

Welcome to EECS 16A!

Designing Information Devices and Systems I



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Fall 2021

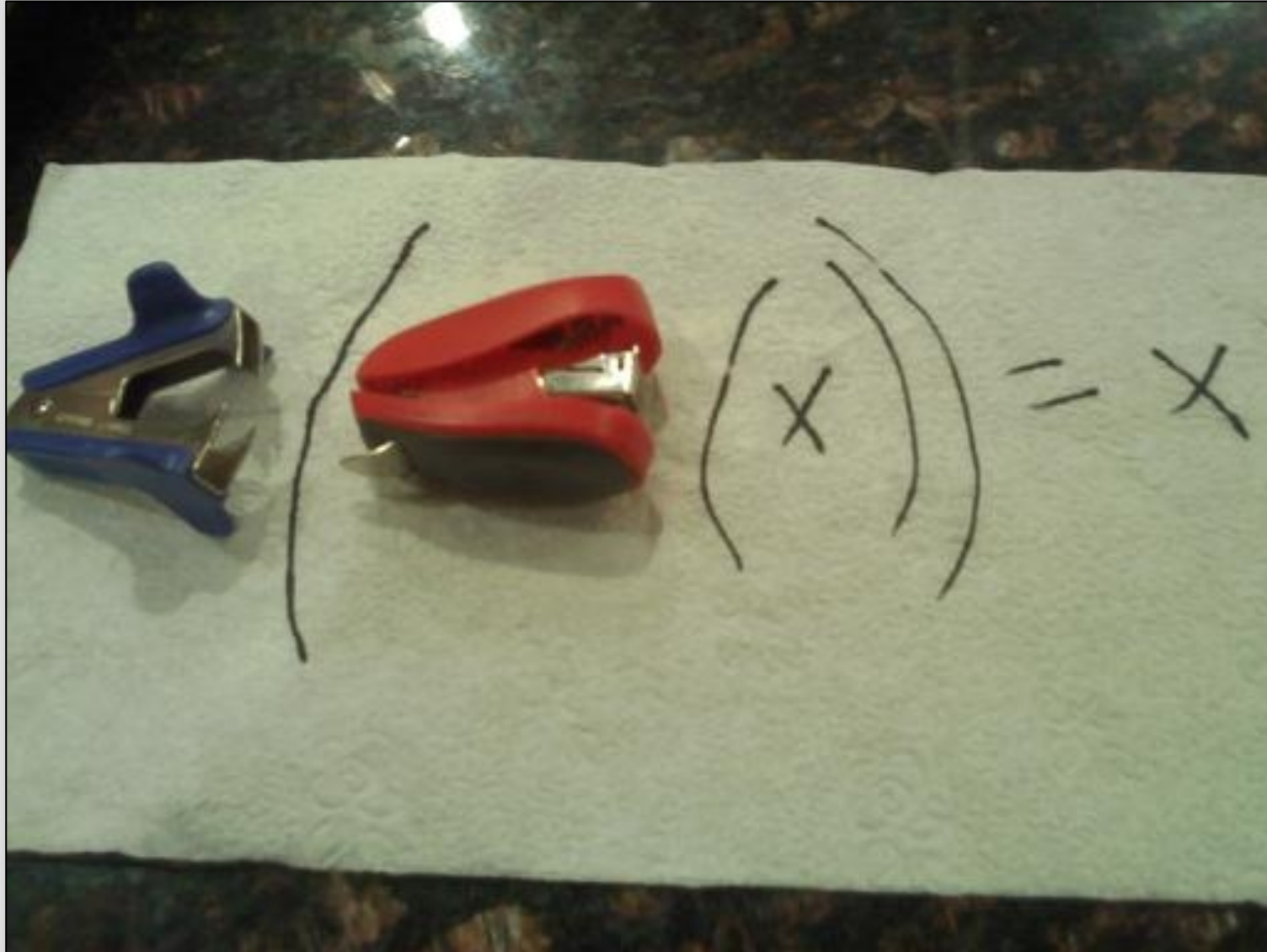
Lecture 3B
Vector Spaces



Announcements

- Last time:
 - Continue with Matrix transformations
 - Matrix Inverse
- Today:
 - Vector spaces
 - Null spaces
 - Subspaces / Row

