EECS 16A   Designing Information Devices and Systems I
Fall 2018   Final Exam

Final Exam Solution

PRINT your student ID: _____

PRINT AND SIGN your name: _____, _____ _____
                              (last name)   (first name)   (signature)

PRINT your discussion section and GSI(s) (the one you attend): _____

Name and SID of the person to your left: _____

Name and SID of the person to your right: _____

Name and SID of the person in front of you: _____

Name and SID of the person behind you: _____

1. What is one of your favorite things to do in Berkeley? (1 Point)

2. What are you looking forward to over the winter break? (1 Point)

Do not turn this page until the proctor tells you to do so. You may work on the questions above.

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3. Death Star Destruction (10 points)

(a) (2 points) Oh no! Darth Vader has just completed the Death Star for Emperor Palpatine, and they're getting ready to destroy Alderaan. The Death Star destroys planets by shooting a beam of charged particles at its target – this evil beam can be modeled as a stream of electrons fired from the reflector dish of the Death Star. You have been taken prisoner and forced to help the Empire design and operate the space station.

Darth Vader uses 3 identical capacitors to charge up his planet-destroying beam, but he wants to maximize the equivalent capacitance. How can you arrange three identical capacitors, each with capacitance C, to maximize the equivalent capacitance? **Draw the circuit, and write down the equivalent capacitance $C_{network}$ in terms of C.**
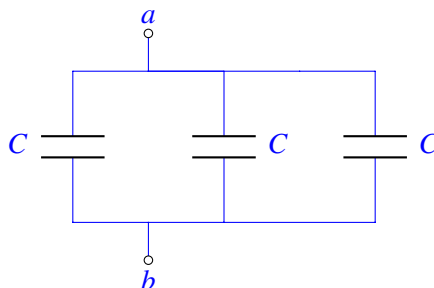
Solution: With 3 identical capacitors, there are only 4 distinct network arrangements. Here, we will only analyze two configurations (all capacitors in series, and all in parallel) – the other 2 are left as an exercise for the reader.

All in series:
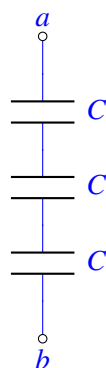
$$C_{series} = (C||C)||C = \left(\frac{C^2}{C+C}\right)||C = \left(\frac{C}{2}\right)||C = \frac{\frac{C^2}{2}}{\frac{C}{2}+C} = \frac{\frac{C^2}{2}}{\frac{3C}{2}} = \frac{C}{3}$$

All in parallel:

$$C_{parallel} = C + C + C = 3C$$

Clearly, $C_{parallel} = 3C > \frac{C}{3} = C_{series}$. In fact, $C_{parallel}$, of the 4 possible configurations, will give the maximum capacitance, and so is our desired layout. (Note that the question does not ask you to justify your choice, so full credit is awarded for writing down the parallel configuration and correct capacitance.) The total capacitance is $3C$.

(b) (2 points) Assume the equivalent capacitance of Darth Vader's arrangement is $C_{network} = 1$kF. (This is a lot – for reference, the capacitance of the Earth is just 700µF!) If 4.5×10^{15}J are needed to destroy Alderaan, **to what voltage** will you have to charge your capacitor system ?

Solution: Using

$$E = \tfrac{1}{2}C_{network}V^2,$$
$$V = \sqrt{\tfrac{2E}{C_{network}}}.$$

In this case, $C_{network} = 1kF = 1000F$. Thus,

$$V = 3 \times 10^6 V.$$

(c) (2 points) Huzzah! You are rescued by the rebels, who need your help to decipher the plans of the Death Star that Princess Leia has helped recover.

With your help, the rebels realize there is one weakness in the design of the Death Star – if they can overload a port on the surface of the Death Star, the entire space station will explode! (Whoever drew up this Death Star design must have gone to Stanford...) Luke Skywalker's X-wing fires its charged beam from this capacitor network, where each capacitor has capacitance C_L:
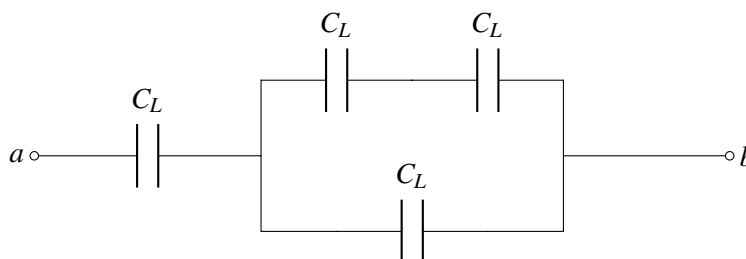

Figure 3.1: Figure for part (c)

What is the total capacitance of this network between nodes a and b, in terms of C_L? Simplify the expression.

