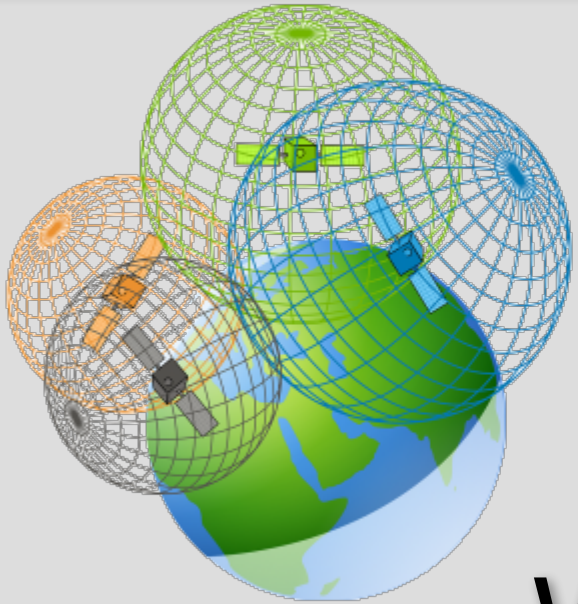
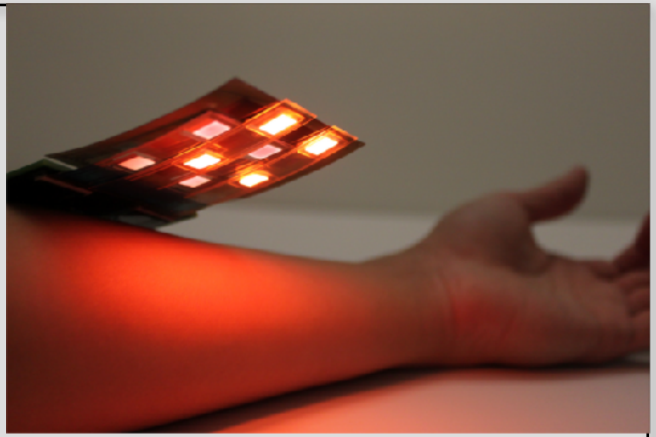
The Quest





Welcome to EECS 16A!

Designing Information Devices and Systems I

Ana Arias and Miki Lustig

Fall 2021

Lecture 4A

EigenVals/Vecs/Spaces

Announcements

- Last time:
  - Vector spaces
  - Null spaces
  - Subspaces
- Today:
  - Computing the determinant
  - Eigen Values and Eigen Vectors of a Matrix
    - Example via page-rank

Equivalent Statements

- Matrix A is **invertible**
- $A\overrightarrow{x} = \overrightarrow{b}$ has a unique solution
- A has linearly independent columns (A is **full rank**)
- A has a **trivial nullspace**
- The **determinant** of A is not zero

Jargon from Last time

- **Rank** a matrix A is the number of linearly independent columns
- **Nullspace** of a matrix A is the set of solutions to $A\overrightarrow{x} = 0$
- A **vector space** is a set of vectors connected by two operators (+,x)
- A vector **subspace** is a subset of vectors that have "nice properties"
- A **basis** for a vector space is a minimum set of vectors needed to represent all vectors in the space
- **Dimension** of a vector space is the number of basis vectors
- **Column space** is the span (range) of the columns of a matrix
- **Row space** is the span of the rows of a matrix

Vector Space

- A vector space, is a set of vectors and scalars ($\mathbb{V} \in \mathbb{R}^N, \mathbb{F} \in \mathbb{R}$) and two operators $\cdot, +$ that satisfy the following:

Axioms of closure

1. $\alpha \vec{x} \in \mathbb{V}$
2. $\vec{x} + \vec{y} \in \mathbb{V}$

Axioms of addition (+)

3. $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$ (associativity)
4. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ (commutativity)
5. $\exists \vec{0} \in \mathbb{V}$ s.t. $\vec{x} + \vec{0} = \vec{x}$ (additive identity)
6. $\exists(-\vec{x}) \in \mathbb{V}$ s.t. $\vec{x} + (-\vec{x}) = \vec{0}$ (additive inverse)

Axioms of scaling (·)

7. $\alpha(\vec{x} + \vec{y}) = \alpha \vec{x} + \alpha \vec{y}$ (distributivity)
8. $\alpha \cdot (\beta \vec{x}) = (\alpha\beta) \cdot \vec{x}$
9. $(\alpha + \beta)\vec{x} = \alpha \vec{x} + \beta \vec{x}$
10. $1 \cdot \vec{x} = \vec{x}$

Subspaces

- A subspace $\mathbb{U}$ consists of a subset of $\mathbb{V}$ in vector space $(\mathbb{V}, \mathbb{F}, +, \cdot)$

  - $\mathbb{U} \subset \mathbb{V}$ and have 3 properties

  1. Contains $\overrightarrow{0}$, i.e., $\overrightarrow{0} \in \mathbb{U}$
  2. Closed under vector addition: $\overrightarrow{v}_1, \overrightarrow{v}_2 \in \mathbb{U}, \Rightarrow \overrightarrow{v}_1 + \overrightarrow{v}_2 \in \mathbb{U}$
  3. Closed under scalar multiplication: $\overrightarrow{v}_1 \in \mathbb{U}, \alpha \in \mathbb{F}, \Rightarrow \alpha \overrightarrow{v} \in \mathbb{U}$

Null Space

- Definition: The null-space of $A \in \mathbb{R}^{N \times M}$ is the set of all vectors $\overrightarrow{x} \in \mathbb{R}^M$ such that: $A\overrightarrow{x} = 0$

$$A\overrightarrow{x} = 0$$

Rank

- $A \in \mathbb{R}^{N \times M}$, $\text{Rank}\{A\} = \dim\left\{\text{Span}\{A\}\right\}$
- $\text{Rank}\{A\} = \dim\left\{\text{Span}\{A\}\right\} \leq \min(M, N)$
- $\text{Rank} = L$, mean the matrix $A \in \mathbb{R}^{N \times M}$ has L independent rows&columns
- $\text{Rank}\{A\} + \dim\left\{\text{Null}\{A\}\right\} = \min(M, N)$

The Determinant

- For $A \in \mathbb{R}^{2\times 2}$

$$\det(A) = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

When $\det(A) \neq 0$, A is invertible

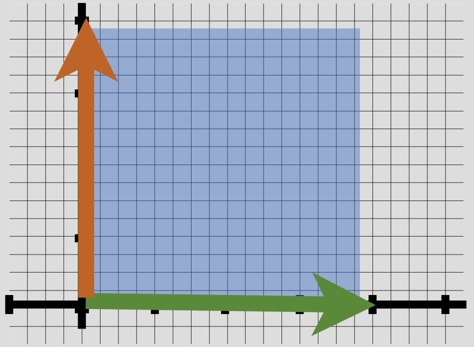
Recall:

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Interpretation of Determinant of a Matrix in $\mathbb{R}^{2\times 2}$

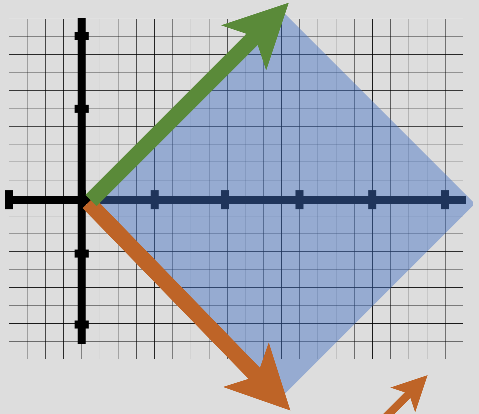
- Area of a parallelogram

$$\det(A) = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$



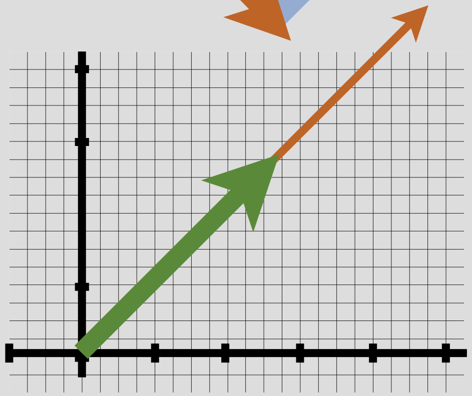
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Area≠0



