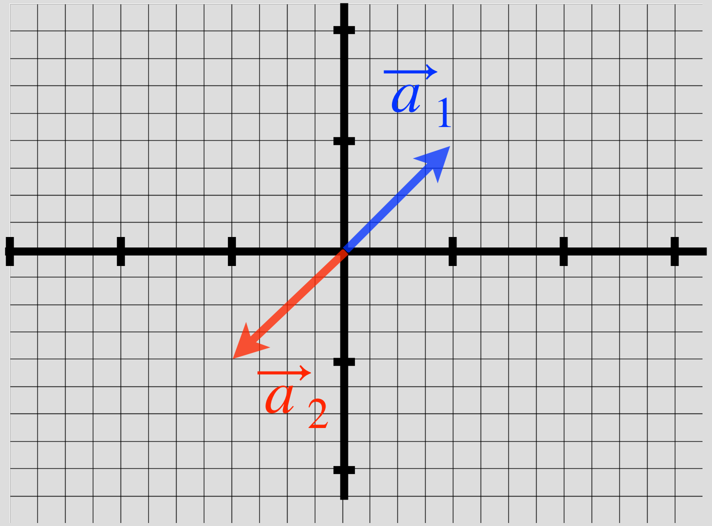
Every $\overrightarrow{b} \in \mathbb{R}^2$ can be written as linear combinations!

So also, $\overrightarrow{b} \in \mathbb{S}$

Linear Dependence

Recall:

$$A = \begin{array}{c}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \\ \begin{matrix} \swarrow & \swarrow \\ \vec{a}_1 & \vec{a}_2 \end{matrix}\end{array}$$



$\vec{a}_1$ and $\vec{a}_2$ are linearly dependent

$$\vec{a}_1 = -\vec{a}_2$$

Linear Dependence

- Definition 1:
  A set of vectors $\{\overrightarrow{a}_1, \overrightarrow{a}_2, \cdots, \overrightarrow{a}_N\}$ are linearly dependent if $\exists \{\alpha_1, \alpha_2, \cdots, \alpha_N\} \in \mathbb{R}$, such that:

$$\overrightarrow{a}_i = \sum_{j \neq i} \alpha_j \overrightarrow{a}_j \qquad 1 \leq i, j \leq M$$

$$\downarrow$$

$\overrightarrow{a}_i$ in the span of all $\overrightarrow{a}_j$s

For example: if $\overrightarrow{a}_2 = 3\overrightarrow{a}_1 - 2\overrightarrow{a}_5 + 6\overrightarrow{a}_7$

Linear Dependence

Are these linearly dependent?

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$$

Need to solve:

Linear Dependence

Are these linearly dependent?

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$$

Are linearly dependent

Need to solve:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

but we showed that….

$$\frac{b_1+b_2}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{b_1-b_2}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

So….

$$\frac{3+1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{3-1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Linear dependence / independence

$$\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}3\\1\end{bmatrix}\right\} \quad \Rightarrow 2\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}1\\-1\end{bmatrix}-\begin{bmatrix}3\\1\end{bmatrix}=0$$

- Definition 2:
  A set of vectors $\left\{\overrightarrow{a}_1, \overrightarrow{a}_2, \cdots, \overrightarrow{a}_N\right\}$ are linearly dependent if $\exists\left\{\alpha_1, \alpha_2, \cdots, \alpha_N\right\} \in \mathbb{R}$, such that:

$$\sum_{i=1}^{N} \alpha_i \overrightarrow{a}_i = 0$$

As long as not all $a_i = 0$

- Definition:
  A set of vectors $\left\{\overrightarrow{a}_1, \overrightarrow{a}_2, \cdots, \overrightarrow{a}_N\right\}$ are linearly independent if they are not dependent