Solution:

$$C_{network} = [(C_L||C_L) + C_L]||C_L = \tfrac{3}{5}C_L$$

(d) (4 points) The port will overload if hit by more than 25kJ of energy. Assume that Luke's X-wing capacitor system from part (c) has 1kF of total capacitance, and that it is charged to 10V. If Luke's X-wing is hit during the raid, and his damaged X-wing can only discharge its capacitor system from 10V to 5V, **will Luke be able to overload the port if he hits it?**

Solution: We know that
$$E = \tfrac{1}{2}CV^2 = \tfrac{1}{2}QV.$$

Because the X-wing can only discharge from 10 V to 5 V,
$$E_{10V,initial} = \tfrac{1}{2} * 1000F * (10V)^2 = 50kJ$$
$$E_{5V,final} = \tfrac{1}{2} * 1000F * (5V)^2 = 12.5kJ$$
$$E_{10V,initial} - E_{5V,final} = E_{discharge} = 50kJ - 12.5kJ = 37.5kJ$$
$$E_{discharge} = 37.5kJ > 25kJ = E_{overload}$$

Yes, Luke can overload the port!

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4. (25 points) A Tool to Help Compute All the Fun You're Having

Starting in the 1950-1960's, the world began a series of missions to get to the moon. Back then though, computers took up entire rooms and could never fit on a spaceship! They needed a better way of computing values on the fly using what they did have: analog circuits.

In class you have seen circuits that can amplify, add, subtract and even integrate voltages, but we're missing a key ingredient to make computational circuits: **multiplication**. Although making a multiplier circuit is not as straightforward as we would like it to be, we can definitely use our now fully-developed EE16A skills to make this a reality.

You may find the following formulas useful throughout the problem:

$$\ln(e^a) = a \qquad\qquad \ln(a) + \ln(b) = \ln(ab)$$
$$e^{\ln(a)} = a \qquad\qquad \ln(a) - \ln(b) = \ln\left(\tfrac{a}{b}\right)$$

(a) (4 points) To start off, your TA Nick suggests that you first draw a block diagram that would do what you want, and then worry about how to implement it later. He starts you off with the following incomplete block diagram and blocks.

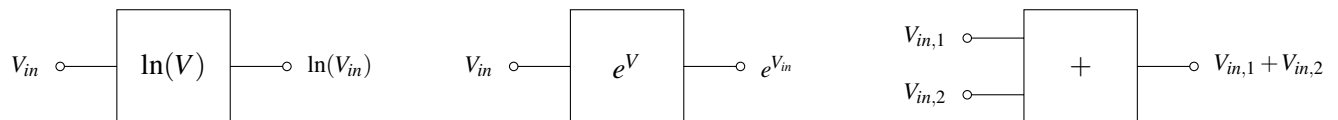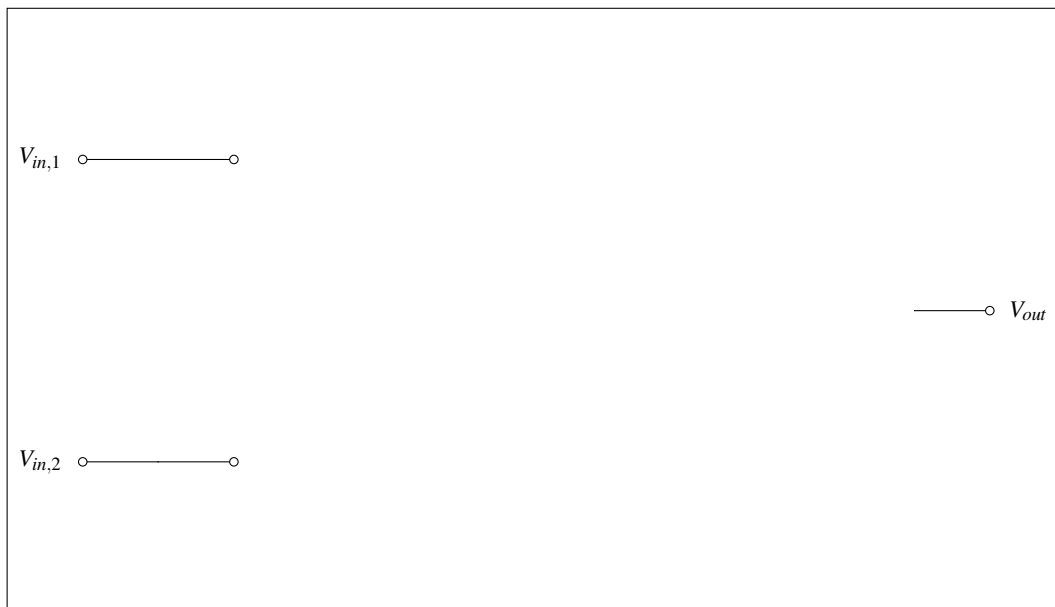You are allowed to use any amount of the following logarithmic, exponential and summer blocks:

V_{in} ○— $\boxed{\ln(V)}$ —○ $\ln(V_{in})$ V_{in} ○— $\boxed{e^V}$ —○ $e^{V_{in}}$ $V_{in,1}$ ○— $\boxed{+}$ —○ $V_{in,1} + V_{in,2}$
$V_{in,2}$ ○—

Figure 4.1: Function blocks you can use

In the box below, finish the **block diagram** so that the output is equal to $V_{out} = V_{in,1} V_{in,2}$. **Also provide a mathematical justification for why your block diagram works for full credit!**

$V_{in,1}$ ○———○

———○ V_{out}

$V_{in,2}$ ○———○

Solution: We want to use the hints at the beginning of the problem. In order to get two voltages multiplied together, we need to get them both inside the same logarithm, then cancel the logarithm by using an exponential block. See the diagram below:
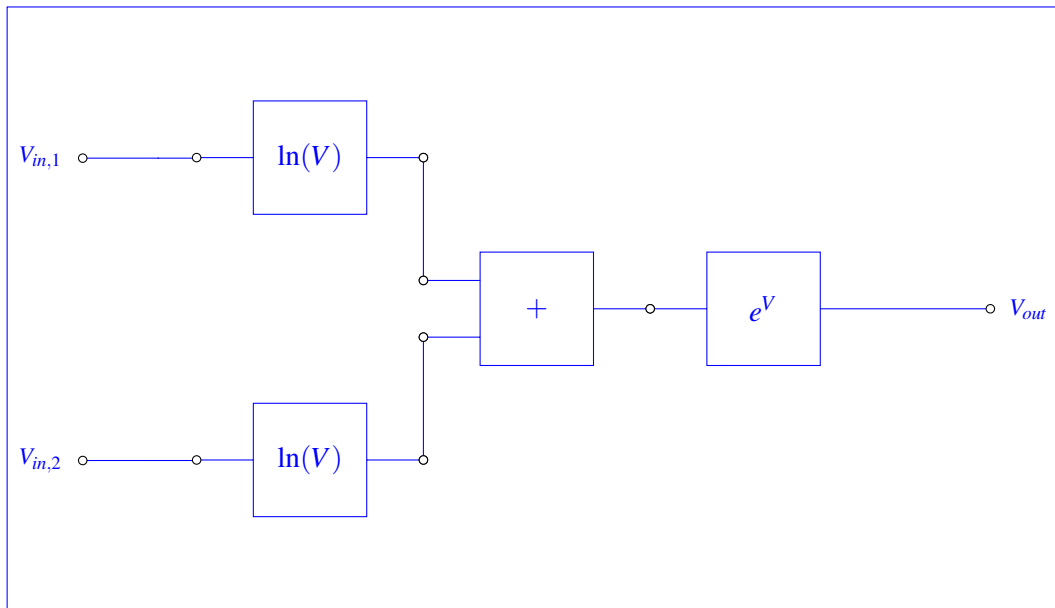
the voltages going into the summer are:

$$\ln(V_{in,1})$$

and,

$$\ln(V_{in,2})$$

The output of the summer is therefore:

$$\ln(V_{in,1}) + \ln(V_{in,2})$$

Using the fact that $\ln(a) + \ln(b) = \ln(ab)$ simplifies this expression into

$$\ln(V_{in,1}V_{in,2})$$

And lastly, passing through the e block gives:

$$e^{\ln(V_{in,1}V_{in,2})}$$

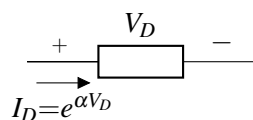Using the fact that $e^{\ln(a)} = a$ gives:

$$V_{in,1}V_{in,2}$$

as desired.

We made a multiplier circuit! Go us!

While you were making your block diagram, your friend Vlad was watching over your shoulder. With his near-infinite knowledge of circuits, he predicts what you are trying to do and gives you a "magic device" that he claims will help you out.

