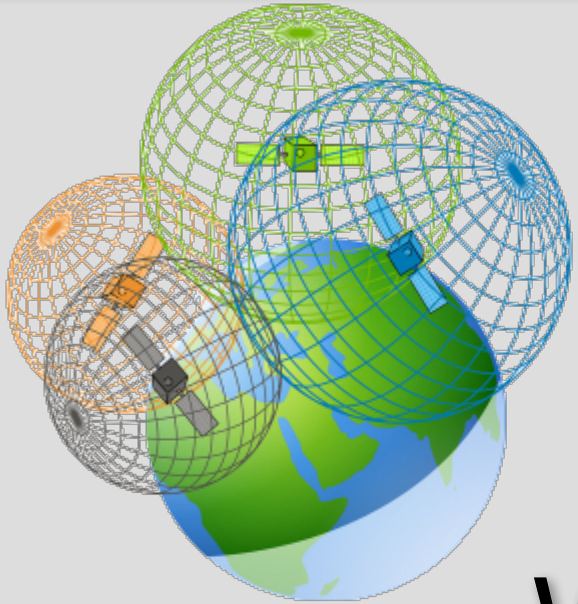
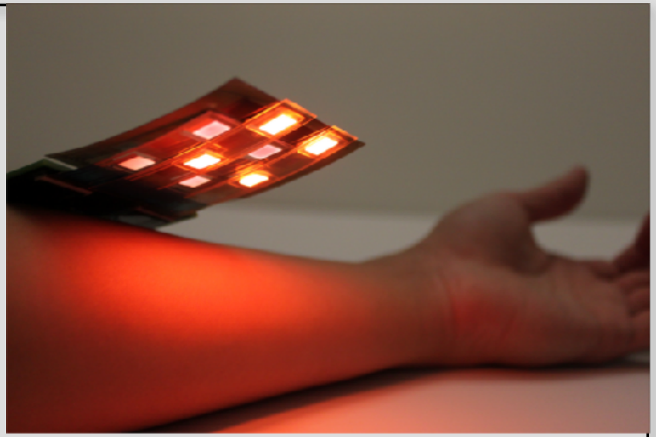




Welcome to EECS 16A!

Designing Information Devices and Systems I

Ana Arias and Miki Lustig

Fall 2021

Lecture 5A
EigenVals/Vecs/Spaces

TLDR

https://www.wnyc.org/story/tldr-hundred-songs/

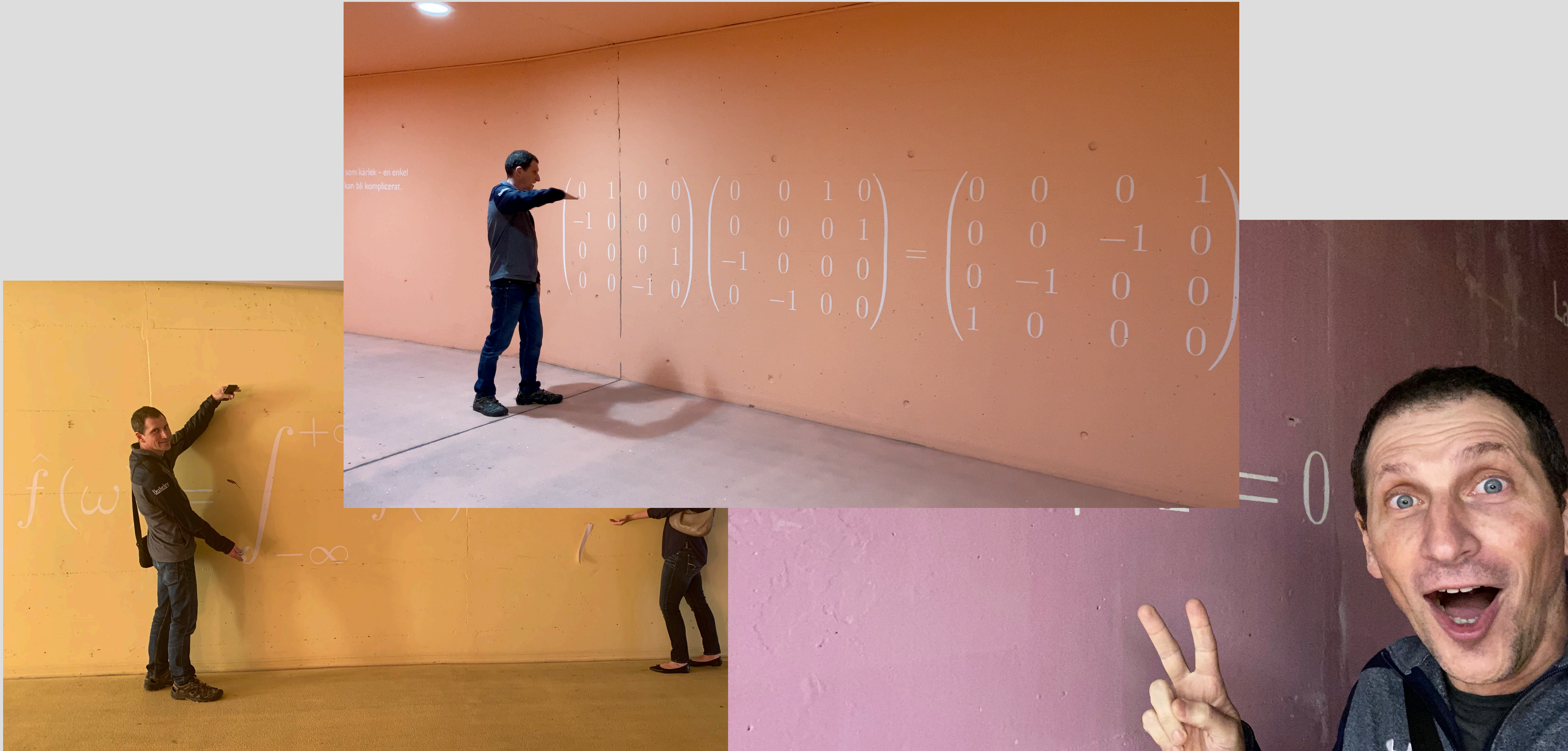
One way to make money making music online is the boring way. Write one song that does incredibly well and live off the royalties for the rest of your life.

Matt Farley is a musician who's gone a different route. He's written over 14,000 songs and he makes a tiny bit of money each time someone plays one on Spotify or iTunes. PJ visited Matt at his home recording studio to see how it all works.

Thanks for listening. Be sure to check out Matt's music. If you like the show, you can subscribe to us on iTunes. Also, please check out all our previous episodes!

https://herchen.com/motern-media-song-search/?sword=

Photos from Switzerland



Announcements

- Midterm 1:
  - We posted on piazza a google form for requesting a room on campus to take midterm 1
  - If you have roommates in the course, you all MUST request
  - Please submit the form by Thursday night (Sep 30th)

- Last time:
  - Computing the determinant
  - Eigen Values and Eigen Vectors of a Matrix
    - Example via page-rank
- Today:
  - More on Eigenvalus, spaces and vectors

Recap

- What have we done in EECS 16A so far?
  1. Set of Equations
  2. Matrix vector multiplication
  3. Gaussian elimination
  4. Span, linear independence
  5. Matrices as transformations
  6. Matrix inversion
  7. Column space, null space
  8. Eigenvalues ; Eigenspace

Eigenvalues and Eigenvectors

$$A\overrightarrow{v} = \lambda\overrightarrow{v}$$

Must be square!

Any number!

Q: What does $\lambda = 0$ mean?

Non-zero!

$\overrightarrow{v} \neq \overrightarrow{0}$

Eigen Values and Eigen Vectors

- Definition: Let $A \in \mathbb{R}^{N \times N}$ be a square matrix, and $^{**}\lambda \in \mathbb{R}$

  if $\exists \overrightarrow{v} \neq \overrightarrow{0}$ such that $A\overrightarrow{v} = \lambda \overrightarrow{v}$,

  then λ is an eigenvalue of A, $\overrightarrow{v}$ is an eigenvector

  and $\mathrm{Null}(A - \lambda I)$ is its eigenspace.