Linear dependence / independence

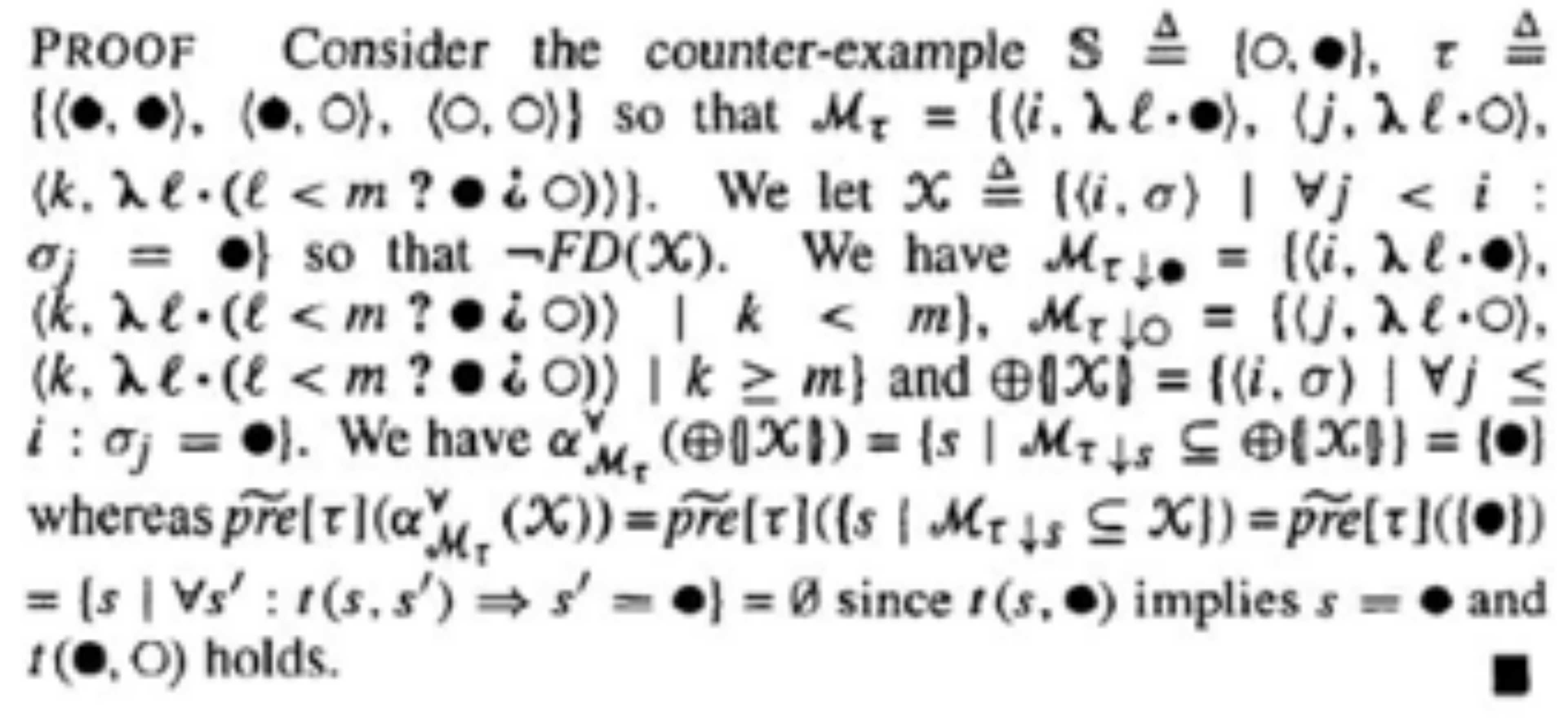
Are these linearly dependent?

$$\left\{ \underbrace{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}}_{\text{span} = \mathbb{R}^2}, \underset{\in \mathbb{R}^2}{\underset{\searrow}{\begin{bmatrix}\pi\\\sqrt{2}\end{bmatrix}}} \right\}$$

linearly dependent!