This device is shown below: This device has the following IV characteristics:
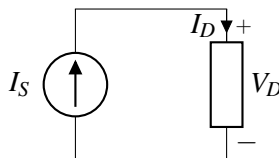

Mystery Device IV Characteristics
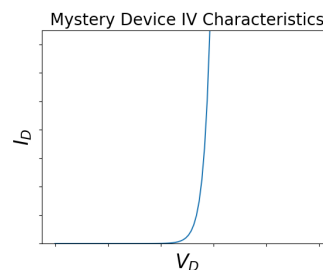
Numerically, this can be stated as:

$$I_D = e^{\alpha V_D}$$

where α is some constant and V_D is the voltage across the device as in the above figure.

IMPORTANT: Note the passive sign convention.

(b) (3 points) Your friend Gireeja sees you looking confused, and reads what you've done so far. She decides to help push you in the right direction. She reminds you that if you know how a device reacts when you put current through it or apply a voltage across it, then you are in a good spot to understand how it would behave in any circuit.

   i. (2 points) **In the following circuit, express the voltage across the device, V_D, as a function of the source current, I_S.**

   ii. (1 point) **In the following circuit, express the current through the device, I_D, as a function of the source voltage, V_S.**
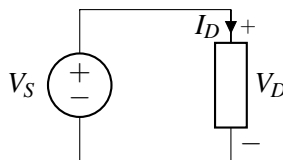
Solution:

i. From the above IV characteristics we know that:

$$I_S = I_D = e^{\alpha V_D}$$
$$\rightarrow \ln(I_S) = \alpha V_D$$
$$\rightarrow V_D = \frac{1}{\alpha} \ln(I_S)$$

Putting a fixed current through the device causes a voltage to appear across the device that is related to the natural log of the current.

ii. This follows directly from the given IV relationship above $V_D = V_S$ therefore,
$$I_D = e^{\alpha V_S}$$

(c) (6 point) By now we have all the tools necessary to make this a reality. Unfortunately, as you may remember from the touchscreen lab, ideal current sources don't exist. As such, we will use an op-amp to create the behavior we are interested in. Let's analyze the circuit below in steps.

Hint: *Your answers for some parts of this problem will not perfectly match with the ideal scenario in part (a). Don't be afraid of getting different answers! After all, a block diagram is just a starting point.*
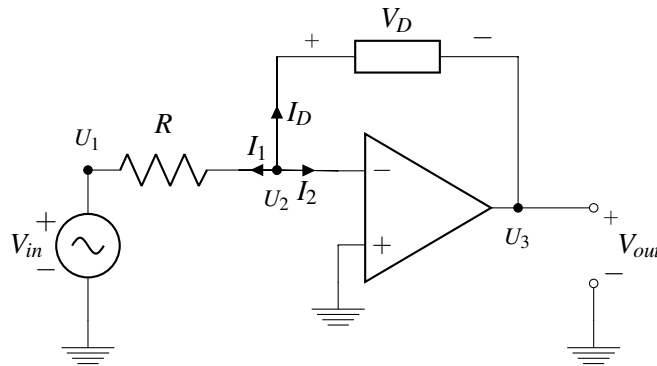
i. (1 point) **write V_D in terms of V_{out} only. You may assume the circuit is in negative feedback.**
Solution: We know the following three things:

$$V_D = U_2 - U_3$$
$$V_2 = U^- - 0$$
$$V_{out} = U_3 - 0$$

Since we can assume negative feedback, then we know that $U^+ = U^-$. Since $U^+ = 0$, then $U^- = 0$. Substituting into the above node equations,

$$V_D = U^- - U_3$$
$$V_D = 0 - U_{out}$$
$$\rightarrow V_D = -V_{out}$$

ii. (1 point) **Now write out the KCL expression at the U_2 node in terms of the three currents I_D, I_1 and I_2.**
Solution: By KCL:

$$I_1 + I_D + I_2 = 0$$

iii. (4 points) **Use the IV relations for each device and the KCL expression above to find V_{out} as a function of V_{in}.**
Solution: We immediately know that $I_2 = 0$ from the golden rules. Substituting $V = IR$ for I_1 and the mystery device IV relationship for I_D gives:

$$\frac{U_2 - U_1}{R} + e^{\alpha(U_2 - U_3)} + 0 = 0$$

We know $U_2 = U^- = U^+ = 0$ from the golden rules. We also know $U_1 = V_{in}$. From part (i) we also know that $U_2 - U_3 = -V_{out}$, thus

$$\frac{0 - V_{in}}{R} + e^{-\alpha V_{out}} = 0$$

$$e^{-\alpha V_{out}} = \frac{V_{in}}{R}$$

$$-\alpha V_{out} = \ln\left(\frac{V_{in}}{R}\right)$$

$$V_{out} = -\frac{1}{\alpha}\ln\left(\frac{V_{in}}{R}\right)$$

(d) (6 points) Just one more piece left to analyze. You may assume the circuit is in negative feedback. **In the circuit below, write V_{out} as a function of V_{in}.**