**In general $\lambda \in \mathbb{C}$

Disciplined Approach:

$$A\overrightarrow{v} = \lambda\overrightarrow{v}$$

1. Form $B_\lambda = A - \lambda I$
2. Find all the λs resulting in a non-trivial null space for B_λ
    - Solve: $\det(B_\lambda) = 0$
    - ➡️ Nth order characteristic polynomial with N solutions
    - Each solution is an eigenvalue!
3. For each λ find the vector space $\mathrm{Null}(B_\lambda)$

Solutions for the Characteristic Polynomial

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix}\right) = (a - \lambda)(d - \lambda) - bc = 0$$

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

- Three cases:
  - Two real distinct eigenvalues
  - Single repeated eigenvalue
  - Two complex-valued eigenvalues

Distinct Eigenvalues

- Theorem: Let $A \in \mathbb{R}^{N \times N}$, with M distinct eigenvalues and corresponding eigenvectors $\lambda_i, \overrightarrow{v}_i \mid 1 \leq i \leq M$. It is the case that all $\overrightarrow{v}_i$ are linearly independent. (Proof 9.6.2 in the notes)

- If $A \in \mathbb{R}^{2 \times 2}$ has two distinct eigenvalues, then:
  - $\overrightarrow{v}_1, \overrightarrow{v}_2$ are linearly independent
  - $\text{Span}\{\overrightarrow{v}_1, \overrightarrow{v}_2\} = \mathbb{R}^2$ – form a basis!

Proof 9.6.1 in the notes

Concept: By contradiction. Assume linear dependence→ This results in either $\lambda_1 = \lambda_2$, or $\overrightarrow{v}_2 = \overrightarrow{0}$

Matrix transformations

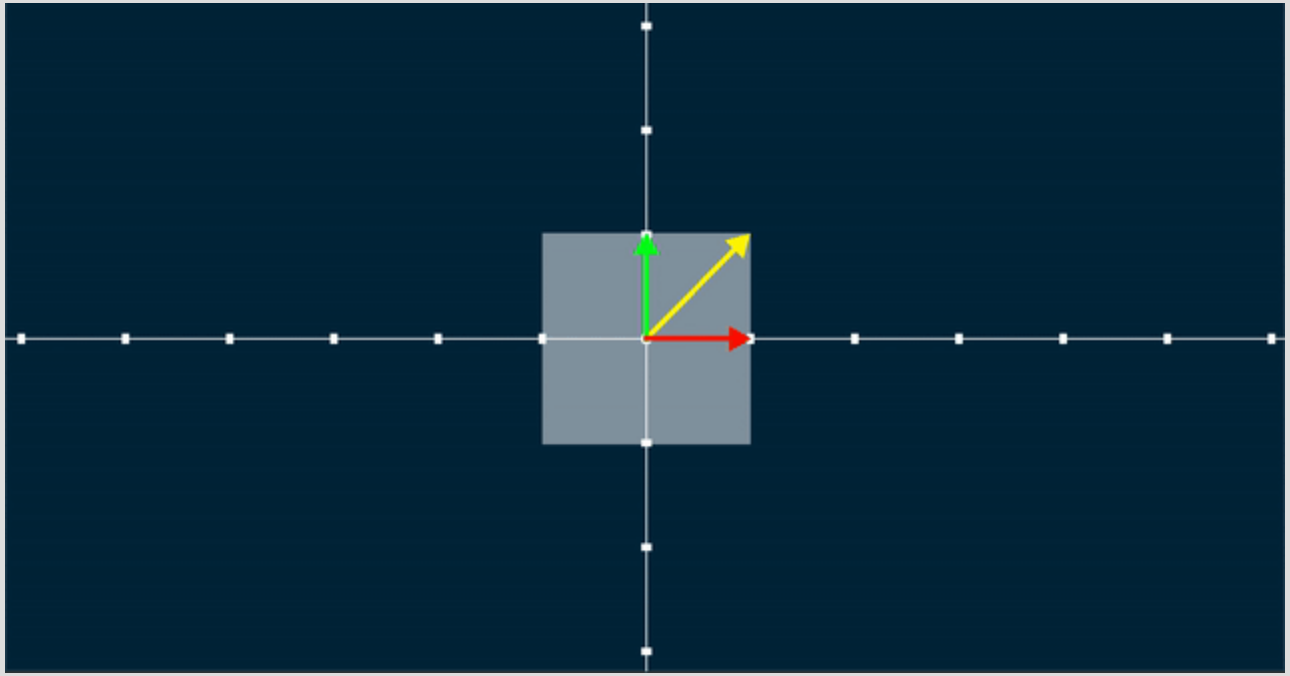
$$A\overrightarrow{v} = \lambda\overrightarrow{v}$$

What does the matrix do?

What is the A matrix?

What are its eigenvectors?

What are its eigenvalues?



Eigen Value Decomposition

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{bmatrix} 1-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} = (1-\lambda)(2-\lambda) - 0 = 0$$

$$\lambda_1 = 1$$

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \vec{v} = 0$$

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} V_2 = 0 \\ V_1 = F.V \end{array}$$

$$\vec{V_1} \in Span\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$

$$\lambda_2 = 2$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \vec{v} = 0$$

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} V_2 = F.V. \\ V_1 = 0 \end{array}$$

$$\vec{V_2} \in Span\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

Matrix transformations

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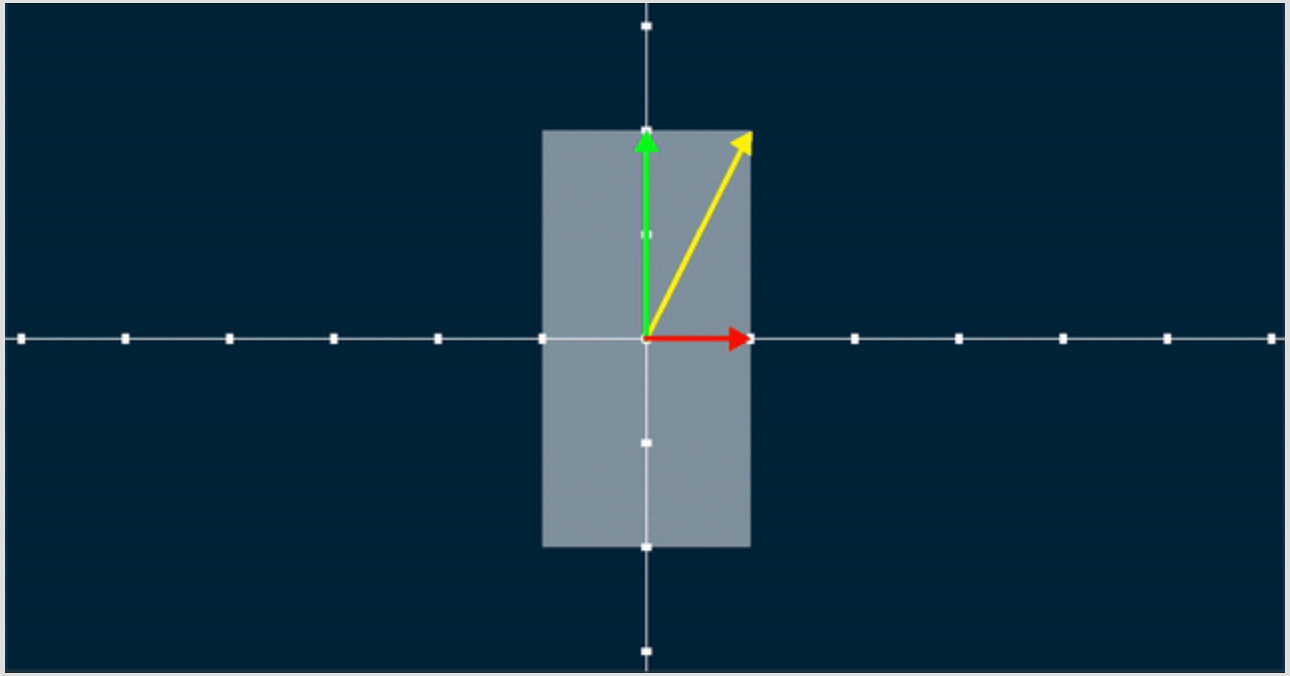

Eigenvectors as a basis

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \lambda_1 = 1,\ \lambda_2 = 2 \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A\vec{v}_1 = 1 \cdot \vec{v}_1 \qquad A\vec{v}_2 = 2 \cdot \vec{v}_2$$

Q: What about $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$?