Matrix Inversion



Invertibility of Linear Transformations

- Theorem: A is invertible, if and only if (iff) the columns of A are linearly independent.
 1. If columns of A are lin. dep. then A^{-1} does not exist
 2. If A^{-1} exists, then the cols. of A are linearly independent

Proof concept: Assume linear dependence and invertibility and show that it is a contradiction

From linear independence: $\exists \vec{\alpha} \neq 0$ such that $A\vec{\alpha} = 0$

Assume A^{-1} exists

$$\begin{aligned} A\vec{\alpha} &= 0 \\ A^{-1}A\vec{\alpha} &= A^{-1}0 \\ I\vec{\alpha} &= 0 \end{aligned}$$

But $\vec{\alpha} \neq 0$! Hence A^{-1} does not exist

Inverse of a 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

1. Flip a and d
2. Negate b and c
3. Divide by $ad - bc$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Derive via Gauss Elimination!

Equivalent Statements

- Matrix A is **invertible**
- $A\vec{x} = \vec{b}$ has a unique solution
- A has linearly independent columns (A is **full rank**)
- A has a **trivial nullspace**
- The **determinant** of A is not zero

Today (and next time's) Jargon

- **Rank** a matrix A is the number of linearly independent columns
- **Nullspace** of a matrix A is the set of solutions to $A\vec{x} = 0$
- A **vector space** is a set of vectors connected by two operators $(+, \cdot)$
- A vector **subspace** is a subset of vectors that have “nice properties”
- A **basis** for a vector space is a minimum set of vectors needed to represent all vectors in the space
- **Dimension** of a vector space is the number of basis vectors
- **Column space** is the span (range) of the columns of a matrix
- **Row space** is the span of the rows of a matrix

<https://www.ncbi.nlm.nih.gov/pmc/articles/PMC4142121/>

- Basis - 3 times
- Rank - 4 times
- Row space - 4 times
- Columns (of a matrix) - 6 times
- Subspace - 17 times
- Null Space - 29 times
- Eigen - 87 times

Vector Space

- From Merriam Webster:

Definition of *vector space*

a set of vectors along with operations of addition and multiplication such that the set is a commutative group under addition, it includes a multiplicative inverse, and multiplication by scalars is both associative and distributive

Vector Space

- A vector space, is a set of vectors and scalars ($\mathbb{V} \in \mathbb{R}^N, \mathbb{F} \in \mathbb{R}$) and two operators $\cdot, +$ that satisfy the following:

Axioms of closure

$$1. \alpha \vec{x} \in \mathbb{V}$$

$$2. \vec{x} + \vec{y} \in \mathbb{V}$$

Axioms of addition
(+)

$$3. \vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z} \text{ (associativity)}$$

$$4. \vec{x} + \vec{y} = \vec{y} + \vec{x} \text{ (commutativity)}$$

$$5. \exists \vec{0} \in \mathbb{V} \text{ s.t. } \vec{x} + \vec{0} = \vec{x} \text{ (additive identity)}$$

$$6. \exists (-\vec{x}) \in \mathbb{V} \text{ s.t. } \vec{x} + (-\vec{x}) = \vec{0} \text{ (additive inverse)}$$

Axioms of scaling
(\cdot)

$$7. \alpha(\vec{x} + \vec{y}) = \alpha\vec{x} + \alpha\vec{y} \text{ (distributivity)}$$

$$8. \alpha \cdot (\beta \vec{x}) = (\alpha\beta) \cdot \vec{x}$$

$$9. (\alpha + \beta) \vec{x} = \alpha \vec{x} + \beta \vec{x}$$


$$10. 1 \cdot \vec{x} = \vec{x}$$

Vector Space

- A vector space \mathbb{V} is a set of vectors and two operators $\cdot, +$ that satisfy the following:


1. $\alpha \vec{x} \in \mathbb{V}$
2. $\vec{x} + \vec{y} \in \mathbb{V}$
3. $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$ (associativity)
4. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ (commutativity)
5. $\exists \vec{0} \in \mathbb{V}$ s.t. $\vec{x} + \vec{0} = \vec{x}$ (additive identity)
6. $\exists (-\vec{x}) \in \mathbb{V}$ s.t. $\vec{x} + (-\vec{x}) = \vec{0}$
7. $\alpha(\vec{x} + \vec{y}) = \alpha\vec{x} + \alpha\vec{y}$ (distributivity)
8. $\alpha \cdot (\beta \vec{x}) = (\alpha\beta) \cdot \vec{x}$
9. $(\alpha + \beta)\vec{x} = \alpha\vec{x} + \beta\vec{x}$
10. $1 \cdot \vec{x} = \vec{x}$

 Is \mathbb{R}^2 a vector space?

 Is $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2 \times 2}$?

 Is $\alpha \in \mathbb{R}, \alpha \geq 0$?

 Is $\text{span} \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$?

 Is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

 Is 0?

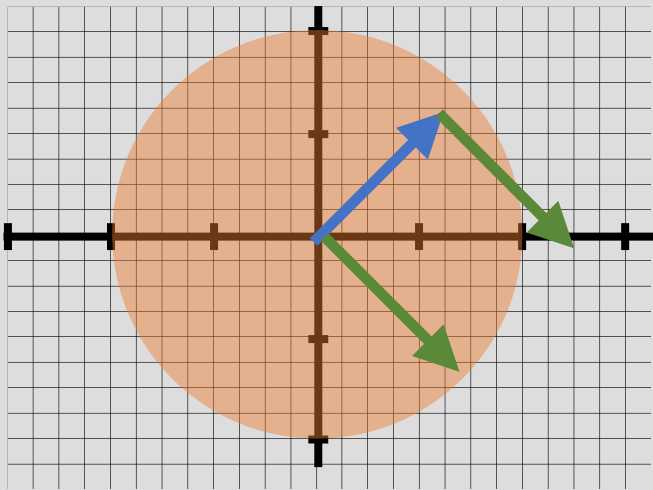
Subspaces

- A subspace \mathbb{U} consists of a subset of \mathbb{V} in vector space $(\mathbb{V}, \mathbb{F}, +, \cdot)$
 - $\mathbb{U} \subset \mathbb{V}$ and have 3 properties
 1. Contains $\vec{0}$, i.e., $\vec{0} \in \mathbb{U}$
 2. Closed under vector addition: $\vec{v}_1, \vec{v}_2 \in \mathbb{U}, \Rightarrow \vec{v}_1 + \vec{v}_2 \in \mathbb{U}$
 3. Closed under scalar multiplication: $\vec{v}_1 \in \mathbb{U}, \alpha \in \mathbb{F}, \Rightarrow \alpha \vec{v} \in \mathbb{U}$

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Q: Consider all vectors \vec{v} who's length < 1 . Is this a subspace?

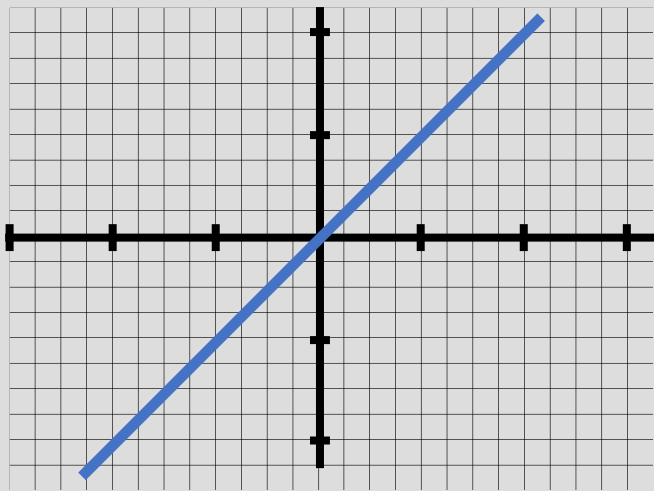


A: not closed under addition,
nor scalar mult.