Solutions for linear equations

- Theorem: if the columns of the matrix A are linearly dependent then, $A\overrightarrow{x} = \overrightarrow{b}$ does <u>not</u> have a unique solution



PROOF Consider the counter-example $\mathbb{S} \triangleq \{\circ, \bullet\}$, $\tau \triangleq \{\langle\bullet, \bullet\rangle, \langle\bullet, \circ\rangle, \langle\circ, \circ\rangle\}$ so that $\mathcal{M}_\tau = \{\langle i, \lambda\ell\cdot\bullet\rangle, \langle j, \lambda\ell\cdot\circ\rangle, \langle k, \lambda\ell\cdot(\ell < m\ ?\ \bullet\ ¿\ \circ)\rangle\}$. We let $\mathcal{X} \triangleq \{\langle i, \sigma\rangle \mid \forall j < i : \sigma_j = \bullet\}$ so that $\neg FD(\mathcal{X})$. We have $\mathcal{M}_{\tau\downarrow\bullet} = \{\langle i, \lambda\ell\cdot\bullet\rangle, \langle k, \lambda\ell\cdot(\ell < m\ ?\ \bullet\ ¿\ \circ)\rangle \mid k < m\}$, $\mathcal{M}_{\tau\downarrow\circ} = \{\langle j, \lambda\ell\cdot\circ\rangle, \langle k, \lambda\ell\cdot(\ell < m\ ?\ \bullet\ ¿\ \circ)\rangle \mid k \geq m\}$ and $\oplus\llbracket\mathcal{X}\rrbracket = \{\langle i, \sigma\rangle \mid \forall j \leq i : \sigma_j = \bullet\}$. We have $\alpha^{\forall}_{\mathcal{M}_\tau}(\oplus\llbracket\mathcal{X}\rrbracket) = \{s \mid \mathcal{M}_{\tau\downarrow s} \subseteq \oplus\llbracket\mathcal{X}\rrbracket\} = \{\bullet\}$ whereas $\widetilde{pre}[\tau](\alpha^{\forall}_{\mathcal{M}_\tau}(\mathcal{X})) = \widetilde{pre}[\tau](\{s \mid \mathcal{M}_{\tau\downarrow s} \subseteq \mathcal{X}\}) = \widetilde{pre}[\tau](\{\bullet\}) = \{s \mid \forall s' : t(s, s') \Rightarrow s' = \bullet\} = \emptyset$ since $t(s, \bullet)$ implies $s = \bullet$ and $t(\bullet, \circ)$ holds. ■

Solutions for linear equations

- Theorem: if the columns of the matrix A are linearly dependent then, $A\overrightarrow{x} = \overrightarrow{b}$ does <u>not</u> have a unique solution

Proof for $A \in \mathbb{R}^{3\times3}$

know: columns are linearly ~~in~~dependent

show: more than 1 solution

Concept: pick some specific solution $\overrightarrow{x}^*$, and show that there's another one

Let: $A\overrightarrow{x}^* = \overrightarrow{b}$ and $A = [\overrightarrow{a}_1 \quad \overrightarrow{a}_2 \quad \overrightarrow{a}_3]$

From linear dependence Def 2:

$$\alpha_1\overrightarrow{a}_1 + \alpha_2\overrightarrow{a}_2 + \alpha_3\overrightarrow{a}_3 = 0$$

Solutions for linear equations

- Theorem: if the columns of the matrix A are linearly dependent then, $A\vec{x} = \vec{b}$ does <u>not</u> have a unique solution

Proof for $A \in \mathbb{R}^{3\times 3}$

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Let: $A\vec{x}^* = \vec{b}$ and $A = [\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3]$

From linear dependence Def 2:

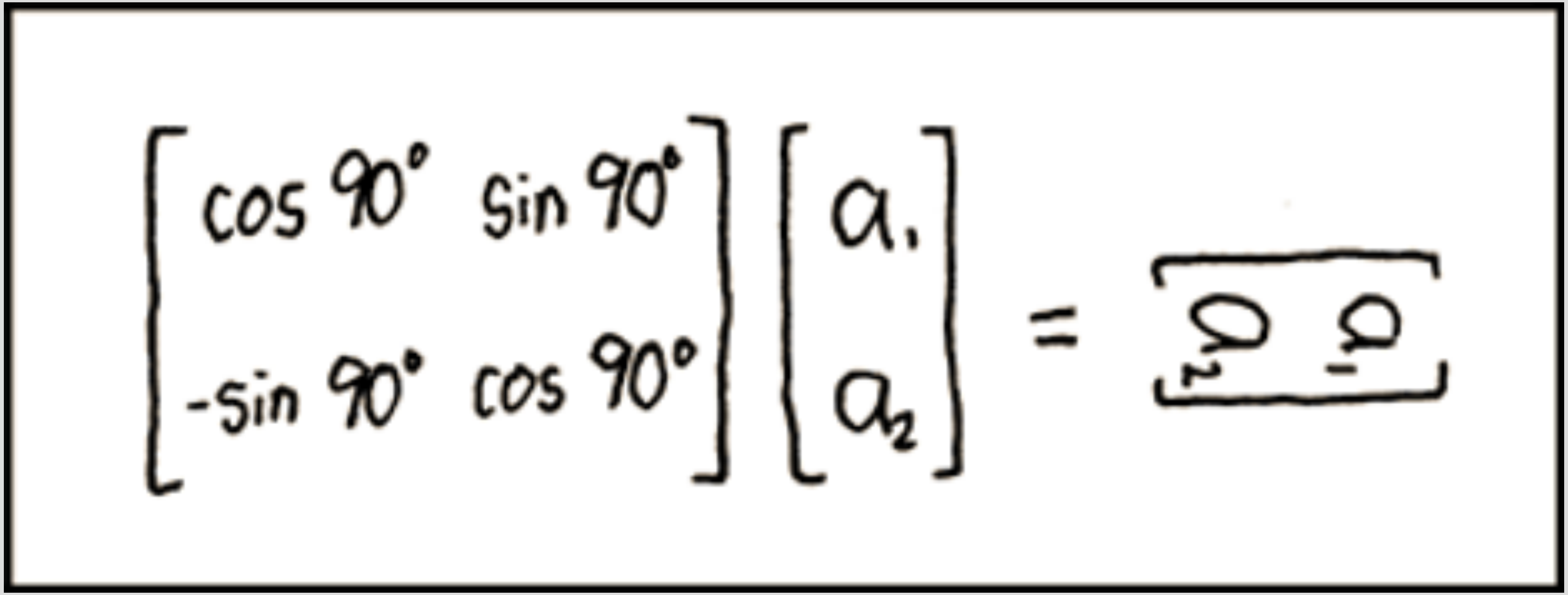
$$\alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \alpha_3\vec{a}_3 = 0 \longrightarrow \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}}_{\vec{\alpha}} = 0 \qquad \Rightarrow A\vec{\alpha} = 0$$

Set $\vec{x}^\dagger = \vec{x}^* + \vec{\alpha}$

$$\Rightarrow A\vec{x}^\dagger = A(\vec{x}^* + \vec{\alpha}) = A\vec{x}^* + A\vec{\alpha} = \vec{b} + 0$$

So $\vec{x}^\dagger$ is another solution!