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2$$

$$\begin{aligned} A\vec{v}_3 = A(1 \cdot \vec{v}_1 + 1 \cdot \vec{v}_2) &= A\vec{v}_1 + A\vec{v}_2 \\ &= \vec{v}_1 + 2\vec{v}_2 \\ &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{aligned}$$

Example from last time:

$$A = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} \qquad \vec{v}_1 \in \mathrm{Span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \qquad \vec{v}_2 \in \mathrm{Span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$\lambda_1 = 1/2 \qquad \lambda_2 = 1$$

- $\vec{v}_1, \vec{v}_2$ are linearly independent → basis for $\mathbb{R}^2$

Q: What about $\vec{v}_3 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$? $\qquad \vec{v}_3 = \alpha \vec{v}_1 + \beta \vec{v}_2$

Example from last time:

$$A = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} \qquad \vec{v}_1 \in \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \qquad \vec{v}_2 \in \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

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$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Longrightarrow$$

Example from last time:

$$A = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} \qquad \vec{v}_1 \in \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \qquad \vec{v}_2 \in \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$\lambda_1 = 1/2 \qquad \lambda_2 = 1$$

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$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ -1 & 1 & 2 \end{array}\right] \Rightarrow$$

Example from last time:

$$A = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} \quad \vec{v}_1 \in \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \quad \vec{v}_2 \in \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$\lambda_1 = 1/2$ $\lambda_2 = 1$

- $\vec{v}_1, \vec{v}_2$ are linearly independent → basis for $\mathbb{R}^2$

Q: What about $\vec{v}_3 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$? $\vec{v}_3 = \alpha \vec{v}_1 + \beta \vec{v}_2$

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ -1 & 1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 4 \end{array}\right] \Rightarrow$$

Example from last time:

$$A = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} \qquad \vec{v}_1 \in \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \qquad \vec{v}_2 \in \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$\lambda_1 = 1/2 \qquad \lambda_2 = 1$$

- $\vec{v}_1, \vec{v}_2$ are linearly independent → basis for $\mathbb{R}^2$

Q: What about $\vec{v}_3 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$? $\qquad \vec{v}_3 = \alpha \vec{v}_1 + \beta \vec{v}_2$

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ -1 & 1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 4 \end{array}\right] \Rightarrow \begin{array}{l} \alpha = 2 \\ \beta = 4 \end{array}$$

Example from last time:

$$A = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1 \end{bmatrix} \quad \vec{v}_1 \in \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \quad \vec{v}_2 \in \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

$$\lambda_1 = 1/2 \qquad \lambda_2 = 1$$

$$\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ -1 & 1 & 2 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 4 \end{array}\right] \Rightarrow \begin{array}{l} \alpha = 2 \\ \beta = 4 \end{array}$$

$$\begin{aligned} A\vec{v}_3 = A(2\vec{v}_1 + 4\vec{v}_2) &= 2A\vec{v}_1 + 4A\vec{v}_2 \\ &= 2\left(\frac{1}{2}\vec{v}_1\right) + 4(1 \cdot \vec{v}_2) \\ &= \vec{v}_1 + 4\vec{v}_2 \\ &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \end{aligned}$$

Matrix transformations

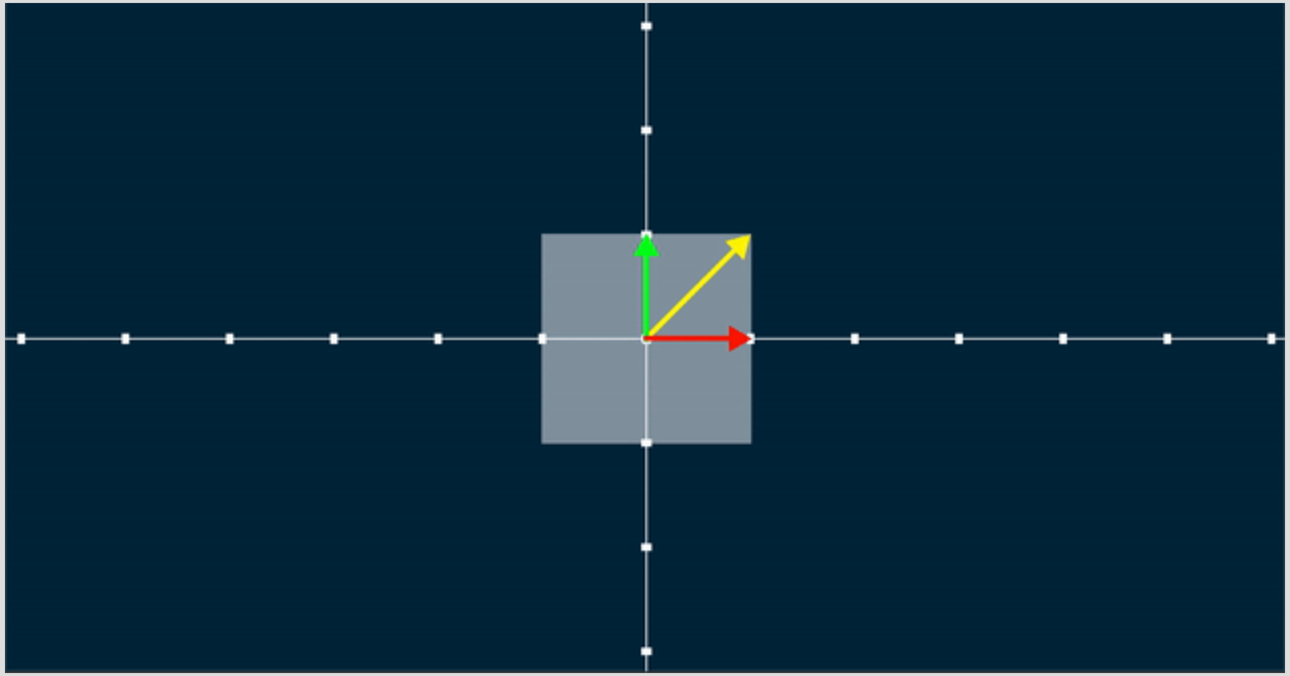
$$A\overrightarrow{v} = \lambda\overrightarrow{v}$$

What does the matrix do?

What is the A matrix?

What are its eigenvectors?

What are its eigenvalues?



Repeated EigenValues

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{bmatrix} 2 - \lambda & 0 \\ 0 & 2 - \lambda \end{bmatrix} = (2 - \lambda)(2 - \lambda) - 0 = 0$$

$$\lambda_{1,2} = 2$$

$$\mathrm{Null}(A - 2I) = \mathrm{Null}(0) = \mathbb{R}^2$$

Eigen space is 2 dimensional!

In general, multiplicity of Eigen-values will result in a multi-dimensional eigenspace

Except is the matrix is defective 😭

Repeated EigenValues

$$A\overrightarrow{v} = \lambda\overrightarrow{v}$$

What does the matrix do?

What is the A matrix?