$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Area≠0



$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

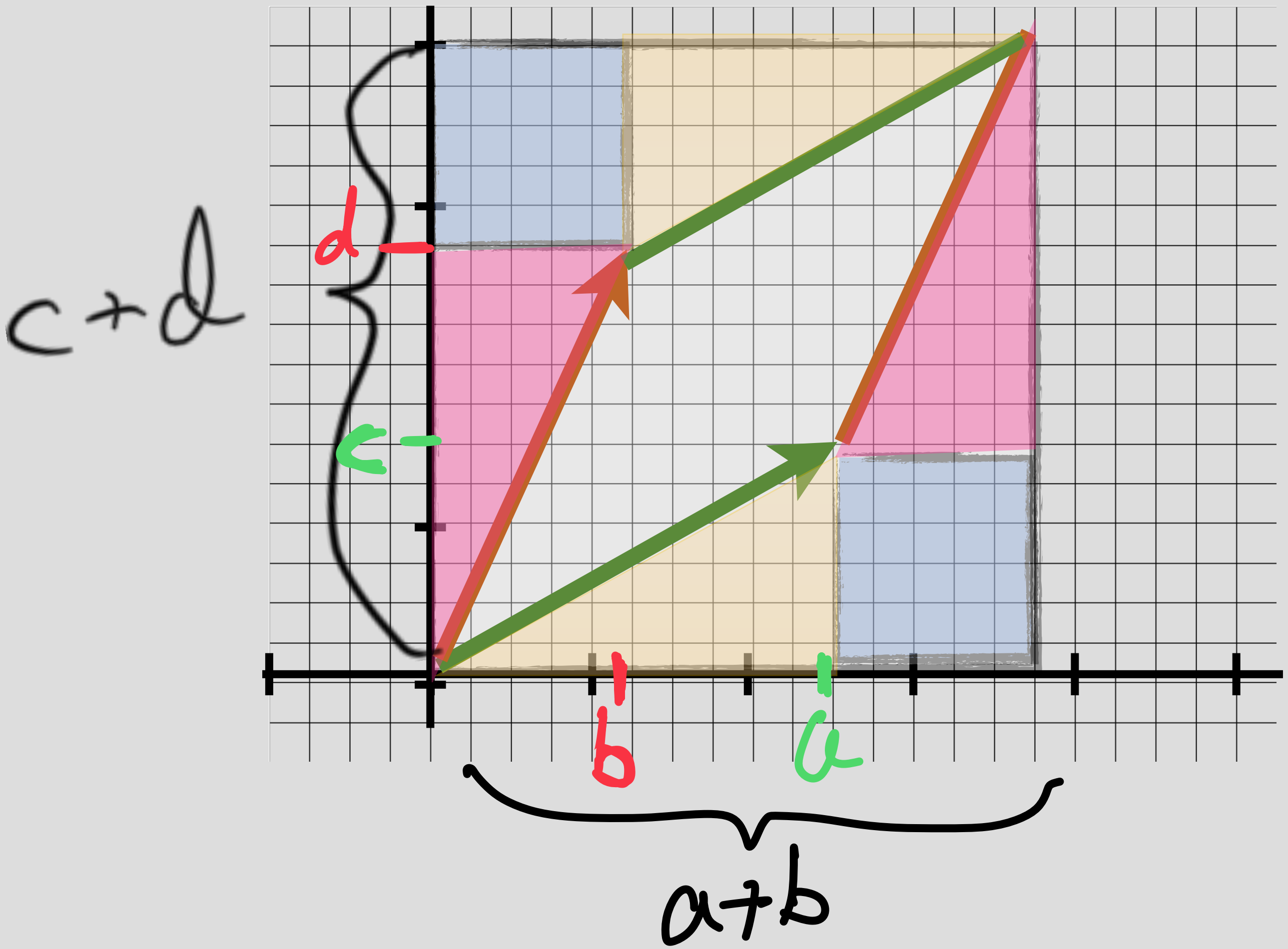
Area=0

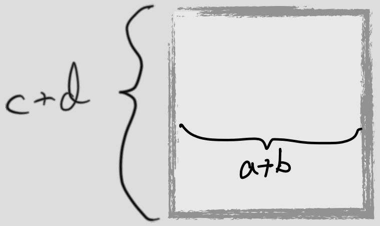
$$\det(A) = 1 \cdot 2 - 1 \cdot 2 = 0$$

Interpretation of Determinant of a Matrix in $\mathbb{R}^{2\times 2}$

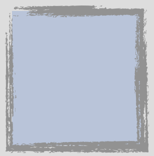
- Area of a parallelogram

$$\det(A) = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = ad - bc$$



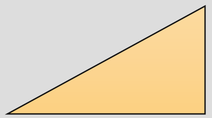


$$(c+d)(a+b)$$



x2

$$bc \times 2$$

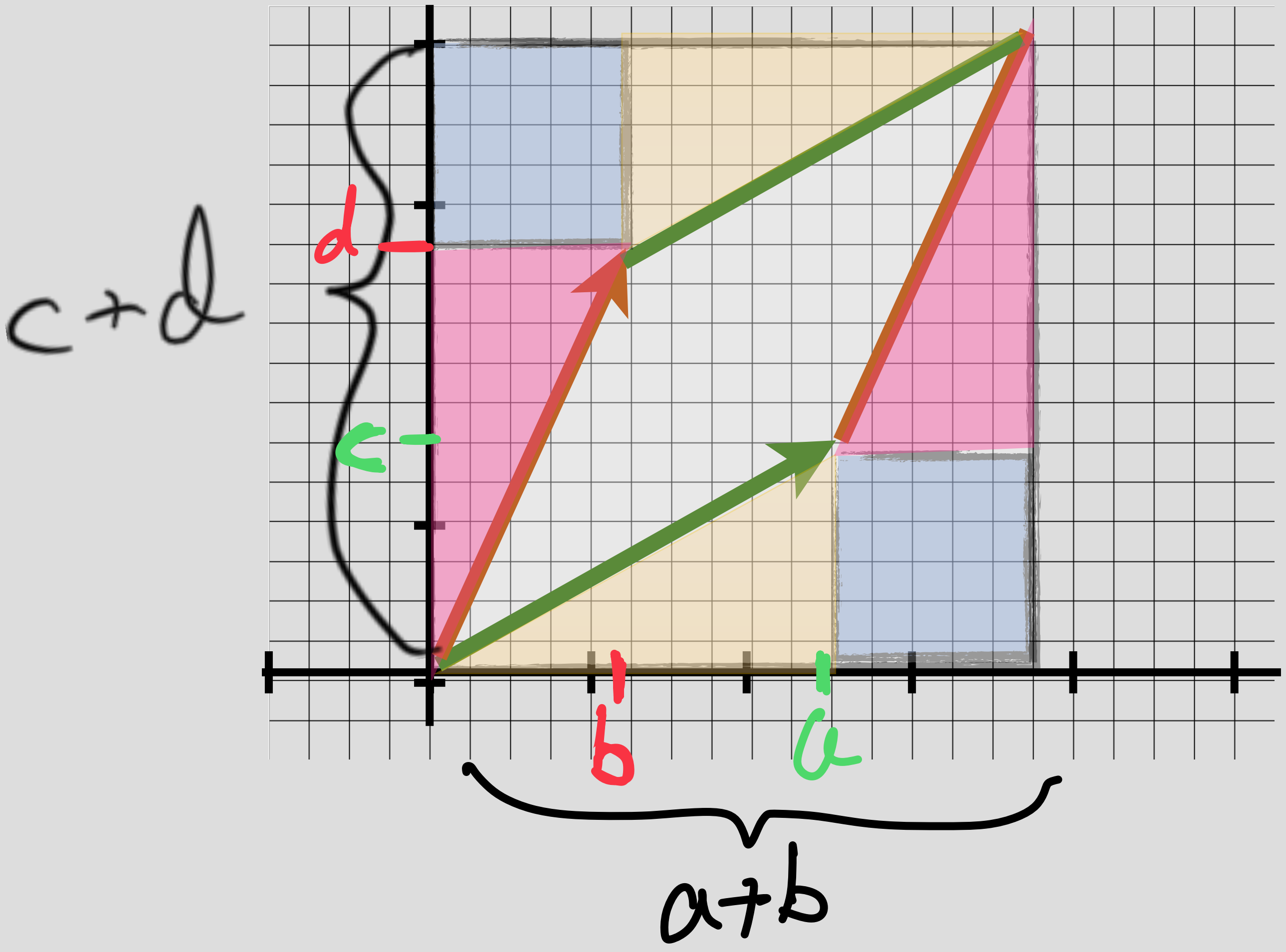