Subspaces

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Q: Is $\text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ a subspace?

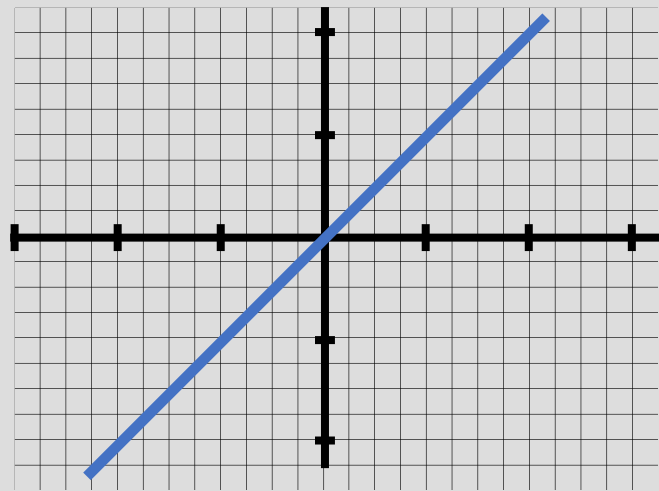


A: Yes!

Subspaces

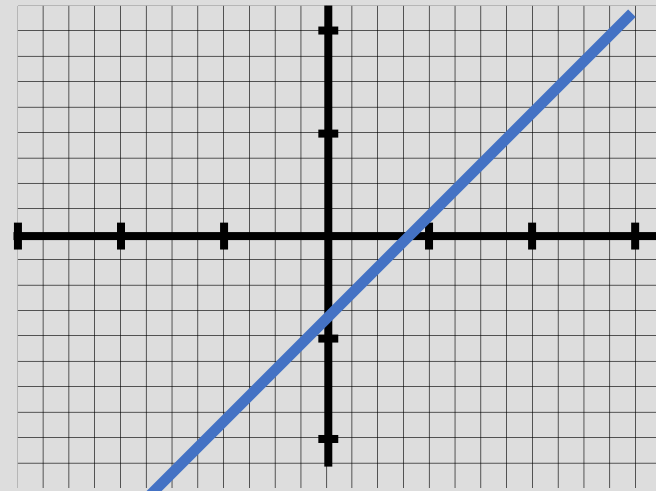
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Q: Is $\text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ a subspace?



A: Yes!

Q: What about this?

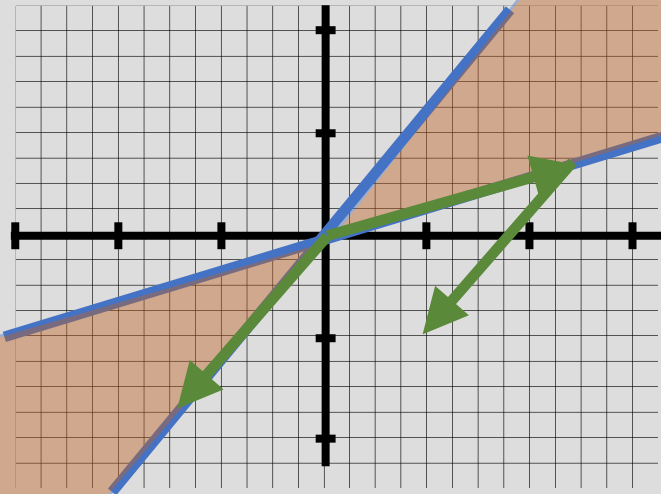


A: $\vec{0} \notin \mathbb{U}$
No!