Hint: *Note the direction of the voltage drop, V_D. Be careful when writing your KCL expressions to account for this!*
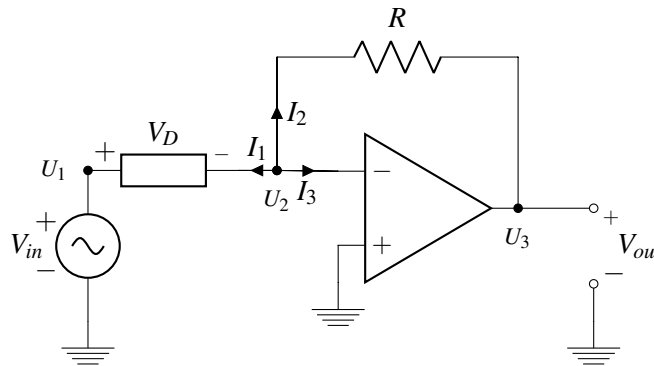
Solution: The strategy here is very similar to part (c):

KCL tells us:

$$I_1 + I_2 + I_3 = 0$$

but we know that $I_3 = 0$ by the golden rules. The current through the mystery device must enter the positive side, so I_1 is in the opposite direction of the defined I_D, so $I_1 = -I_D$. Therefore:

$$-I_D = -I_2$$

$$I_D = I_2$$

Substitute in IV relationships and simplify:

$$e^{\alpha(U_1 - U_2)} = \frac{U_2 - U_3}{R}$$

$$e^{\alpha V_{in}} = \frac{0 - V_{out}}{R}$$

The first line uses the fact that $V_D = U_1 - U_2$. The second line follows from the golden rules. We know $U_2 = U^- = U^+ = 0$ and we also know $U_1 = V_{in}$ and $V_{out} = U_3 - 0$. Continuing:

$$V_{out} = -Re^{\alpha V_{in}}$$

(e) (5 points) Given the polarity of circuits in parts (c) and (d), to put everything together, you need to implement **an inverting summer circuit, where** $V_{out} = -(V_{in_1} + V_{in_2})$.