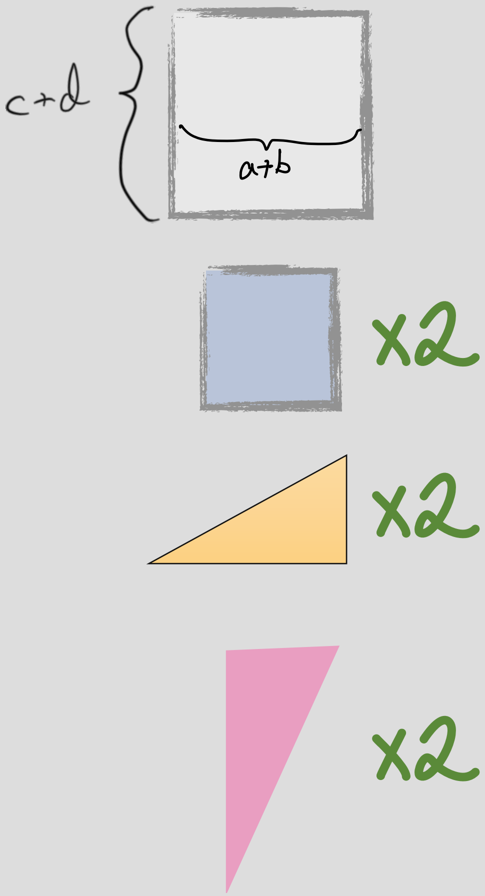
x2

$$\frac{1}{2}ac \times 2$$

x2

$$\frac{1}{2}bd \times 2$$



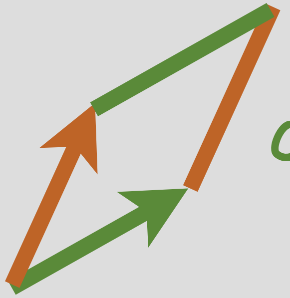


$(c+d)(a+b)$

$2bc$

ac

bd



area $= (c+d)(a+b) - 2bc - ac - bd$

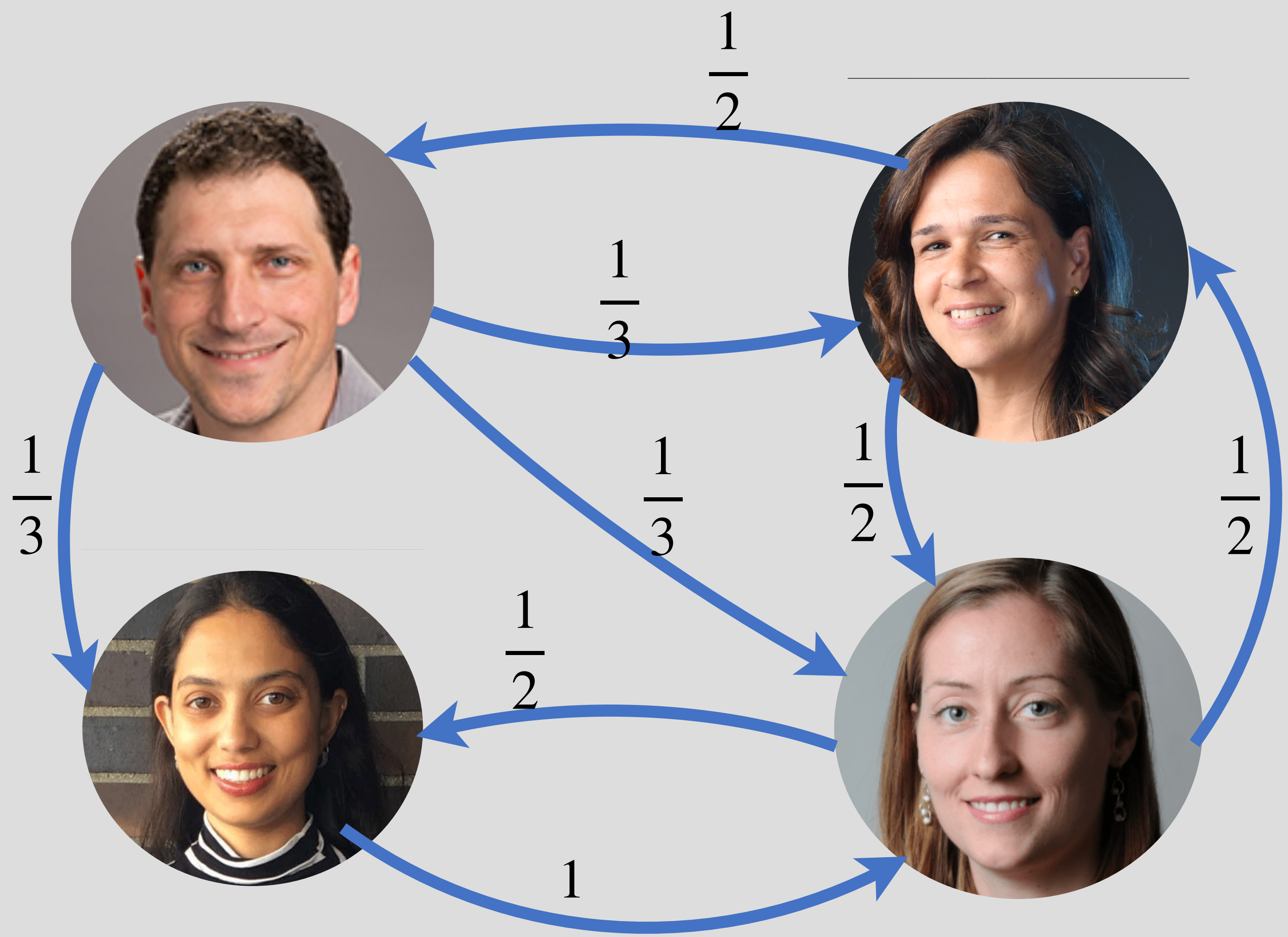
$= ca + cb + da + db - 2bc - ac - bd = ad - bc$

Determinant in $\mathbb{R}^3$

$$\det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = \left[a \times \begin{vmatrix} e & f \\ h & i \end{vmatrix} \right] - \left[b \times \begin{vmatrix} d & f \\ g & i \end{vmatrix} \right] + \left[c \times \begin{vmatrix} d & e \\ g & h \end{vmatrix} \right]$$