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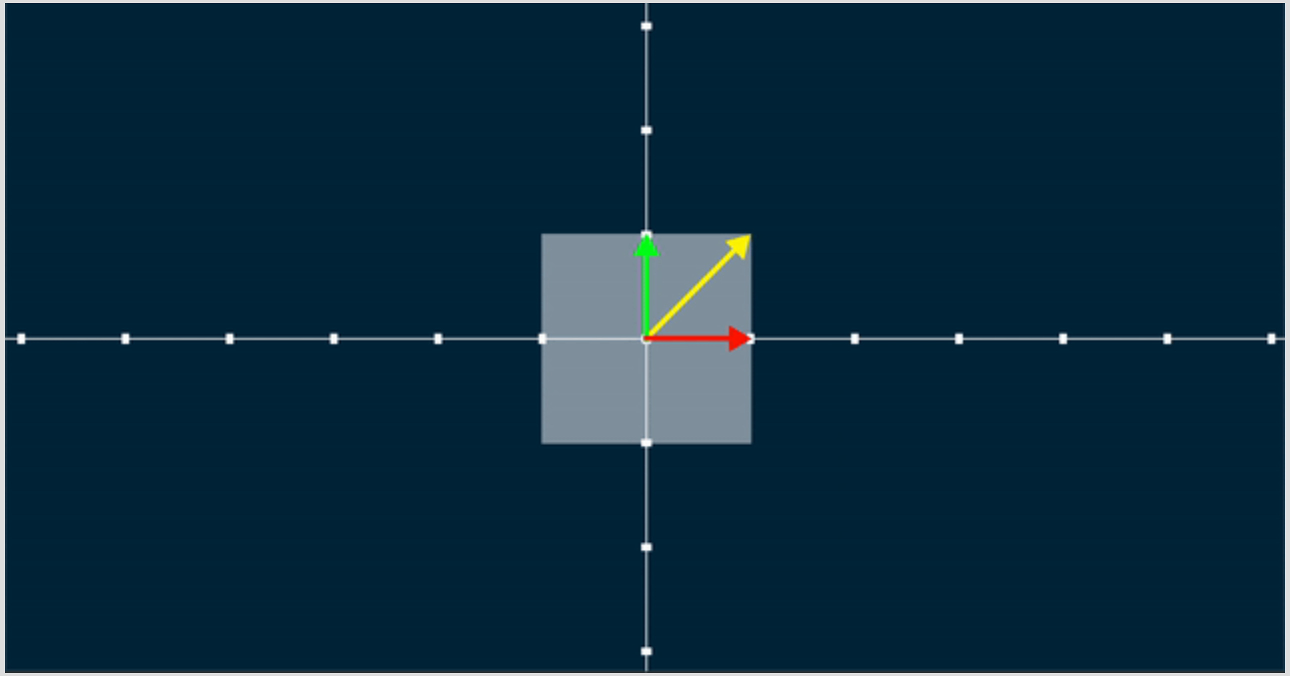


Defective Matrices

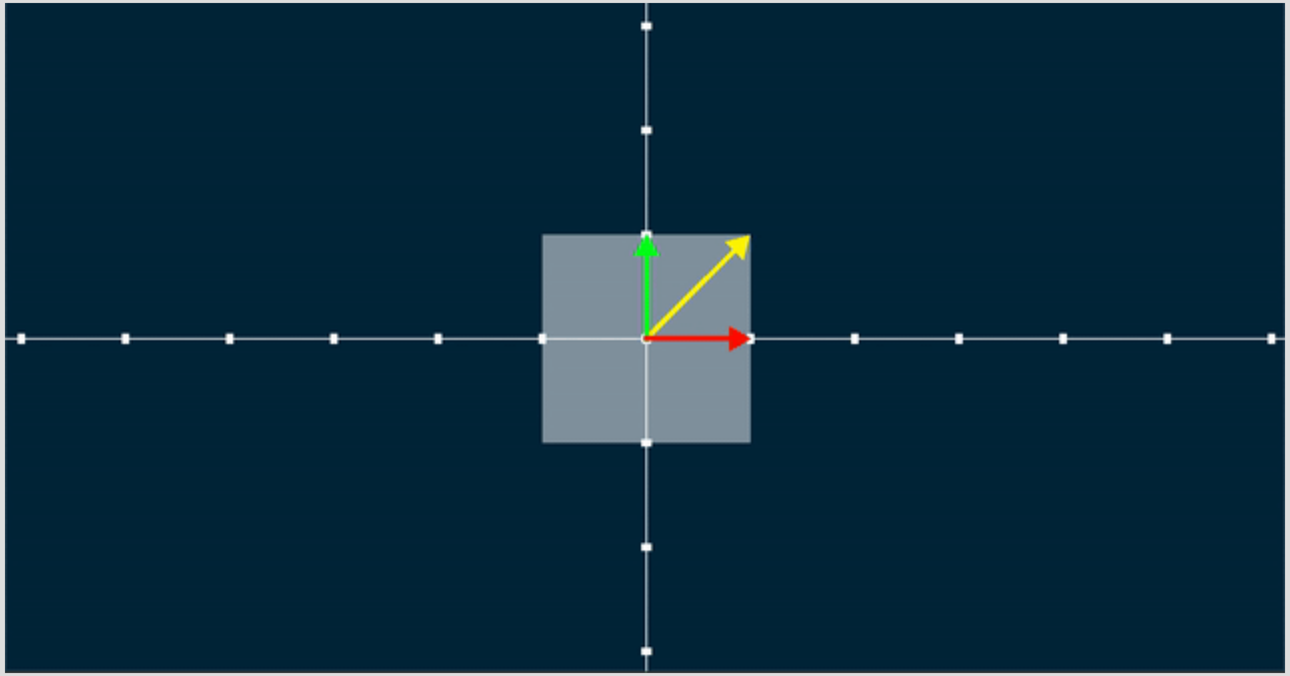
$$A\overrightarrow{v} = \lambda\overrightarrow{v}$$

What does the matrix do?

What is the A matrix?

What are its eigenvectors?

What are its eigenvalues?



Defective Matrix

Outside of class scope 🤨

$$A = \begin{bmatrix} 1 & 1/4 \\ 0 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{bmatrix} 1-\lambda & 1/4 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)(1-\lambda) - 0 = 0$$

$$\lambda_{1,2} = 1$$

$$\text{Null}(A - I) = \text{Null}\left\{ \begin{bmatrix} 0 & 1/4 \\ 0 & 0 \end{bmatrix} \right\}$$

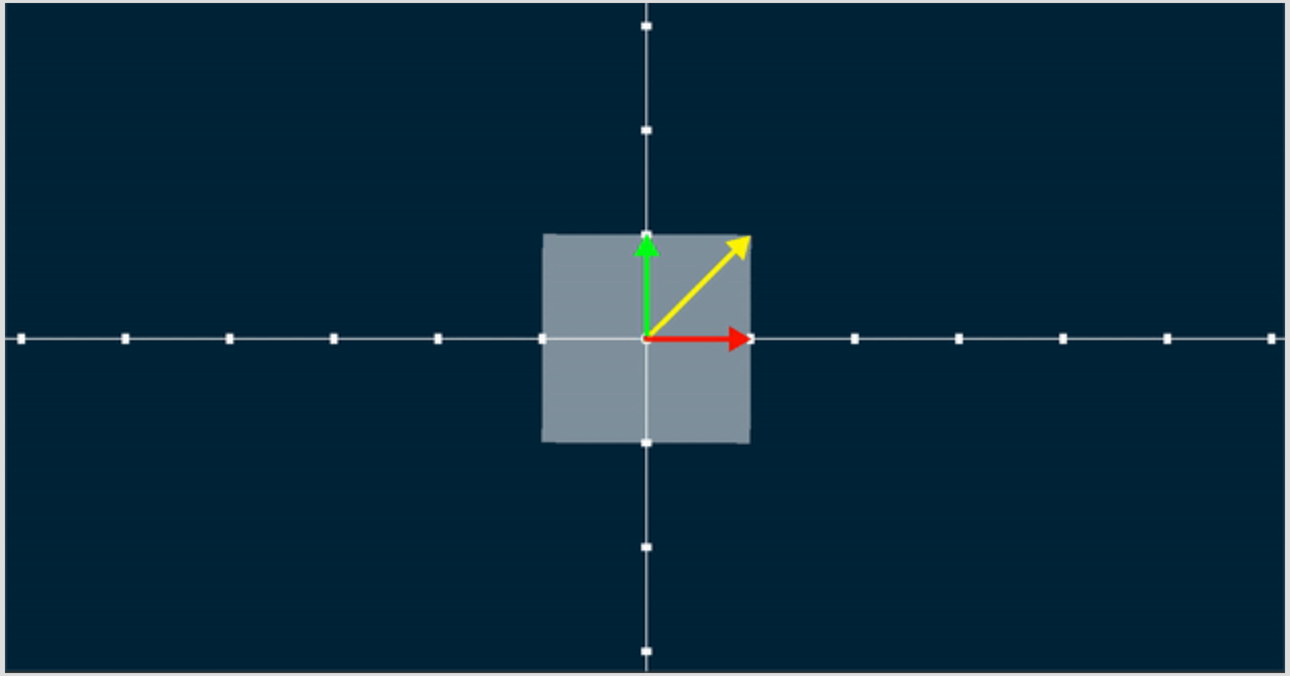
$$\vec{v}_1 \in \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

Eigen space is only 1 dimensional!