You may use up to 1 op-amp (you do not need to label the power supplies) and as many resistors as you wish. **Any resistors used must be assigned a value. You must mathematically justify your circuit's behavior for full credit.**
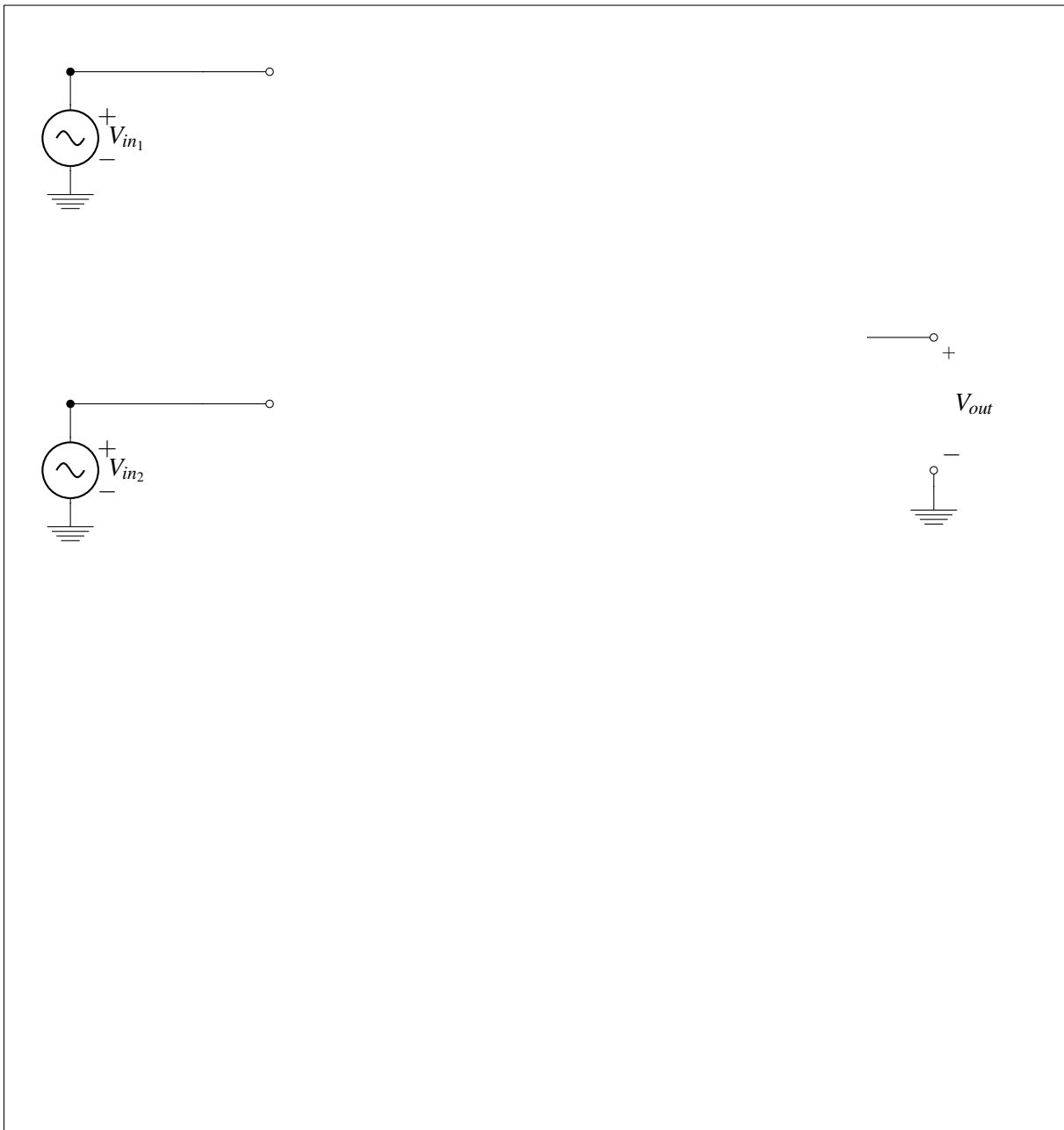
Solution: There are many ways to implement summation with a negation. A non-inverting summer into an inverting amplifier, passive summer circuit with inverting amplifier, etc. Many of these topologies however, require multiple op-amps or do not output exactly $-(V_{in,1} + V_{in,2})$. Since we can only use a single op-amp, we must use the inverting summer configuration.
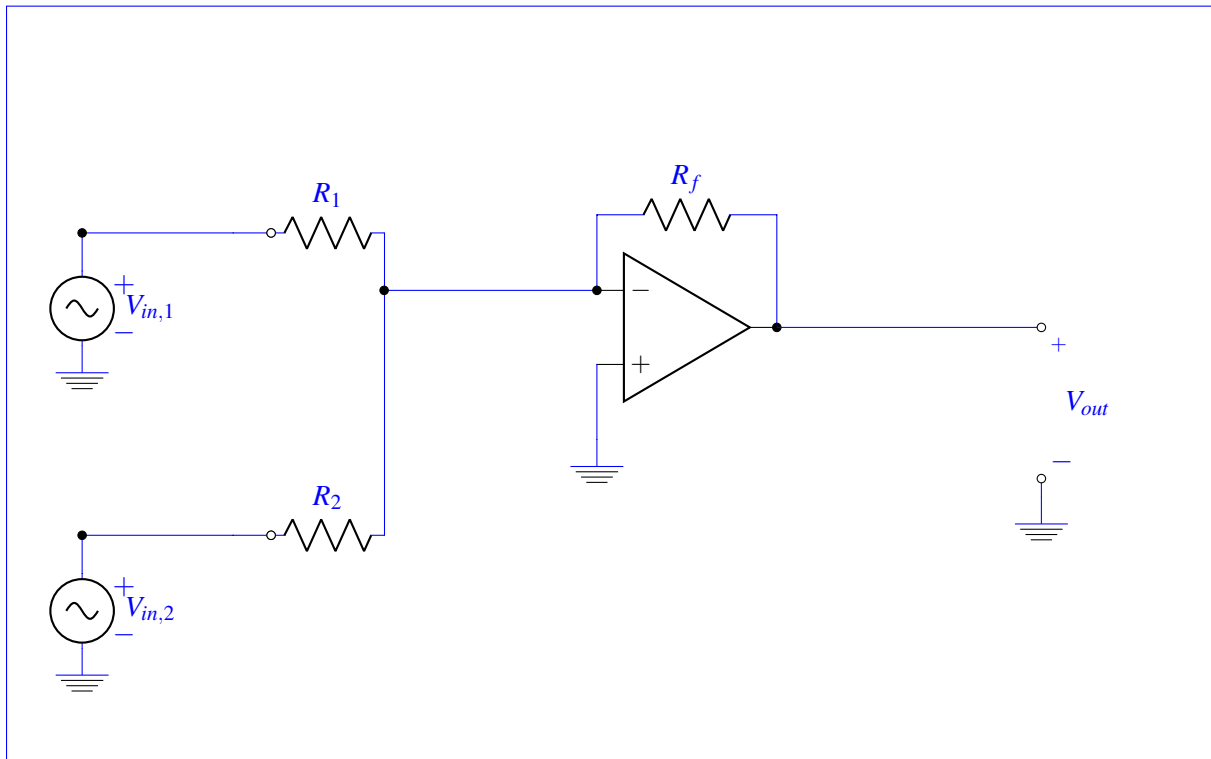
The output of this amplifier is