PageRank

- Ranks websites based on how many high-ranked pages link to them

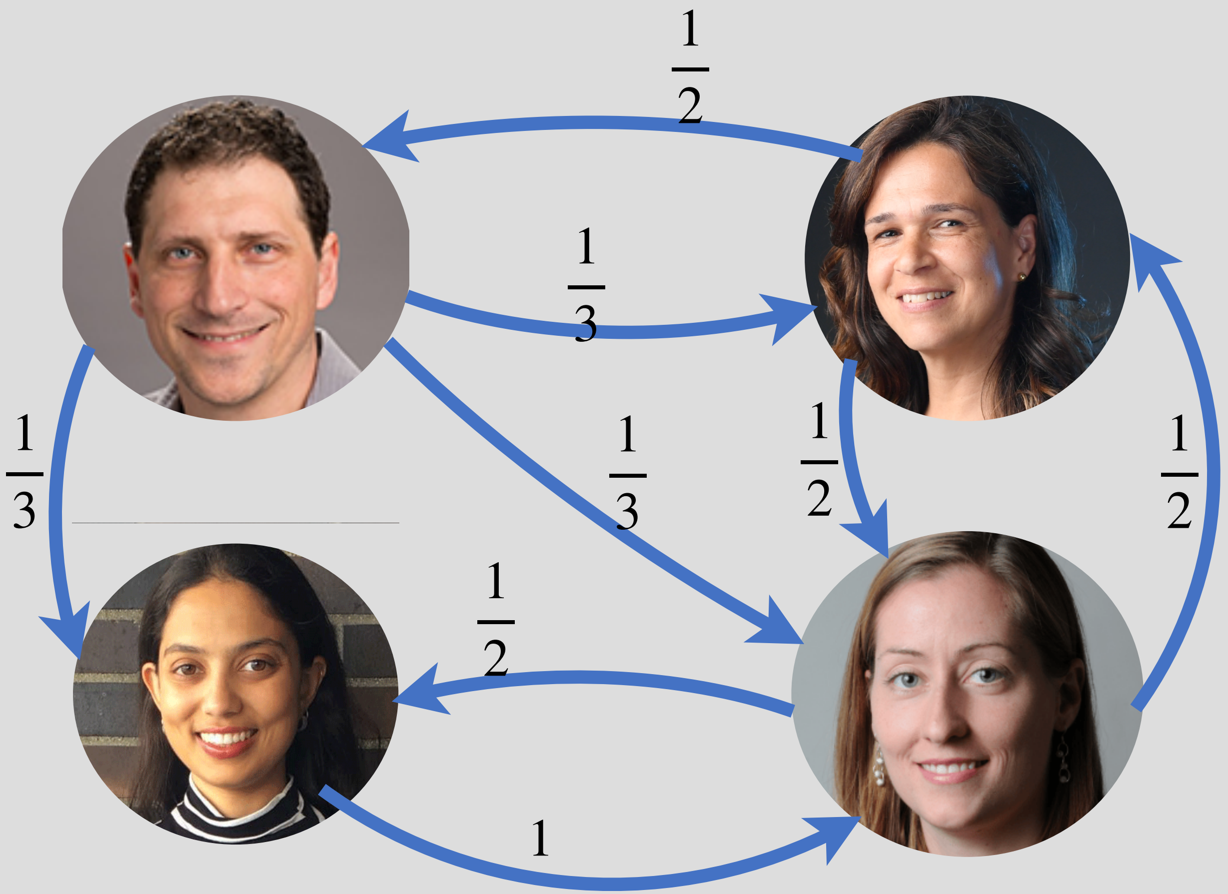


PageRank

From

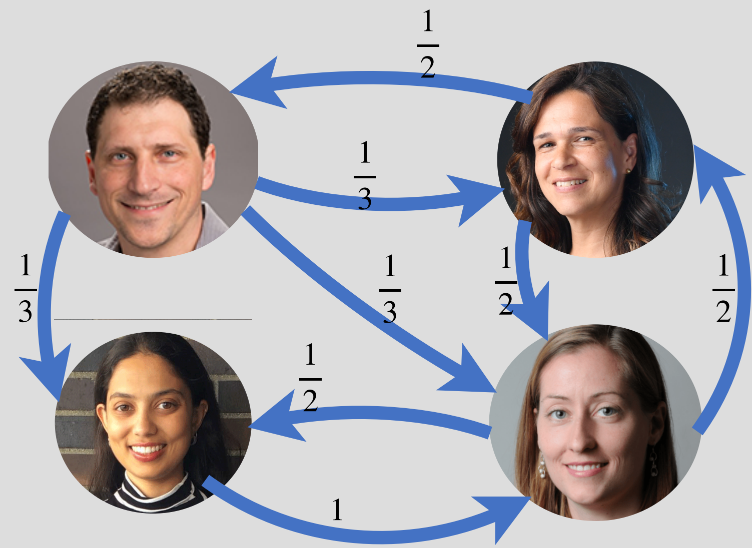
To

$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 \end{bmatrix}$$



PageRank

$$\vec{x}(t+1) = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 1 & 0 \end{bmatrix} \vec{x}(t)$$



$\vec{x}(t) \Rightarrow$ Page ranking

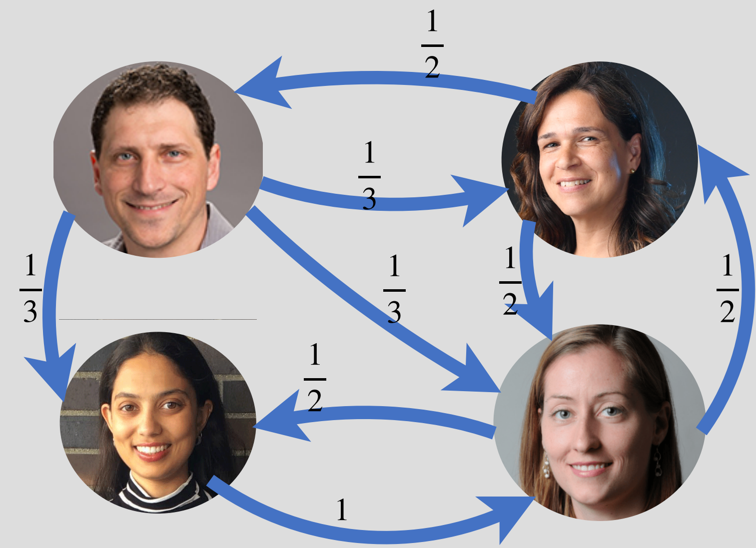
$$\vec{x}(0) = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

$\Downarrow$

equal Ranking

PageRank

$$\vec{x}(t+1) = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 1 & 0 \end{bmatrix} \vec{x}(t)$$