Matrix is called defective 😭

Matrix transformations - Complex Eigenvalues

What does the matrix do?



What is the A matrix?

What are its eigenvectors?

What are its eigenvalues?

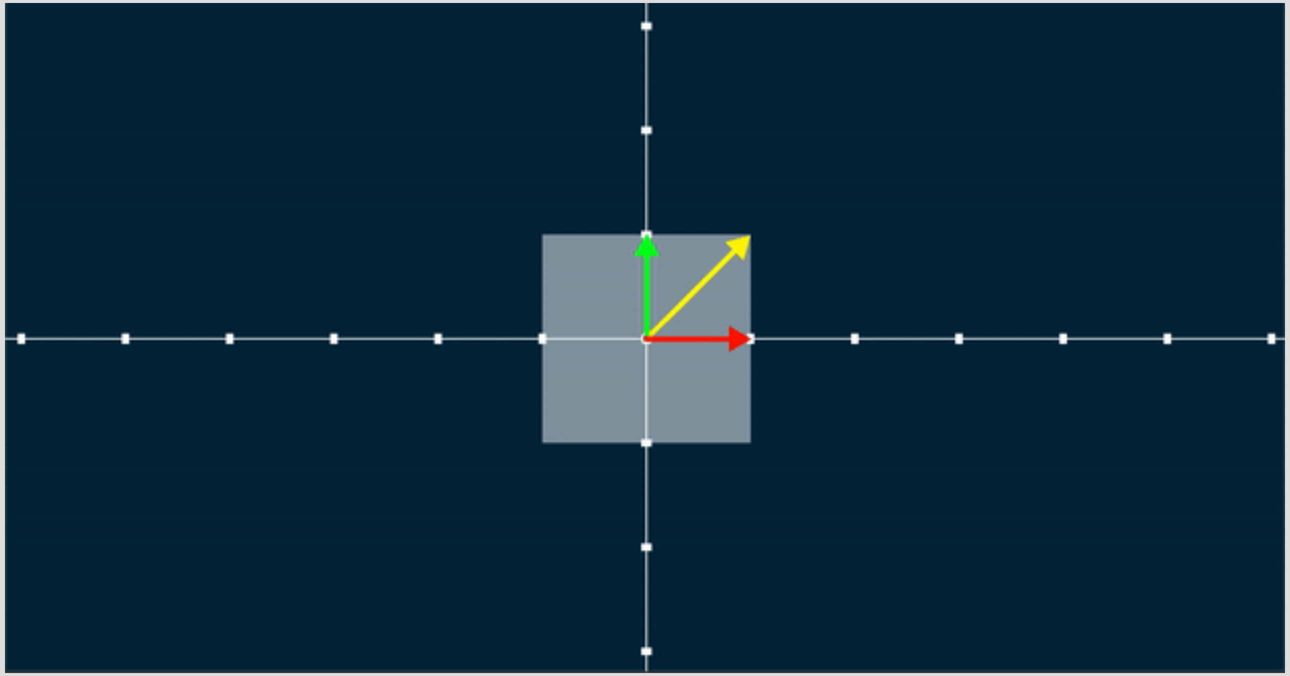
Matrix transformations

What does the matrix do?

What is the A matrix?

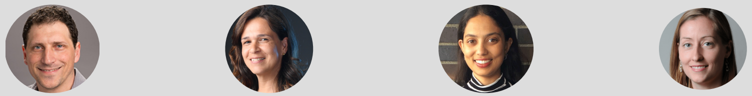
What are its eigenvectors?

What are its eigenvalues?

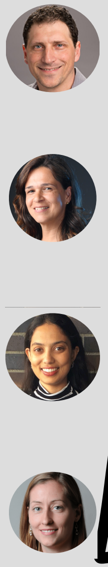


Back 2 PageRank

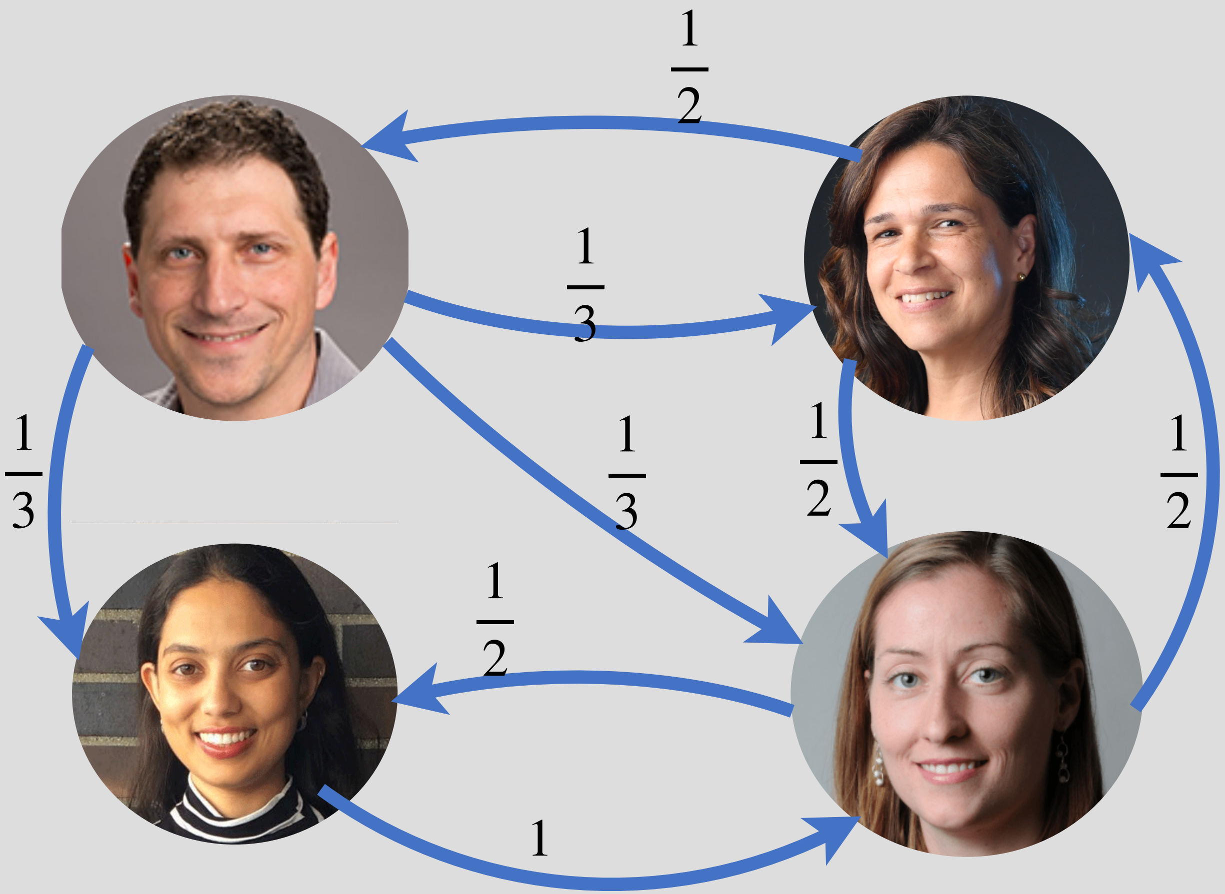
From



To



$$
\begin{bmatrix}
0 & \frac{1}{2} & 0 & 0 \\
\frac{1}{3} & 0 & 0 & \frac{1}{2} \\
\frac{1}{3} & 0 & 0 & \frac{1}{2} \\
\frac{1}{3} & \frac{1}{2} & 1 & 0
\end{bmatrix}
$$