Subspaces

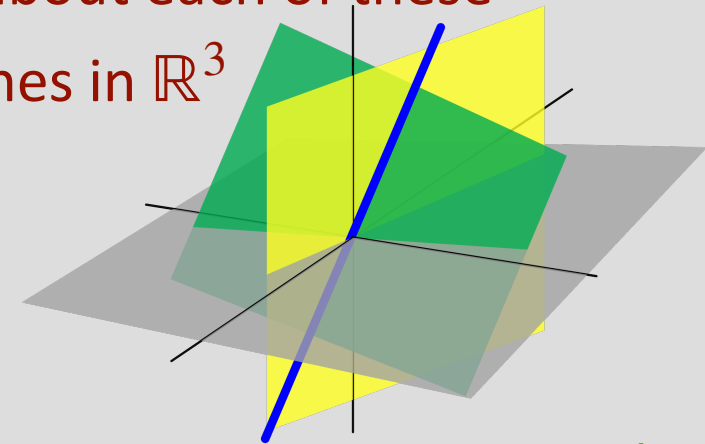
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Q: What about this?



A: Not closed under addition!

Q: What about each of these 2D planes in \mathbb{R}^3



A: yes, as long as passing through 0

Subspaces

Example:

$$W = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mid a, b, d \in \mathbb{R} \right\}, \quad V = \mathbb{R}^{2 \times 2}$$

Is $W \subset V$?



1. Zero vector?



2. Closed under addition?



3. Closed under scalar multiplication?

Bases

- In words: Minimum set of vectors that spans a vector space
- Definition: given \mathbb{V} , a set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_N\}$ is a basis of the vector space, if it satisfies:
 - $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_N\}$ are linearly independent
 - $\forall \vec{v} \in \mathbb{V}, \exists \alpha_1, \alpha_2, \dots, \alpha_N \in \mathbb{R}^N$ such that $\vec{v} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_N \vec{v}_N$

Bases examples

Q: Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ a basis for $V = \mathbb{R}^3$?




Q: Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ a basis for $V = \mathbb{R}^3$?



Q: Is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ a basis for $V = \mathbb{R}^3$?



Bases examples

Q: Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ a basis for $V = \mathbb{R}^3$? 

Q: Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix} \right\}$ a basis for $V = \mathbb{R}^3$? 

Column Space

- The range/span/columnspace of a set of vectors is a set of all possible linear combinations:

$$\text{span} \{ \vec{a}_1, \vec{a}_2, \dots, \vec{a}_M \} = \triangleq \left\{ \sum_{m=1}^M \alpha_m \vec{a}_m \mid \alpha_1, \alpha_2, \dots, \alpha_M \in \mathbb{R} \right\}$$

Consider:

$$A = \begin{bmatrix} a & 0 \\ 0 & b \\ 0 & 0 \end{bmatrix}$$

Q: Are the columns of A , a basis? 

Q: Is the column space of A , a subspace?

Column Space

Consider:

$$A = \begin{bmatrix} a & 0 \\ 0 & b \\ 0 & 0 \end{bmatrix} \quad \vec{v}_1 = A\vec{u}_1, \quad \vec{v}_2 = A\vec{u}_2$$

1. Zero vector?
2. Closed under addition?
3. Closed under scalar multiplication?

Q: Is the column space of A , a subspace?

$$A\vec{0} = \vec{0}$$

$$\vec{v}_1 + \vec{v}_2 = A\vec{u}_1 + A\vec{u}_2 = A(\vec{u}_1 + \vec{u}_2)$$

$$\alpha\vec{v}_1 = \alpha A\vec{u}_1 = A(\alpha\vec{u}_1)$$



Rank

- USA Today University Ranking for Cal:
 - #1 (joint) in Computer Science
 - #2 in Electrical Engineering (joint)
 - #3 in Computer Engineering

Rank

- $A \in \mathbb{R}^{N \times M}$, $\text{Rank} \{A\} = \dim \{ \text{Span} \{A\} \}$

$$A = \begin{bmatrix} a & 0 \\ 0 & b \\ 0 & 0 \end{bmatrix}$$

2

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2

$$A = \begin{bmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{bmatrix}$$

1

- $\text{Rank} \{A\} = \dim \{ \text{Span} \{A\} \} \leq \min(M, N)$

Null Space

- Definition: The null-space of $A \in \mathbb{R}^{N \times M}$ is the set of all vectors $\vec{x} \in \mathbb{R}^M$ such that: $A \vec{x} = 0$

$$A \vec{x} = 0$$

How many solutions for \vec{x} satisfy the above?