$\vec{x}(t) \Rightarrow$ Page ranking

$$\vec{x}(0) = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

$\Downarrow$

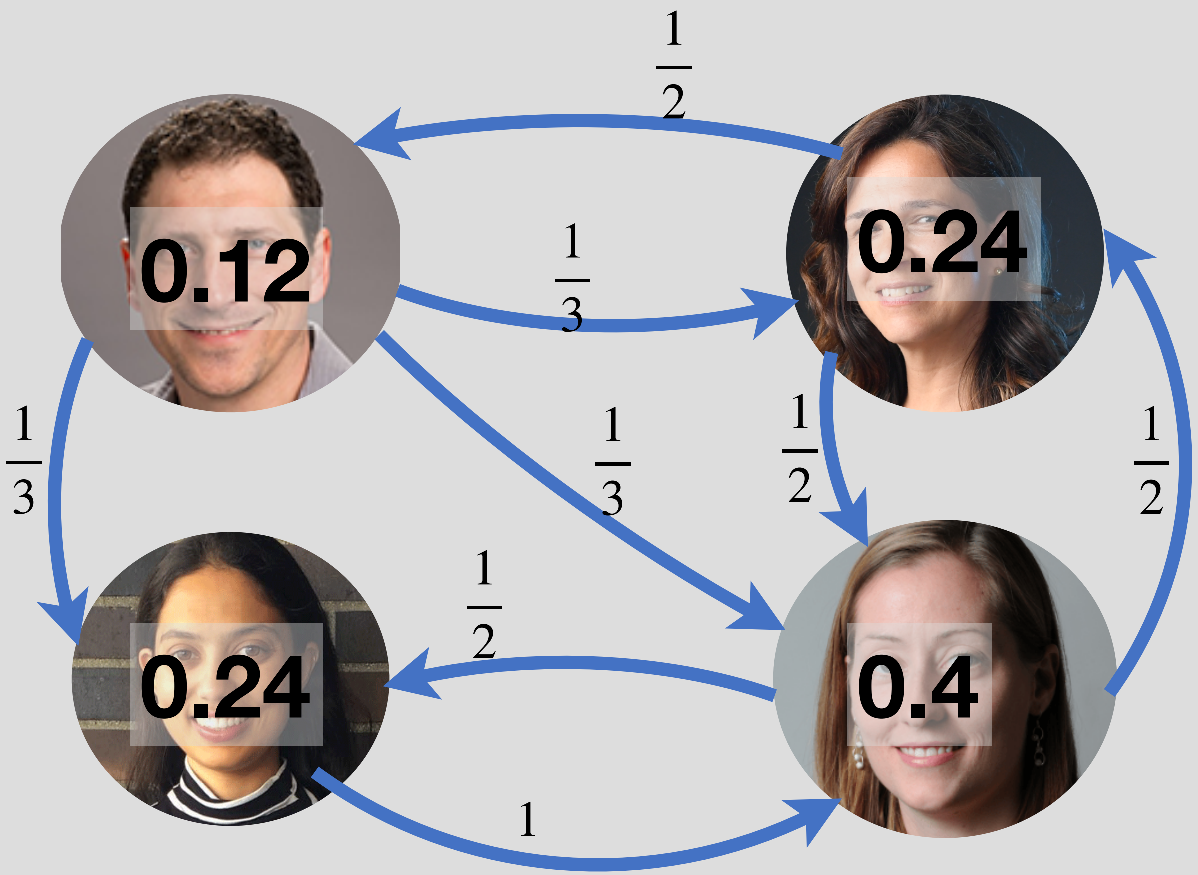
equal Ranking

$t=1$: $\begin{bmatrix} 0.125 \\ 0.208 \\ 0.208 \\ 0.458 \end{bmatrix}$ $t=2$: $\begin{bmatrix} 0.104 \\ 0.271 \\ 0.271 \\ 0.354 \end{bmatrix}$ $t=3$: $\begin{bmatrix} 0.135 \\ 0.212 \\ 0.212 \\ 0.441 \end{bmatrix}$ $\cdots$ $t=100$: $\begin{bmatrix} 0.12 \\ 0.24 \\ 0.24 \\ 0.4 \end{bmatrix}$

Page Rank

$$\begin{bmatrix} 0.12 \\ 0.24 \\ 0.24 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.12 \\ 0.24 \\ 0.24 \\ 0.4 \end{bmatrix}$$

steady state!




Judge me by my PageRank, do you?
Pirillo & Fitz
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General Steady-state solution

$$\overrightarrow{x}_{ss} = Q \cdot \overrightarrow{x}_{ss}$$

$$I\overrightarrow{x}_{ss} = Q \cdot \overrightarrow{x}_{ss}$$

$$Q \cdot \overrightarrow{x}_{ss} - I\overrightarrow{x}_{ss} = \overrightarrow{0}$$

$$(Q - I)\overrightarrow{x}_{ss} = \overrightarrow{0}$$

The $\mathrm{Null}(Q - I)$ is the steady state solution

Find via Gauss elimination!

Eigen Values

We saw an example for a steady-state vector

$$Q \cdot \overrightarrow{x}_{\mathrm{ss}} = 1 \cdot \overrightarrow{x}_{\mathrm{ss}}$$

Direction, and size of the vector did not change!

We will now look at the more general case

$$Q \cdot \overrightarrow{x} = \lambda \cdot \overrightarrow{x}$$

In this case, we say that

$\overrightarrow{x}$ is an Eigen Vector of Q with Eigen Value λ

and $\mathrm{span}\{\overrightarrow{x}\}$ is the associated Eigen-space

Eigen Values

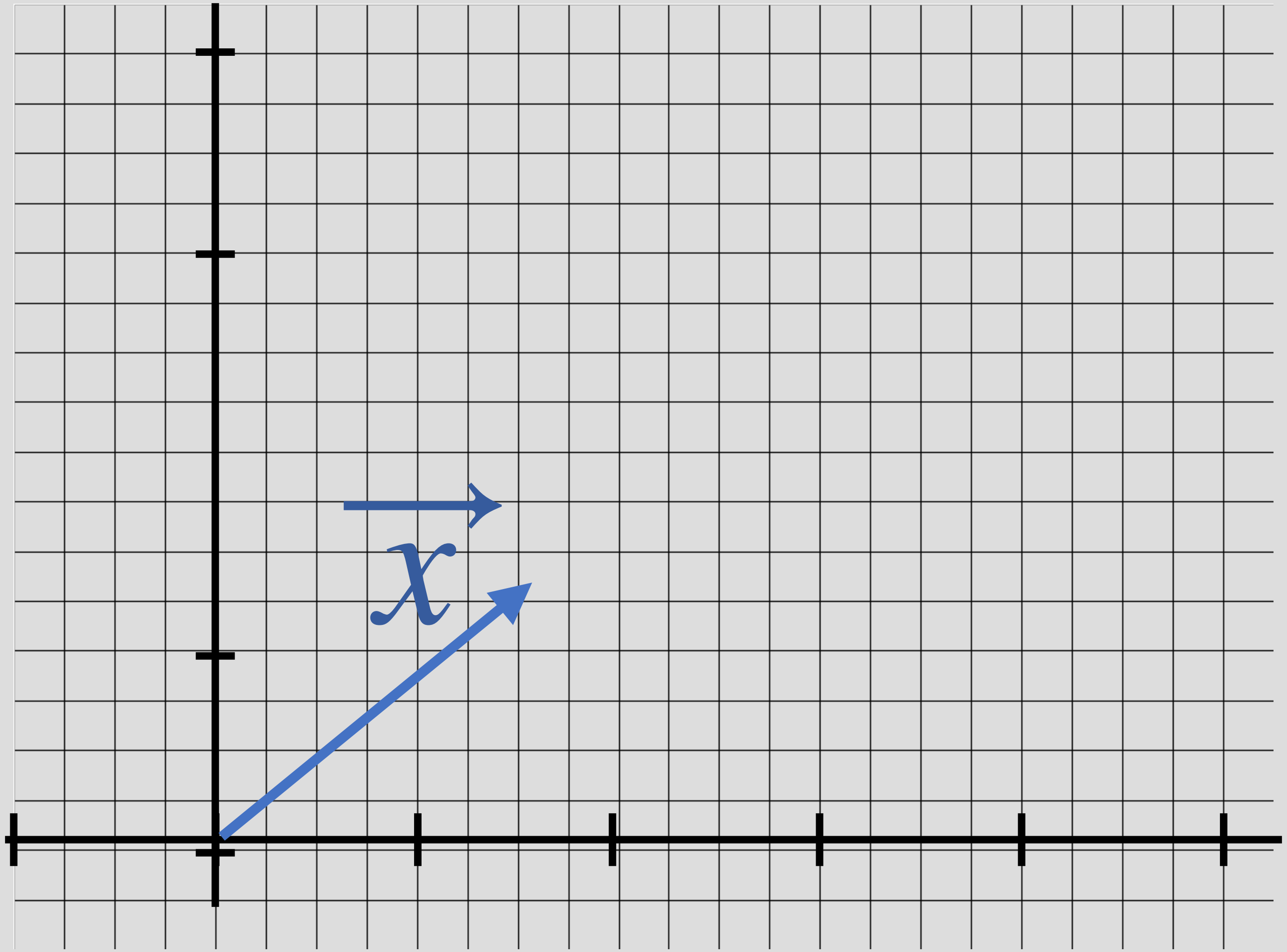
$$Q \cdot \overrightarrow{x} = \lambda \cdot \overrightarrow{x}$$

What happens if,

$\lambda = 1$?