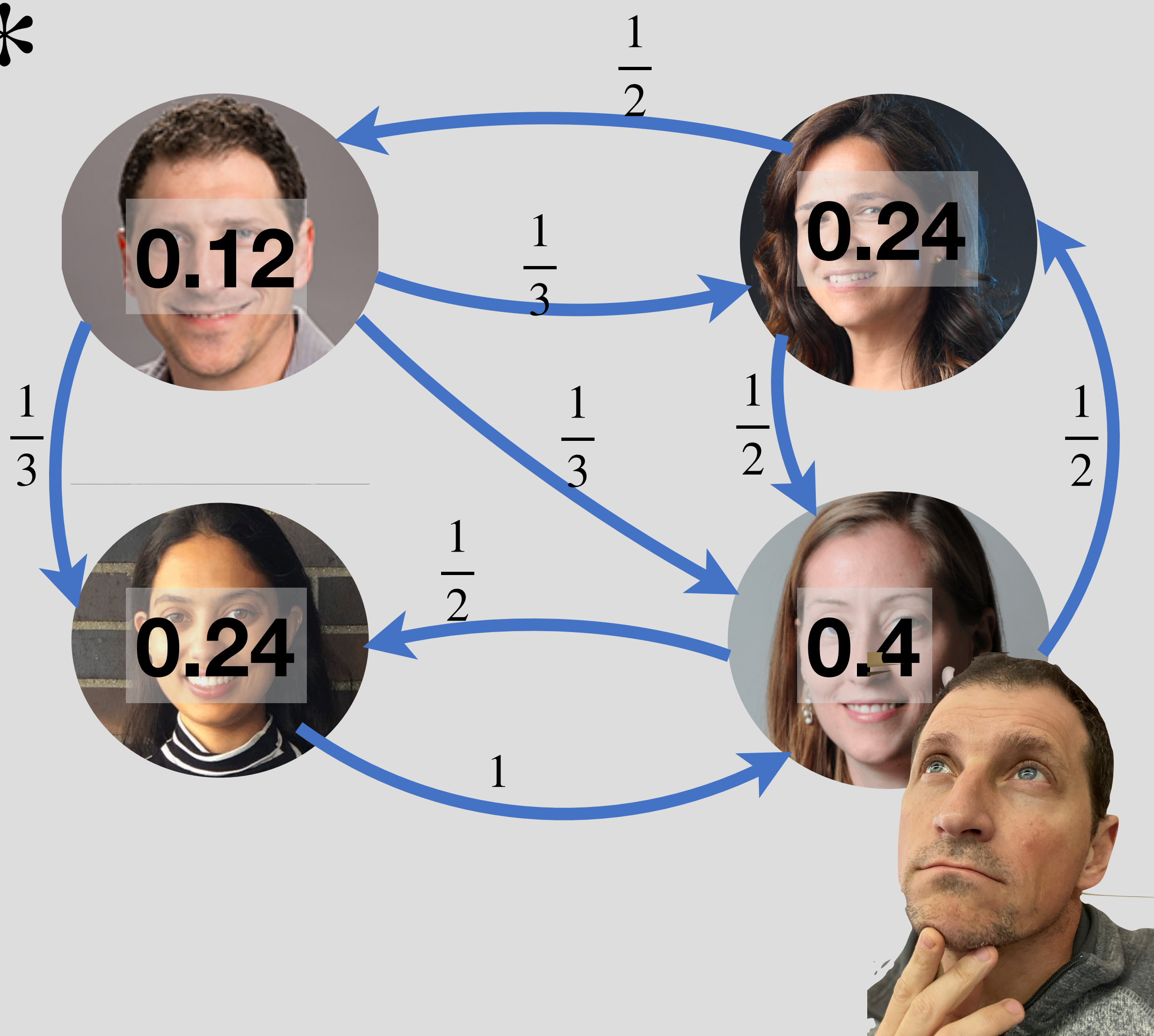


Back 2 Page Rank

$$\overrightarrow{x}^* = A\,\overrightarrow{x}^*$$

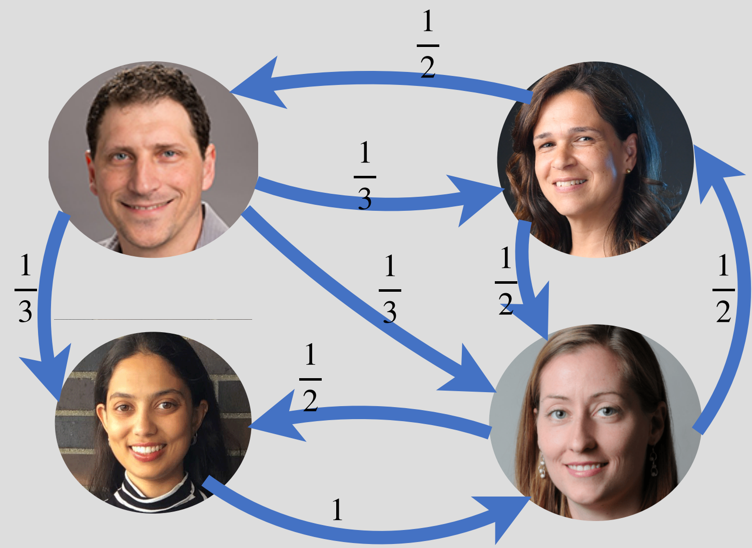
$$\begin{bmatrix} 0.12 \\ 0.24 \\ 0.24 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.12 \\ 0.24 \\ 0.24 \\ 0.4 \end{bmatrix}$$

steady state!



Back 2 PageRank

$$\vec{x}(t+1) = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 1 & 0 \end{bmatrix} \vec{x}(t)$$



$\vec{x}(t) \Rightarrow$ Page ranking

$$\vec{x}(0) = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$$

$\Downarrow$

equal Ranking

$t=1$: $\begin{bmatrix} 0.125 \\ 0.208 \\ 0.208 \\ 0.458 \end{bmatrix}$ $t=2$: $\begin{bmatrix} 0.104 \\ 0.271 \\ 0.271 \\ 0.354 \end{bmatrix}$ $t=3$: $\begin{bmatrix} 0.135 \\ 0.212 \\ 0.212 \\ 0.441 \end{bmatrix}$ ... $t=100$: $\begin{bmatrix} 0.12 \\ 0.24 \\ 0.24 \\ 0.4 \end{bmatrix}$

General Initialization for a Transition Matrix System

$$\overrightarrow{x}(t+1) = A\overrightarrow{x}(t)$$

Assume $\lambda_i \,|\, 1 \le i \le N$ are distinct $\Rightarrow \text{Span}\{\overrightarrow{v}_i \,|\, 1 \le i \le N\} = \mathbb{R}^N$

$$\begin{aligned}
\overrightarrow{x}(1) &= A\overrightarrow{x}(0) \\
&= A(\alpha_1\overrightarrow{v}_1 + \alpha_2\overrightarrow{v}_2 + \cdots + \alpha_N\overrightarrow{v}_N) \\
&= \alpha_1 A\overrightarrow{v}_1 + \alpha_2 A\overrightarrow{v}_2 + \cdots + \alpha_N A\overrightarrow{v}_N \\
&= \alpha_1\lambda_1\overrightarrow{v}_1 + \alpha_2\lambda_2\overrightarrow{v}_2 + \cdots + \alpha_N\lambda_N\overrightarrow{v}_N \\
\overrightarrow{x}(2) &= A\overrightarrow{x}(1) \\
&= A(\alpha_1\lambda_1\overrightarrow{v}_1 + \alpha_2\lambda_2\overrightarrow{v}_2 + \cdots + \alpha_N\lambda_N\overrightarrow{v}_N) \\
&= \alpha_1\lambda_1^2\overrightarrow{v}_1 + \alpha_2\lambda_2^2\overrightarrow{v}_2 + \cdots + \alpha_N\lambda_N^2\overrightarrow{v}_N
\end{aligned}$$