Matrix Transformations

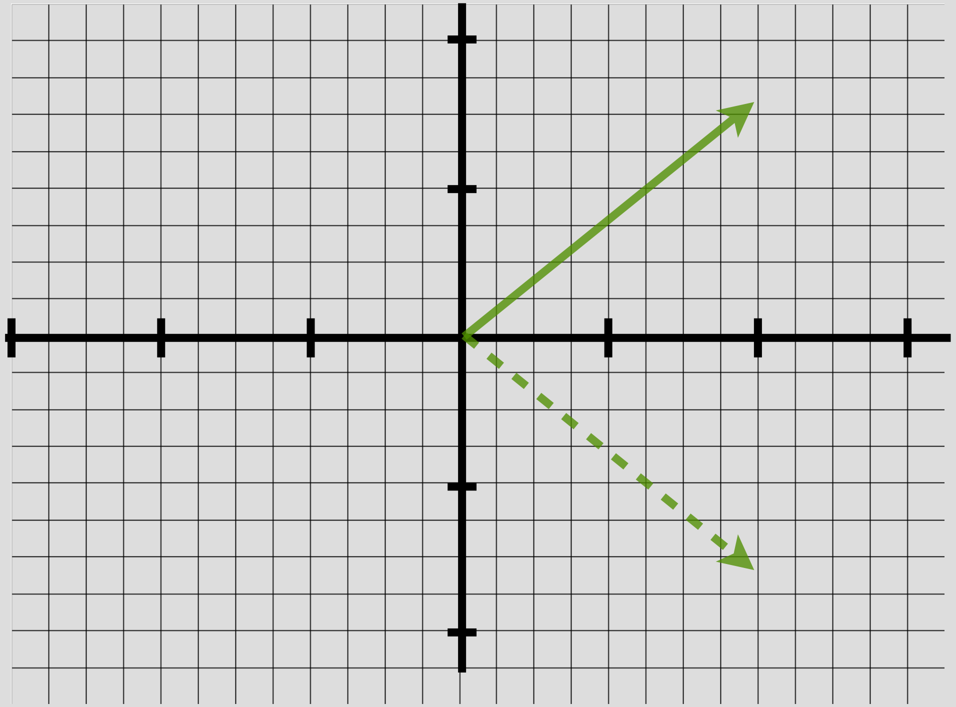


Matrices are operators that transform vectors

$$A\overrightarrow{x} = \overrightarrow{b}$$

Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$



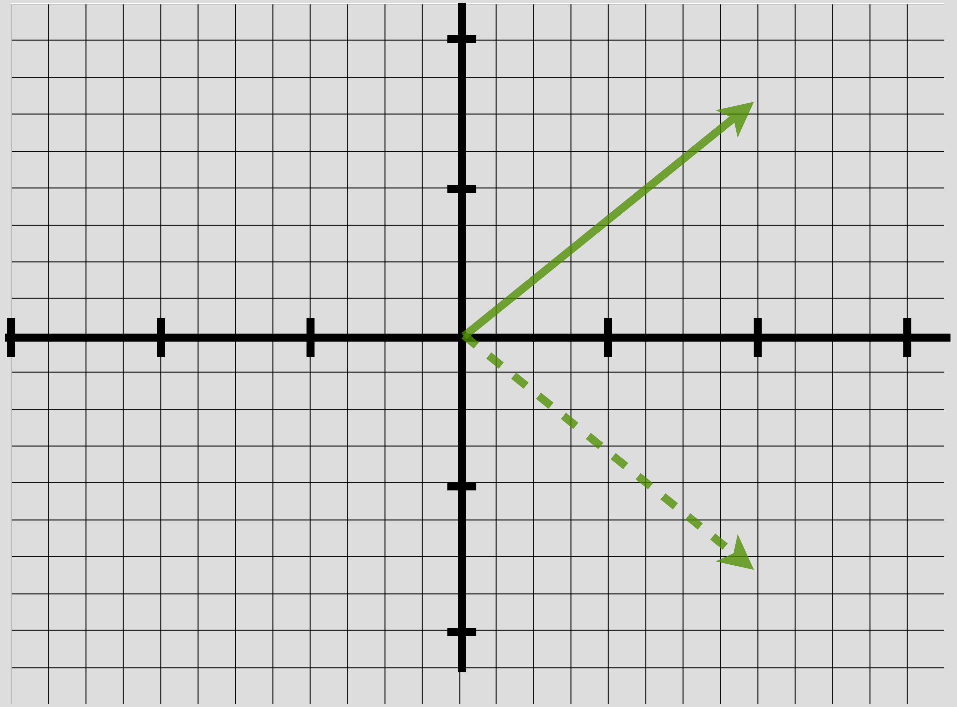
Matrices are operators that transform vectors

$$A\overrightarrow{x} = \overrightarrow{b}$$

Example:

$$\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_1\\-x_2\end{bmatrix}$$

https://www.youtube.com/watch?v=LhF_56SxrGk



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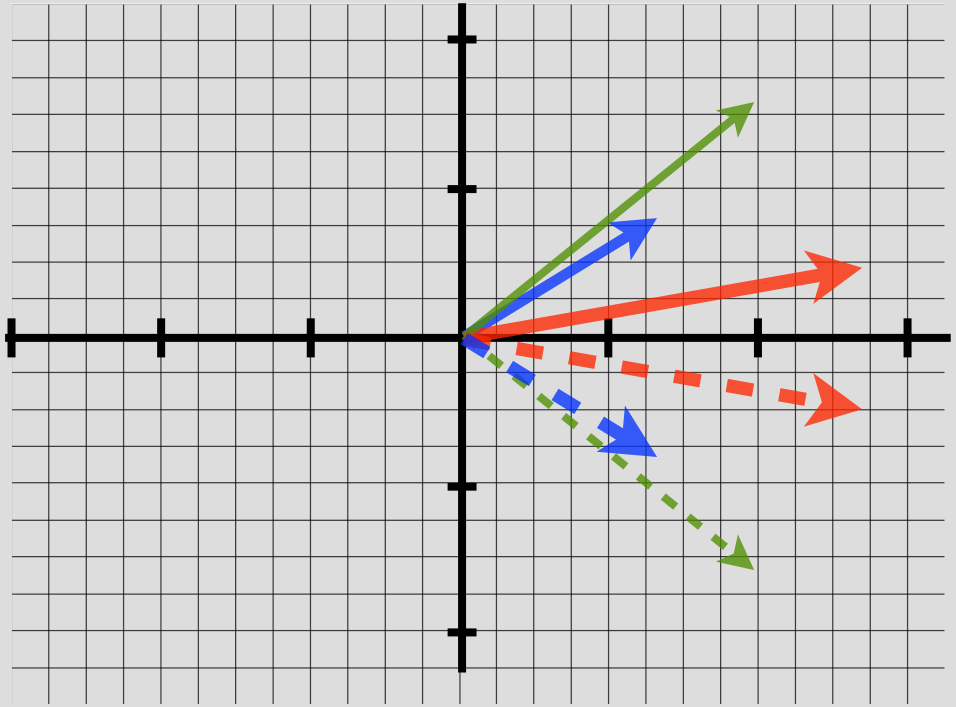
$$A\overrightarrow{x} = \overrightarrow{b}$$

Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

Reflection Matrix!



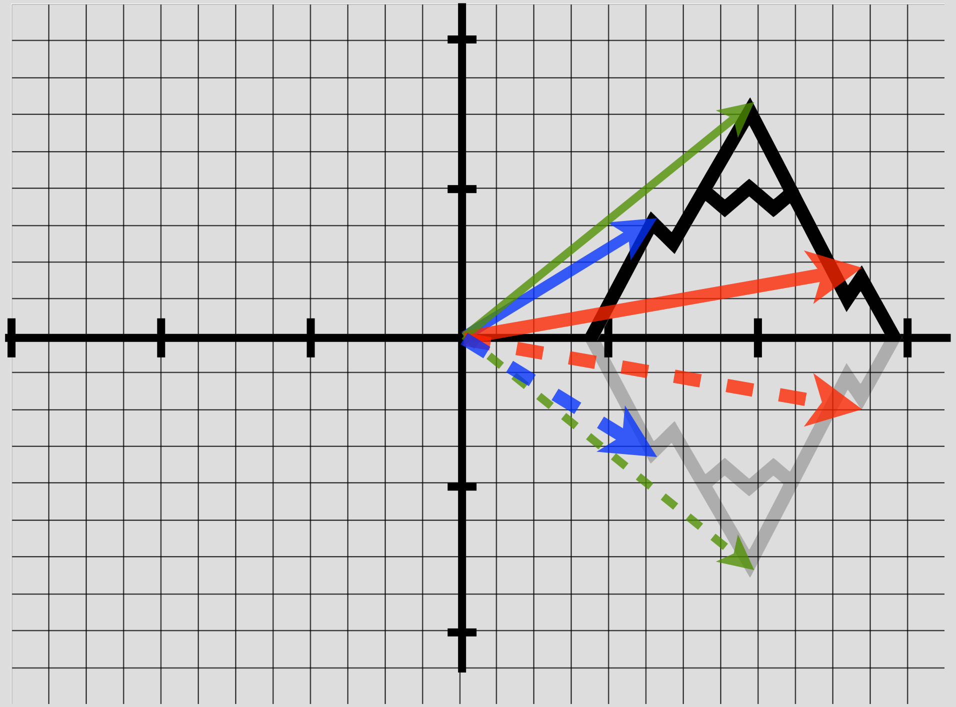
Matrices are operators that transform vectors

$$A\vec{x} = \vec{b}$$

Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

Reflection Matrix!