$\lambda > 1$?

$\lambda < 1$?



Eigen Values and Eigen Vectors

- Definition: Let $Q \in \mathbb{R}^{N \times N}$ be a square matrix, and $\lambda \in \mathbb{R}$ if $\exists \overrightarrow{x} \neq \overrightarrow{0}$ such that $Q\overrightarrow{x} = \lambda \overrightarrow{x}$,
then λ is an eigenvalue of Q, $\overrightarrow{x}$ is an eigenvector and $\mathrm{Null}(Q - \lambda I)$ is its eigenspace.

Computing eigenvalues and vectors via determinant

Consider :

$$Q = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{bmatrix}$$, we want to find λ, $\vec{x}$ such that $Q\vec{x} = \lambda\vec{x}$

$$Q\vec{x} - \lambda\vec{x} = \vec{0}$$

$$(Q\vec{x} - \lambda I)\vec{x} = \vec{0}$$

Find $\vec{x} \in \text{Null}(Q - \lambda I)$:

$$Q - \lambda I = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1/2 - \lambda & 0 \\ 1/2 & 1-\lambda \end{bmatrix}$$

① find λ

② find $\vec{x}$

Computing eigenvalues and vectors via determinant

Find $\vec{x} \in \text{Null}(Q - \lambda I)$:

$$Q - \lambda I = \begin{bmatrix} 1/2 - \lambda & 0 \\ 1/2 & 1 - \lambda \end{bmatrix}$$

① find λ

② find $\vec{x}$

Find λ that results in a non-trivial null space

$$\det(Q - \lambda I) = 0$$

$$(1/2 - \lambda)(1 - \lambda) - (0) \cdot 1/2 = 0$$

$$(1/2 - \lambda)(1 - \lambda) = 0$$

$$\lambda_1 = 1/2,\ \lambda_2 = 1$$

Computing eigenvalues and vectors via determinant

Find $\vec{x} \in \text{Null}(Q - \lambda I)$:

$$Q - \lambda I = \begin{bmatrix} 1/2 - \lambda & 0 \\ 1/2 & 1 - \lambda \end{bmatrix}$$

① find λ $\quad \lambda_1 = 1/2,\ \lambda_2 = 1$

② find $\vec{x}$

$\lambda_1 = 1/2$

$$\begin{bmatrix} 1/2 - 1/2 & 0 \\ 1/2 & 1 - 1/2 \end{bmatrix} \vec{x} = 0$$

$$\begin{bmatrix} 0 & 0 \\ 1/2 & 1/2 \end{bmatrix} \vec{x} = 0$$

$$\left[\begin{array}{cc|c} 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad x_1 = -x_2$$

$$\Downarrow$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \vec{x}_1 \in \text{Span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

Computing eigenvalues and vectors via determinant

Find $\vec{x} \in \mathrm{Null}(Q - \lambda I)$:

$Q - \lambda I = \begin{bmatrix} 1/2-\lambda & 0 \\ 1/2 & 1-\lambda \end{bmatrix}$

① find λ $\quad \lambda_1 = 1/2,\ \lambda_2 = 1$

② find $\vec{x}$

$\lambda_1 = 1/2$

$$\begin{bmatrix} 1/2 - 1/2 & 0 \\ 1/2 & 1 - 1/2 \end{bmatrix} \vec{x} = 0$$

$$\begin{bmatrix} 0 & 0 \\ 1/2 & 1/2 \end{bmatrix} \vec{x} = 0$$

$$\left[\begin{array}{cc|c} 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right] \Downarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = -x_2$

$\vec{x}_1 \in \mathrm{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$

$\lambda_2 = 1$

$$\begin{bmatrix} 1/2 - 1 & 0 \\ 1/2 & 1 - 1 \end{bmatrix} \vec{x} = 0$$

$$\begin{bmatrix} -1/2 & 0 \\ 1/2 & 0 \end{bmatrix} \vec{x} = 0$$

$$\left[\begin{array}{cc|c} 1/2 & 0 & 0 \\ -1/2 & 0 & 0 \end{array}\right] \Downarrow \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = 0$

$\vec{x}_2 \in \mathrm{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

Eigen-vals/vectors/spaces

$$Q = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}$$

The matrix Q has the Eigen-vector

$$\vec{x}_1 \in \text{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

eigenspace

and,

has the Eigen-vector

$$\vec{x}_2 \in \text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

eigenspace

Associated with eigenvalue $\lambda_1 = 1/2$

$$\vec{v} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/2 \cdot 2 + 0(-2) \\ 1/2 \cdot 2 + 1(-2) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

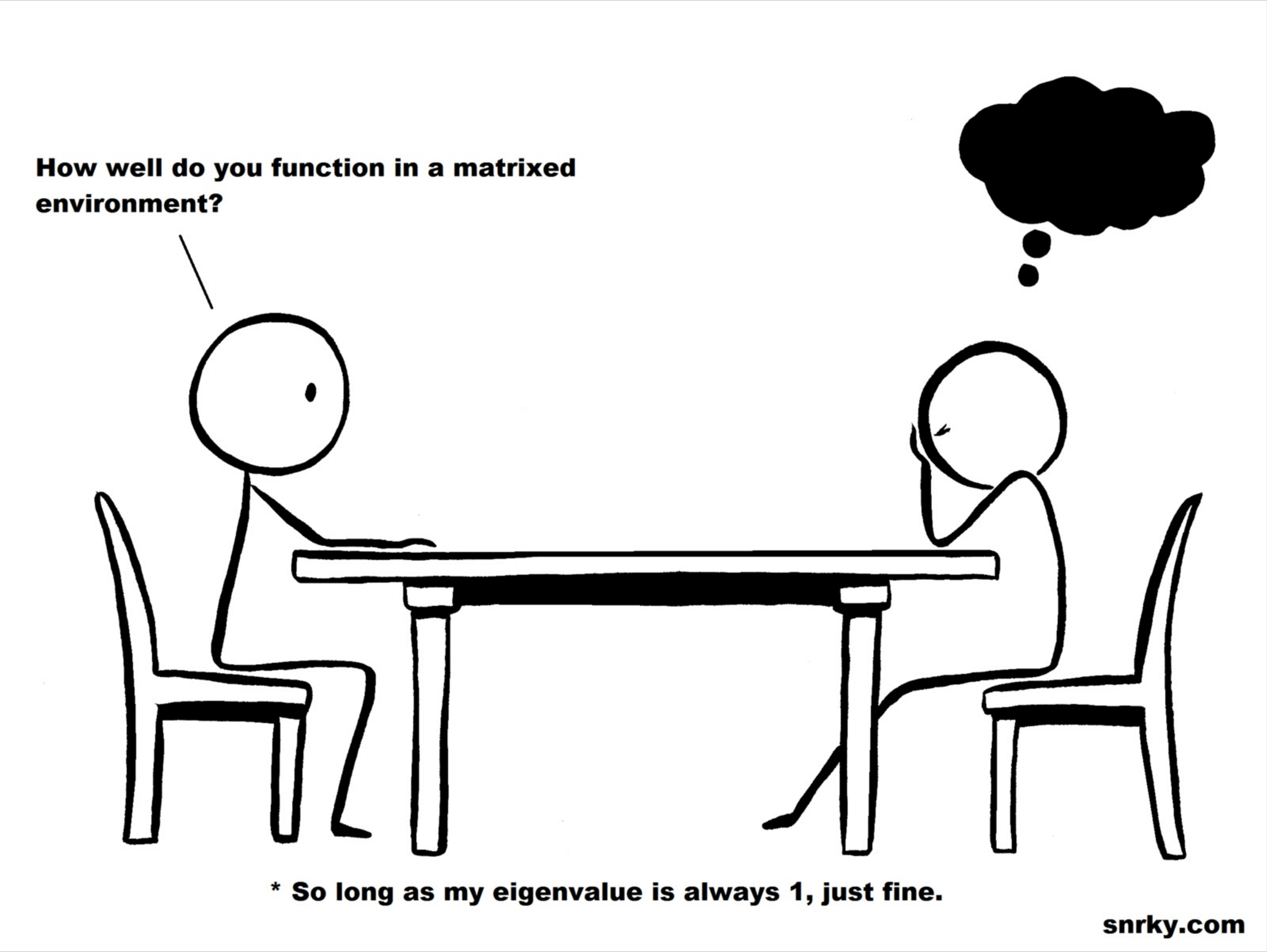
$$Q\vec{v} = 1/2\vec{v}$$

Associated with eigenvalue $\lambda_2 = 1$

$$\vec{u} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/2 \cdot 0 + 0(2) \\ 1/2 \cdot 0 + 1 \cdot 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$Q\vec{u} = 1 \cdot \vec{u}$$