General Initialization for a Transition Matrix System

$$\overrightarrow{x}(t+1) = A\overrightarrow{x}(t)$$

Assume $\lambda_i \,|\, 1 \leq i \leq N$ are distinct $\Rightarrow \mathrm{Span}\{\overrightarrow{v}_i \,|\, 1 \leq i \leq N\} = \mathbb{R}^N$

$$\begin{aligned}\overrightarrow{x}(2) &= A\overrightarrow{x}(1)\\ &= A(\alpha_1\lambda_1\overrightarrow{v}_1 + \alpha_2\lambda_2\overrightarrow{v}_2 + \cdots + \alpha_N\lambda_N\overrightarrow{v}_N)\\ &= \alpha_1\lambda_1^2\overrightarrow{v}_1 + \alpha_2\lambda_2^2\overrightarrow{v}_2 + \cdots + \alpha_N\lambda_N^2\overrightarrow{v}_N\end{aligned}$$

$$\vdots$$

$$\overrightarrow{x}(t) = \alpha_1\lambda_1^t\overrightarrow{v}_1 + \alpha_2\lambda_2^t\overrightarrow{v}_2 + \cdots + \alpha_N\lambda_N^t\overrightarrow{v}_N$$

$$\lim_{t\to\infty} \overrightarrow{x}(t) = ?$$

$$\lim_{t\to\infty} \lambda_i^t = \begin{cases} 0 & , \quad |\lambda| < 1 \\ 1 & , \quad \lambda = 1 \\ (-1)^t & , \quad \lambda = -1 \\ \infty & , \quad |\lambda| > 1 \end{cases}$$

Back 2 PageRank

$$A = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1/2 & 1 & 0 \end{bmatrix}$$

$$\lambda_1 = 1 \qquad \lambda_2 = -0.092 \qquad \lambda_3 = -0.91 \qquad \lambda_4 = 0$$

$$\vec{v}_1 = \begin{bmatrix} 0.12 \\ 0.24 \\ 0.24 \\ 0.4 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 0.44 \\ -0.08 \\ -0.08 \\ -0.28 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} -0.14 \\ 0.26 \\ 0.26 \\ -0.37 \end{bmatrix} \qquad \vec{v}_4 = \begin{bmatrix} 0.43 \\ 0 \\ -0.14 \\ -0.29 \end{bmatrix}$$

$$\overrightarrow{x}(t) = A^t \overrightarrow{x}(0)$$

Back 2 PageRank

$\lambda_1 = 1 \qquad \lambda_2 = -0.092 \qquad \lambda_3 = -0.091 \qquad \lambda_4 = 0$

$$\vec{v}_1 = \begin{bmatrix} 0.12 \\ 0.24 \\ 0.24 \\ 0.4 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 0.44 \\ -0.08 \\ -0.08 \\ -0.28 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} -0.14 \\ 0.26 \\ 0.26 \\ -0.37 \end{bmatrix} \qquad \vec{v}_4 = \begin{bmatrix} 0.43 \\ 0 \\ -0.14 \\ -0.29 \end{bmatrix}$$

$$\vec{x}_0 = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.25 \\ 0.25 \end{bmatrix} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \alpha_3 \vec{v}_3 + \alpha_4 \vec{v}_4$$

$$\underbrace{\begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}}_{\vec{\alpha}} = \underbrace{\begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}}_{x_0}$$

$V\vec{\alpha} = \vec{x}_0$

(1) Solve via G.E

(2) $\vec{\alpha} = V^{-1}\vec{x}_0$

$$\vec{\alpha} = \begin{bmatrix} 1 \\ 0.34 \\ 0.15 \\ 0 \end{bmatrix}$$

Back 2 PageRank

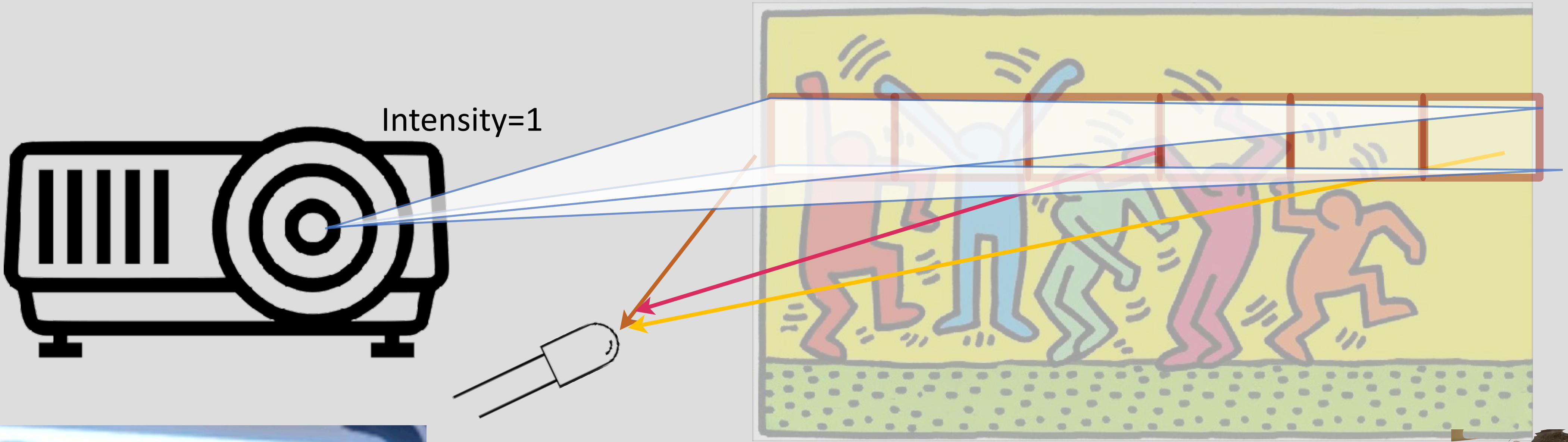
$$\overrightarrow{x}(t) = A^t \overrightarrow{x}(0)$$

$$\begin{aligned} A^t \overrightarrow{x}(0) &= A(1\overrightarrow{v}_1 + 0.34\overrightarrow{v}_2 + 0.15\overrightarrow{v}_3 + 0\overrightarrow{v}_4) \\ &= 1 \cdot 1^t \overrightarrow{v}_1 + 0.34(-0.092)^t \overrightarrow{v}_2 + 0.15(-0.91)^t \overrightarrow{v}_3 + 0 \cdot 0^t \overrightarrow{v}_4 \end{aligned}$$

$$\lim_{t \to \infty} A^t \overrightarrow{x}(0) = \overrightarrow{v}_1$$

Back to Lab — Single Pixel Camera

- What are the best patterns?