$$V_{out} = -R_f \left(\frac{V_{in,1}}{R_1} + \frac{V_{in,2}}{R_2} \right)$$

We want no fractional terms, which tells us we need to figure out how to cancel the resistor terms. If we make $R_1 = R_2 = R$ then the overall output becomes:

$$V_{out} = -\frac{R_f}{R}(V_{in,1} + V_{in,2})$$

We need this fraction to become 1, so we pick $R_f = R$. The actual value is arbitrary, so we'll pick $1k\Omega$ for all resistors. Then we get

$$V_{out} = -(V_{in,1} + V_{in,2})$$

as desired.

And that's it! Now that you have all the individual parts, you can string them together and start selling calculators. You may need to make a few small changes to your original design, but you can do it!

(f) (1 points) Controversial question: Do you wet the toothbrush before or after applying toothpaste? *This problem is for fun. You can write whatever you want or nothing at all and you won't be penalized.* **Solution:** There is no solution. This has been, and likely always will be, one of life's greater mysteries.

PRINT your name and student ID: _____

5. *Gold*en Positioning System (20 points)

In this problem we will explore how real GPS systems work, and touch on a few aspects of implementing GPS receivers.

A Gold code is a sequence of 1's and -1's that has a high autocorrelation at a shift of 0, and small autocorrelations otherwise. Every GPS satellite has a unique Gold code assigned to it, and users are aware of the Gold code used by each satellite. The plot below shows a Gold code of length 5.
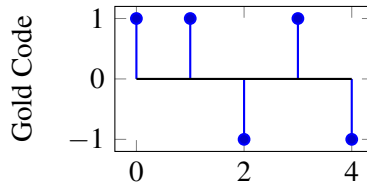
Each GPS satellite has a message that it transmits by modulating the Gold code. When the satellite is transmitting a 1, it sends just the Gold code sequence. When the satellite is transmitting a -1, it sends -1 times the Gold code. For example, if a satellite were transmitting the message [1, -1, 1], it would transmit the following (just as you have seen in the GPS homework problem):
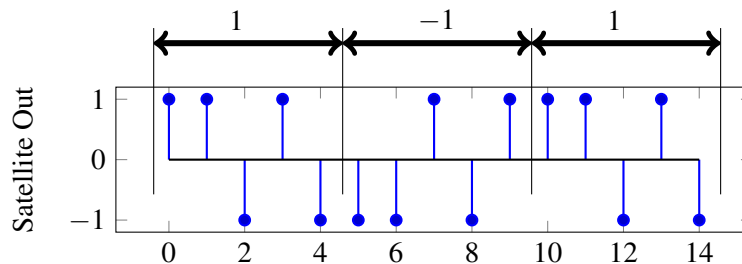
(a) (2 Points) Suppose you receive the following from a GPS satellite that has the same Gold code as above. **What message is the satellite transmitting?**
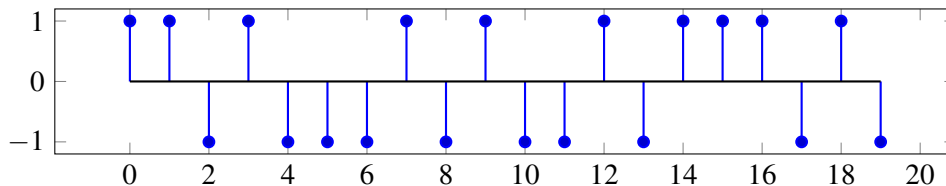
**Solution:** The data sent by the satellite is [1 -1 -1 1]

(b) (4 points) In order to find the message being sent by the satellite, the receiver will find the linear cross-correlation of the received signal with a replica of the satellite Gold code.