How well do you function in a matrixed environment?
* So long as my eigenvalue is always 1, just fine.
snrky.com

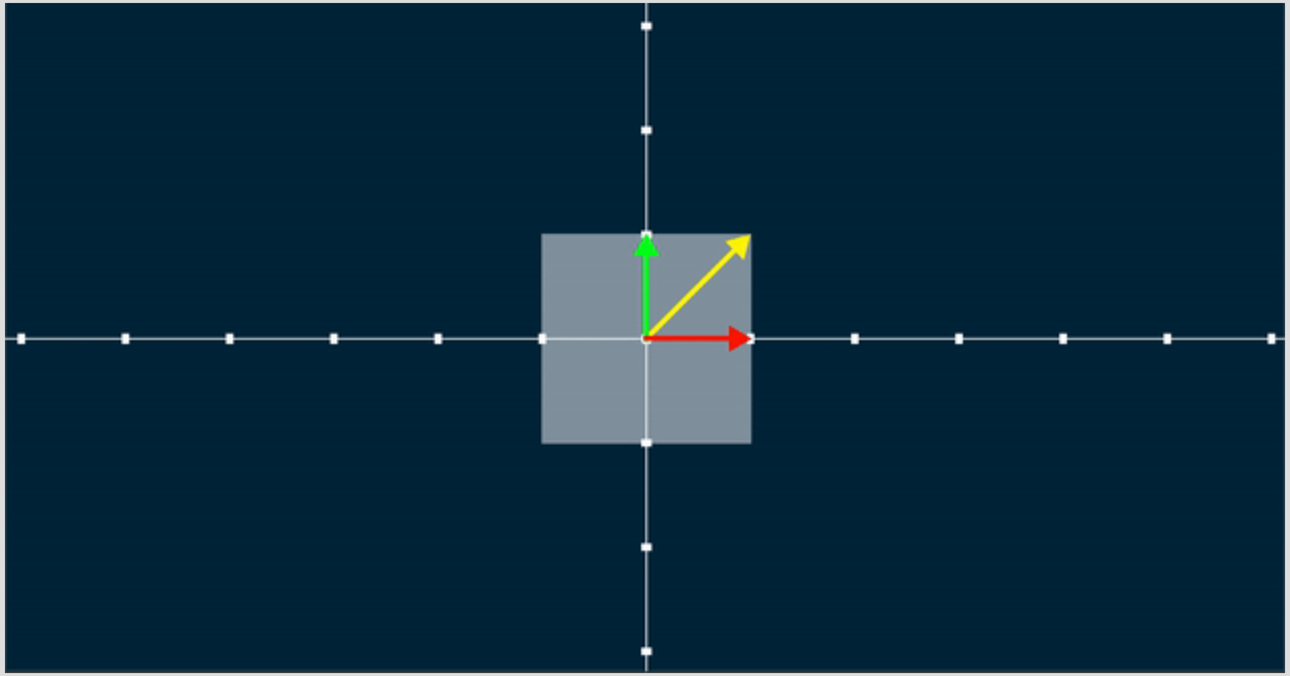
Matrix transformations

What does the matrix do?

What is the A matrix?

What are its eigenvectors?

What are its eigenvalues?



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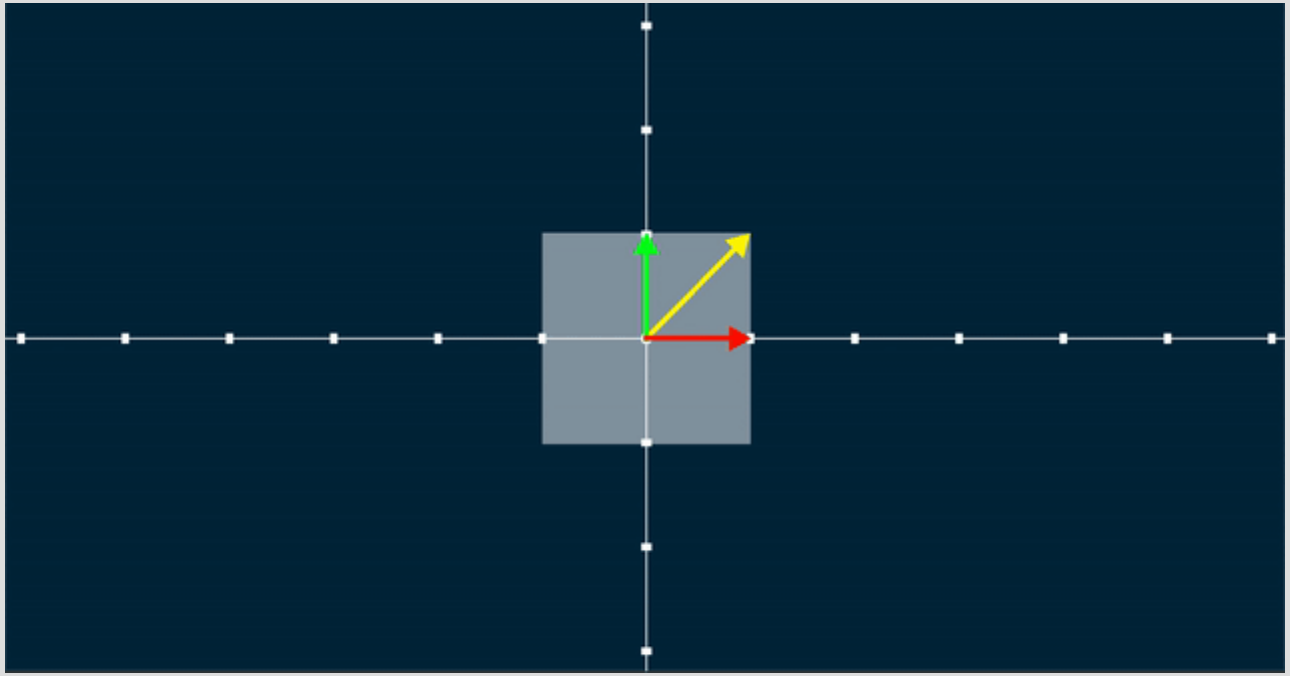