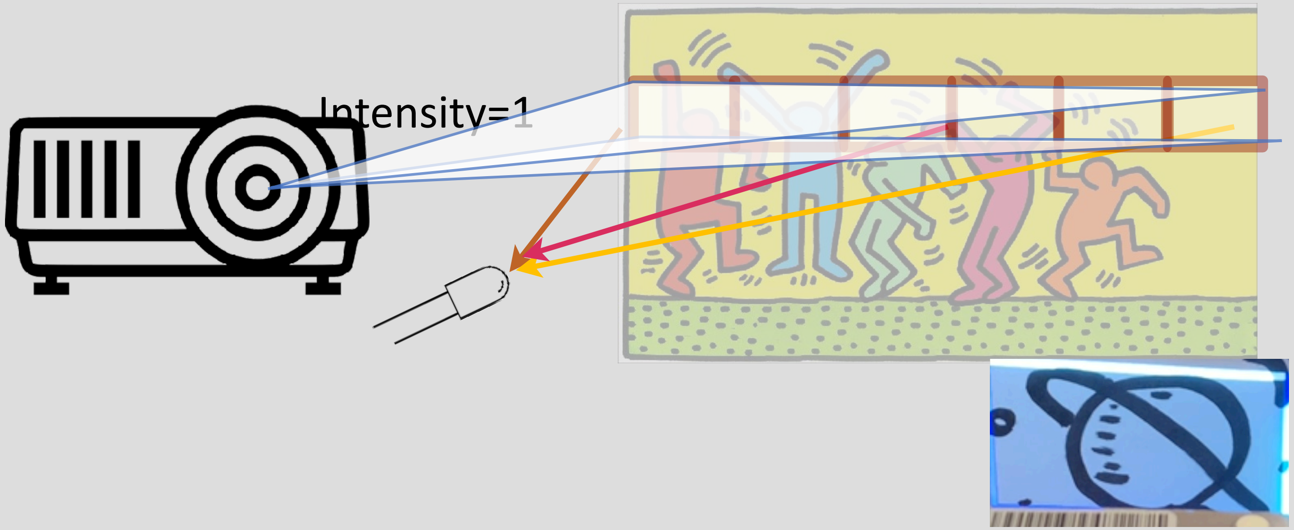
Imaging Model and Reconstruction

Measurement mask/matrix

$$\vec{y} = A\,\vec{x}$$

Sensor data

Unknown image



We saw that it is possible to come up with a system that has A^{-1}

So, $$\vec{x} = A^{-1}\vec{y}$$

Non-ideal imaging

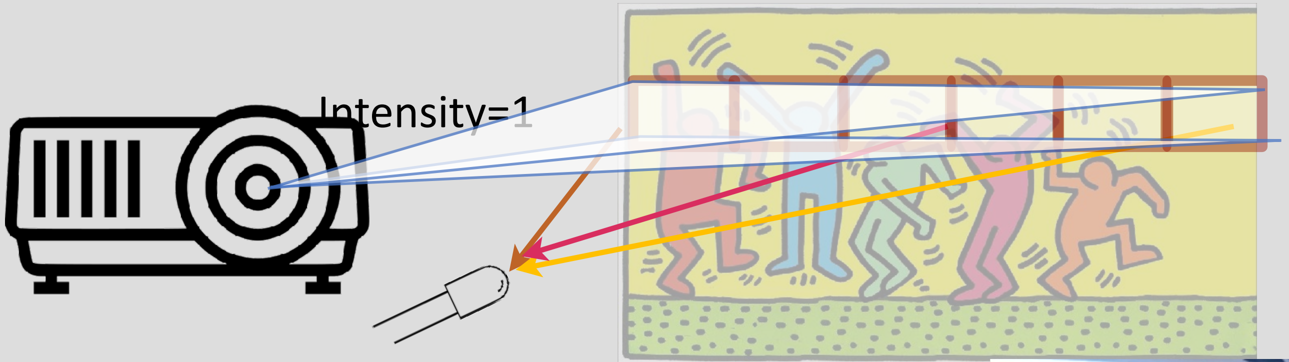
Measurement mask/matrix

$$\vec{y} = A\vec{x} + \vec{w}$$

Sensor data

Unknown image

Noise/interference



We saw that it is possible to come up with a system that has A^{-1}

So,
$$\vec{x} = A^{-1}\vec{y} - A^{-1}\vec{w}$$

Reconstruction error

$$A^{-1}\vec{w} = \alpha_1\lambda_1\vec{v}_1 + \alpha_2\lambda_2\vec{v}_2 + \cdots + \alpha_N\lambda_N\vec{v}_N$$

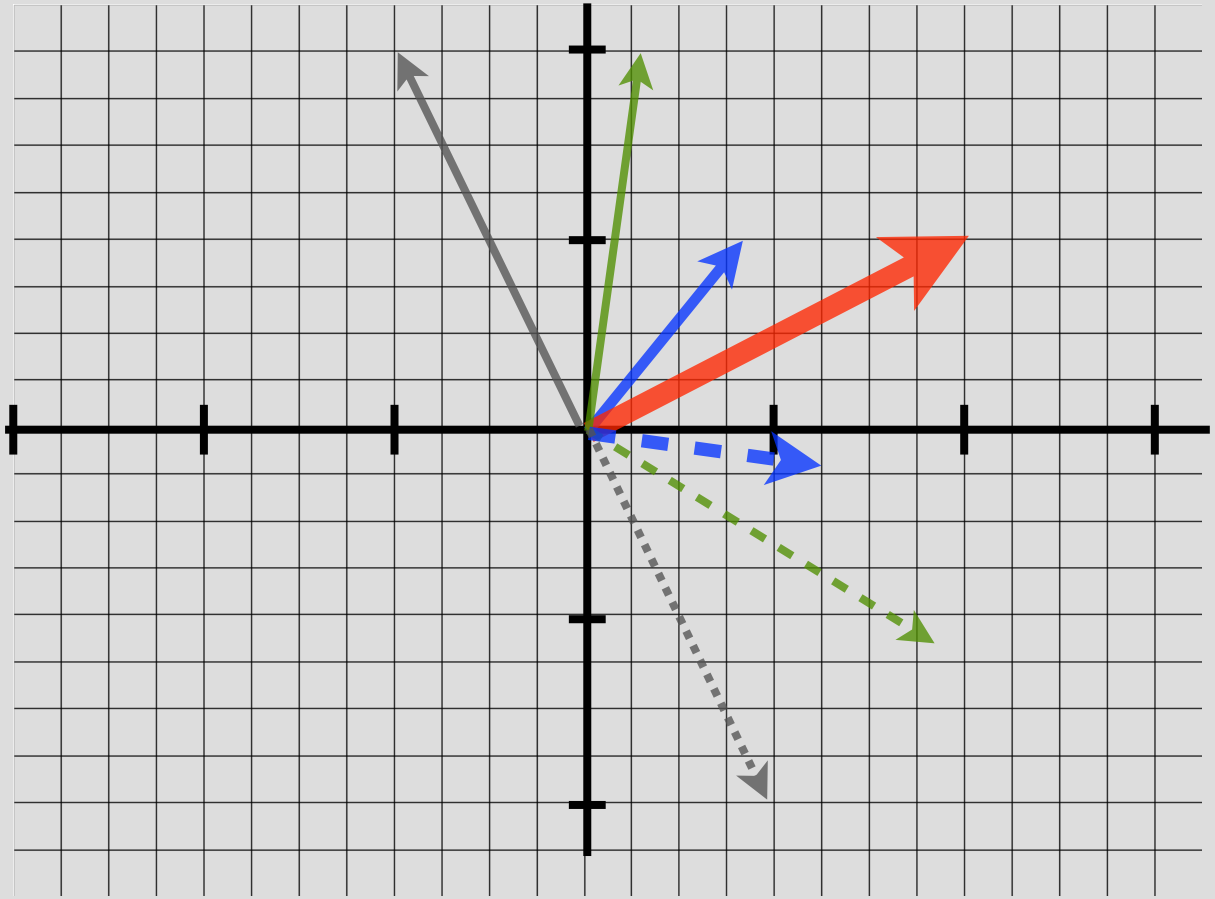
Want to design A, such that A^{-1} has small eigenvalues!

Design of a Reflection matrix

Design a reflection matrix around the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$?

Q: What are the eigenvectors?

A: $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$



Q: What are the eigenvalues?

A: $\lambda_1 = 1, \lambda_2 = -1$

Designing a matrix with specific Eigenvals/vecs

We know:

$$A\overrightarrow{v} = \lambda\overrightarrow{v}$$

$$\overrightarrow{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \overrightarrow{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\lambda_1 = 1, \lambda_2 = -1$$

Set linear equations:

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \cancel{-}\begin{bmatrix} +1 \\ -2 \end{bmatrix}$$

unknown

$$\begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

Designing a matrix with specific Eigenvals/vecs

We know:

$$A\vec{v} = \lambda\vec{v}$$

$$\vec{v}_1 = \begin{bmatrix}2\\1\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}-1\\2\end{bmatrix}$$

$$\lambda_1 = 1, \lambda_2 = -1$$

Set linear equations:

$$\begin{bmatrix}a_1 & a_2\\a_3 & a_4\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}2\\1\end{bmatrix}$$

$$\begin{bmatrix}a_1 & a_2\\a_3 & a_4\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix} = \cancel{-}\begin{bmatrix}+1\\-2\end{bmatrix}$$

unknown

$$\begin{bmatrix}2 & 1 & 0 & 0\\0 & 0 & 2 & 1\\-1 & 2 & 0 & 0\\0 & 0 & -1 & 2\end{bmatrix}\begin{bmatrix}a_1\\a_2\\a_3\\a_4\end{bmatrix} = \begin{bmatrix}2\\1\\1\\-2\end{bmatrix}$$

G.E. $\Rightarrow$ $A = \begin{bmatrix}0.6 & 0.8\\0.8 & -0.6\end{bmatrix}$