Examples

$$\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Linearly
independent!

$$\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\vec{0}$ is always in the null space — trivial Null space

Examples

Gaussian elimination:

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \Rightarrow x_1 = 2x_2$$
$$\Rightarrow \vec{x} = \begin{bmatrix} 2\alpha \\ \alpha \end{bmatrix}$$

Linearly dependent!

$$\vec{x} = \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

A has a non-trivial null-space, span $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

Example

$$A \vec{x} = \vec{b}$$

We know that $\vec{v}_0 \in \text{Null}(A)$

$$\rightarrow A \vec{v}_0 = \vec{0}$$

We know 1 solution: \vec{x}_0

$$\rightarrow A \vec{x}_0 = \vec{b}$$

Example

$$A\vec{x} = \vec{b}$$

We know that $\vec{v}_0 \in \text{Null}(A)$

$$\rightarrow A\vec{v}_0 = \vec{0}$$

We know 1 solution: \vec{x}_0

$$\rightarrow A\vec{x}_0 = \vec{b}$$

Then: $\vec{x}_0 + \alpha\vec{v}_0$ is also a solution

$$\begin{aligned}\rightarrow A(\vec{x}_0 + \alpha\vec{v}_0) &= A\vec{x}_0 + A(\alpha\vec{v}_0) \\ &= \vec{b} + \alpha A\vec{v}_0 \\ &= \vec{b}\end{aligned}$$

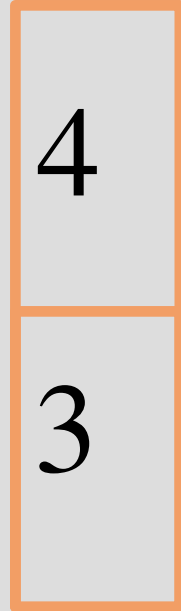
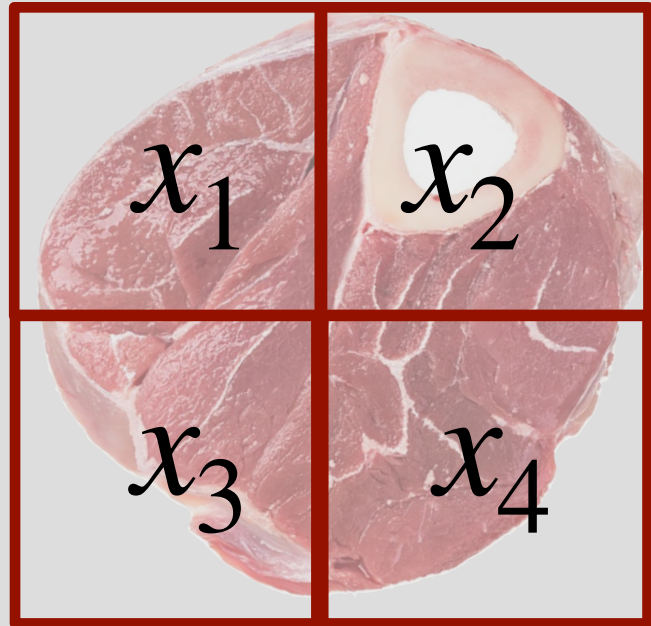
Back to Tomography

$$1 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 4$$

$$0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 1 \cdot x_4 = 3$$

$$1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 = 2$$

$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 5$$



$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \end{array} \right]$$

Null Space of the Tomography System (4 measur.)

Step I

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array} \right]$$

Step II

(3) - (1)

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array} \right]$$

Step III

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array} \right]$$

Step IV

(3) + (2)

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array} \right]$$

Step V

(4) - (3)

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Step VI

(1) - (2)

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Null Space of the Tomography System (4 measur.)

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

x_4 is the free variable:

$$\Rightarrow \vec{x} = \alpha \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Possible reconstruction

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} + \alpha \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$