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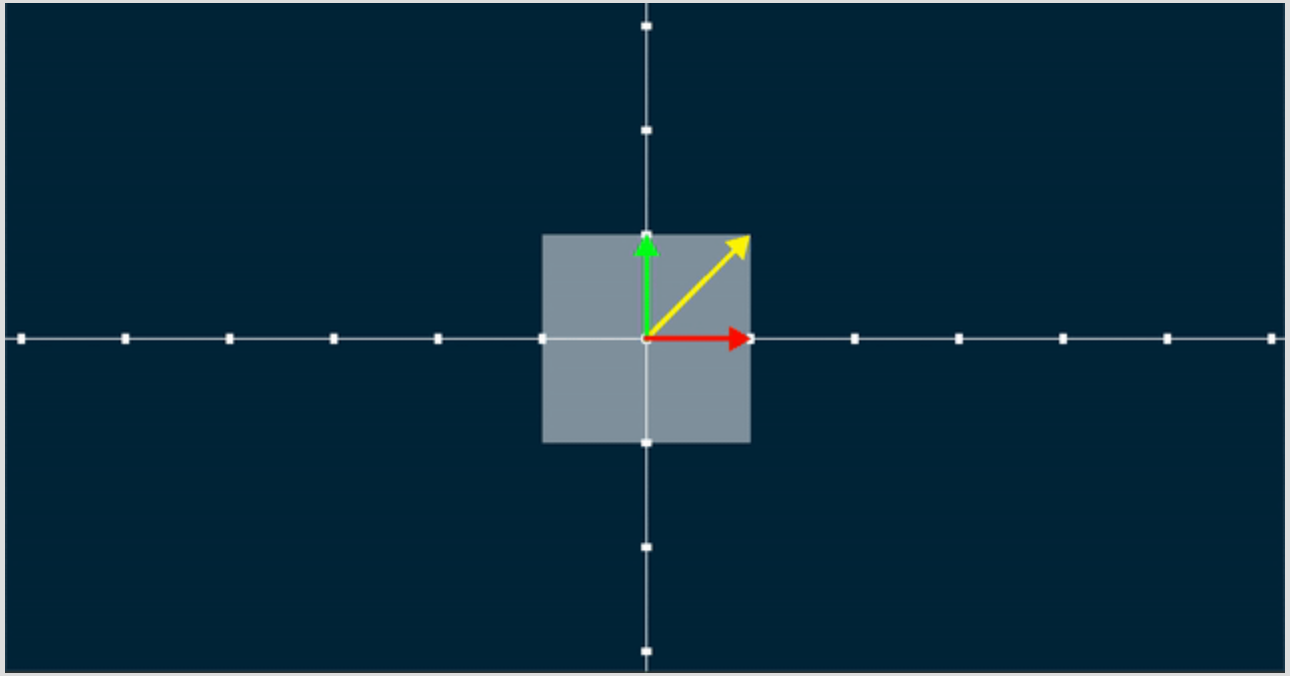

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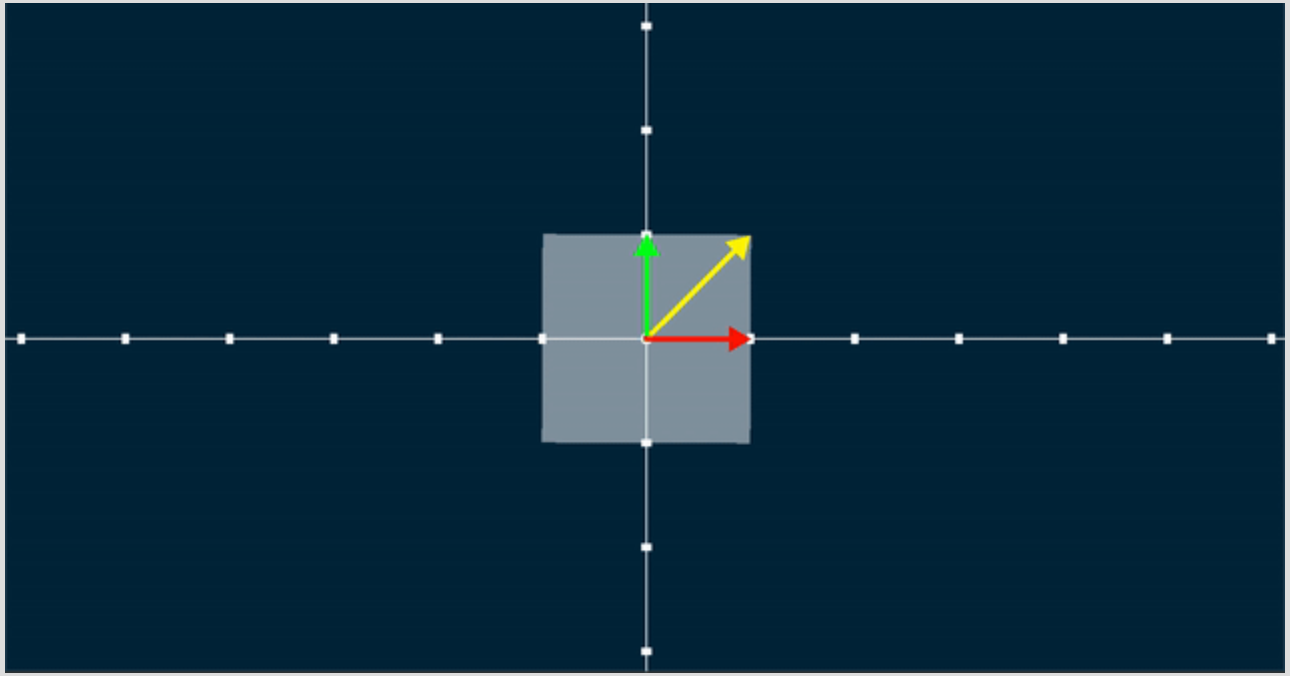

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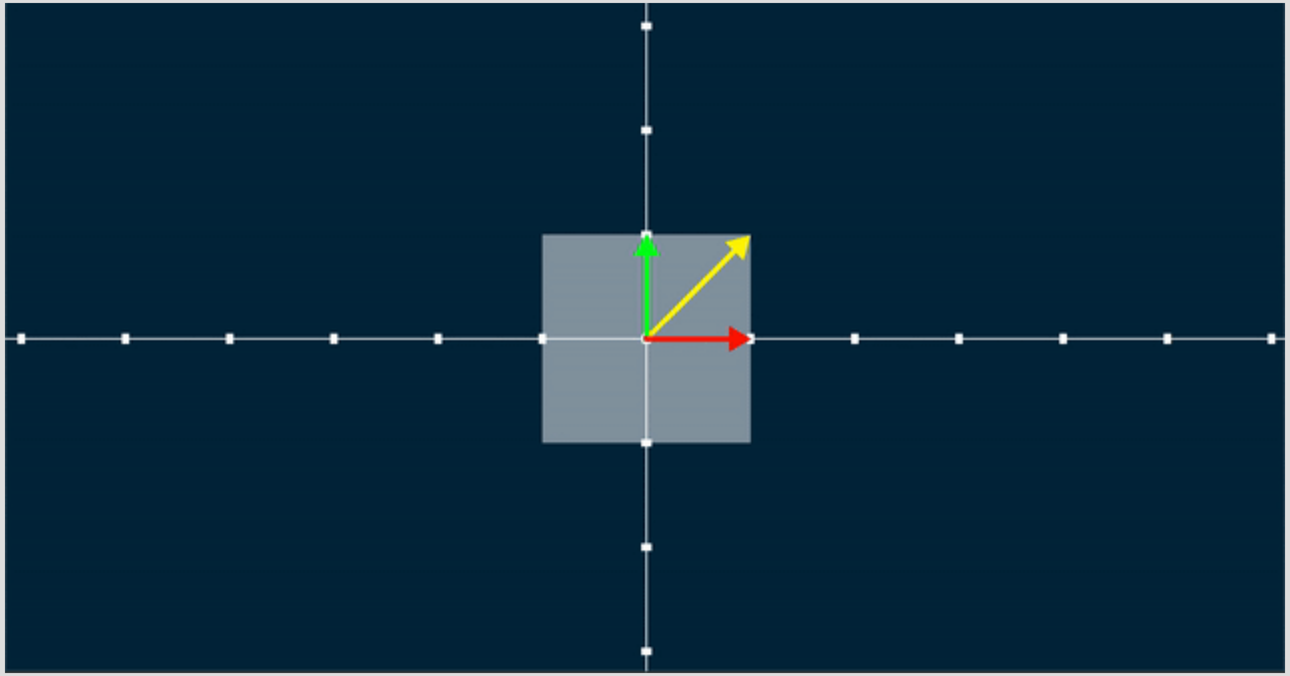

Matrix transformations

For a matrix that flips (reflects) vectors along a line:

What is the A matrix?

What are its eigenvectors?

What are its eigenvalues?