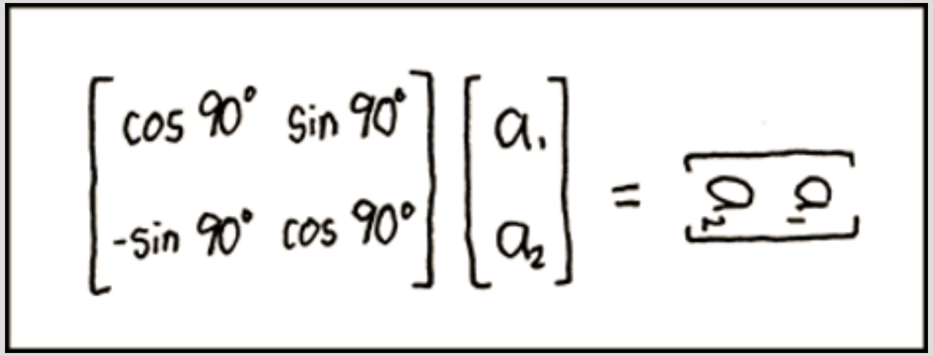
Matrices are operators that transform vectors

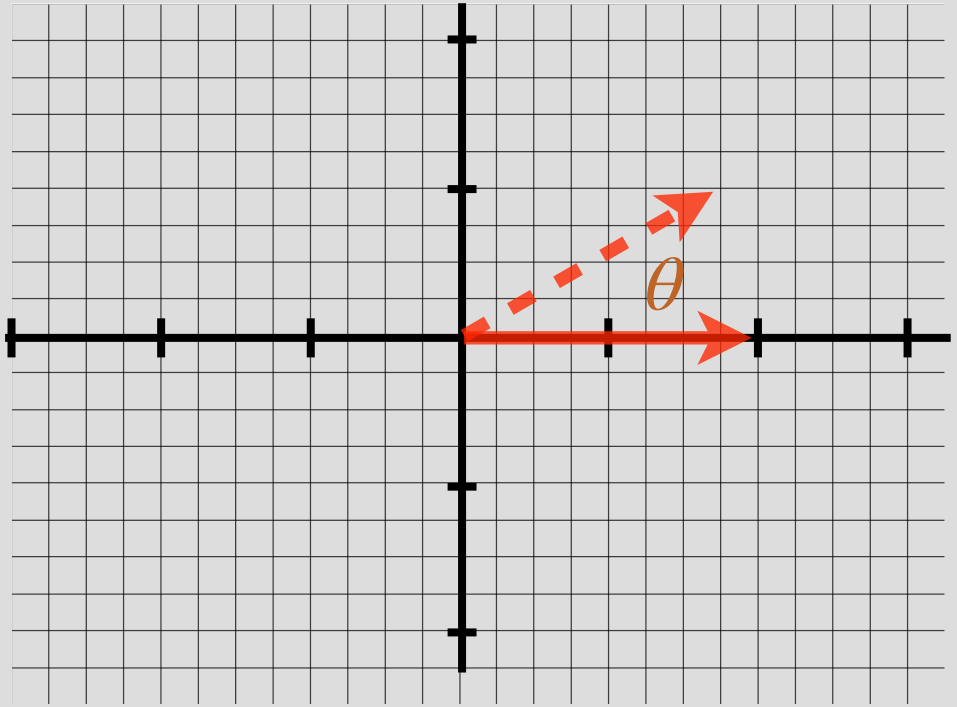
Example 2:

$$\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}\cos(\theta)x_1-\sin(\theta)x2\\ \sin(\theta)x_1+\cos(\theta)x_2\end{bmatrix}$$

$$\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}$$

Rotation Matrix!





Linear Transformation of vectors

f: is a linear transformation if:

$$f(\alpha \overrightarrow{x}) = \alpha f(\overrightarrow{x}) \qquad \alpha \in \mathbb{R}$$

$$f(\overrightarrow{x} + \overrightarrow{y}) = f(\overrightarrow{x}) + f(\overrightarrow{y})$$

Claim: Matrix-vector multiplications satisfy linear transformation

$$A \cdot (\alpha \overrightarrow{x}) = \alpha A \overrightarrow{x}$$

Proof via explicitly writing the elements

$$A \cdot (\overrightarrow{x} + \overrightarrow{y}) = A \overrightarrow{x} + A \overrightarrow{y}$$