We need to find the **linear cross-correlation** of the signals shown below given by

$$\text{corr}(\vec{r}, \vec{g})[k] = \sum_{i=-\infty}^{\infty} r[i]g[i-k]$$

where $r[n]$ is the received signal and $g[n]$ is the Gold code sequence. *Note that neither of these signals is periodic in this part.*

Plot the values of corr$(\vec{r}, \vec{g})[k]$ for $-1 \le k \le 7$. What is the significance of the peaks in the linear cross-correlation?
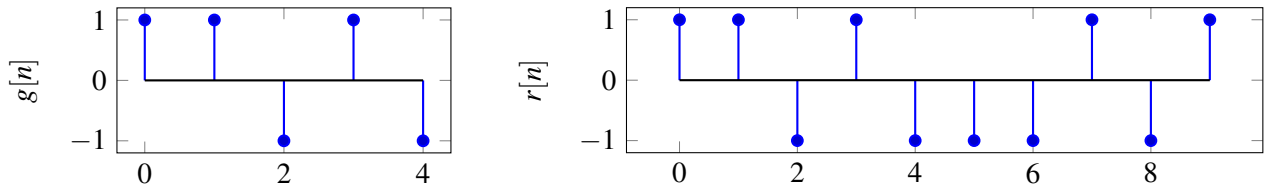

Solution: The cross-correlation of the two signals is shown below. Note that we show the full cross-correlation for completeness, you were only required to show the values between [-1, 7].
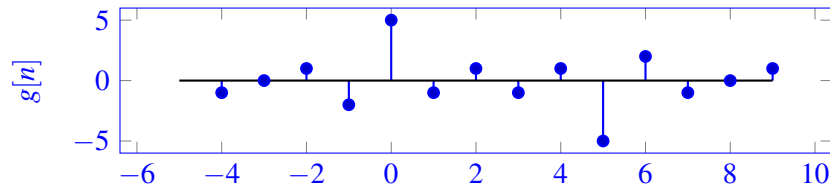
In the above plots, the peaks signify where the gold code is in the received signal. The sign of the peak can be used to determine whether the transmitted data was $+1$ or -1.

(c) (3 points) Real GPS receivers have specialized hardware to perform cross-correlation using circuits. However, since these transmissions are continuous signals instead of discrete values, we will model the received signal $r(t)$ and the Gold code signal $g_2(t)$ as square waves, as shown in the plot below. Notice that $g_2(t)$ shows two periods of the Gold code.
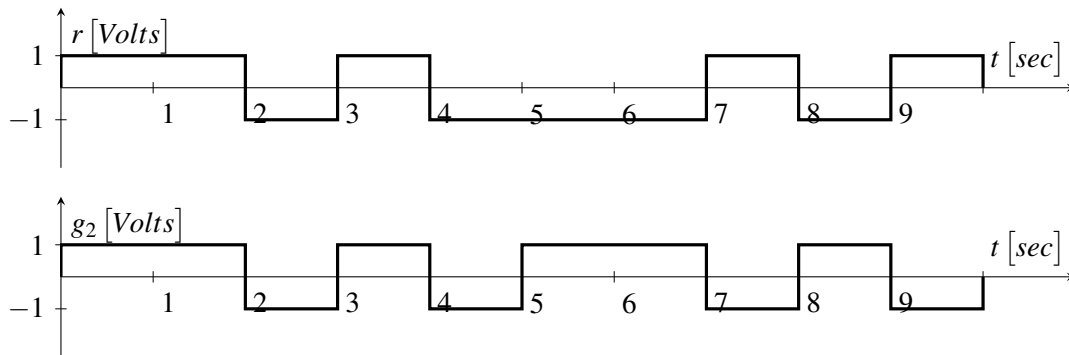
An essential hardware block to implementing a GPS correlator is *Multiply and Integrate*. This circuit multiplies its two inputs, then integrates the product over time. For example, the output of the *Multiply and Integrate* block given the above two inputs would be :

$$y(t) = \int_0^t r(\tau) g_2(\tau) d\tau$$

where $y(t)$ is the circuit output at time t. **Draw $y(t)$ as a function of time, for $t = 0$ to $t = 10$ sec.**
Solution:

Notice that for the first 5 seconds, $r(t)$ and $g_2(t)$ are equal, then for the next 5 seconds $r(t) = -g(t)$.
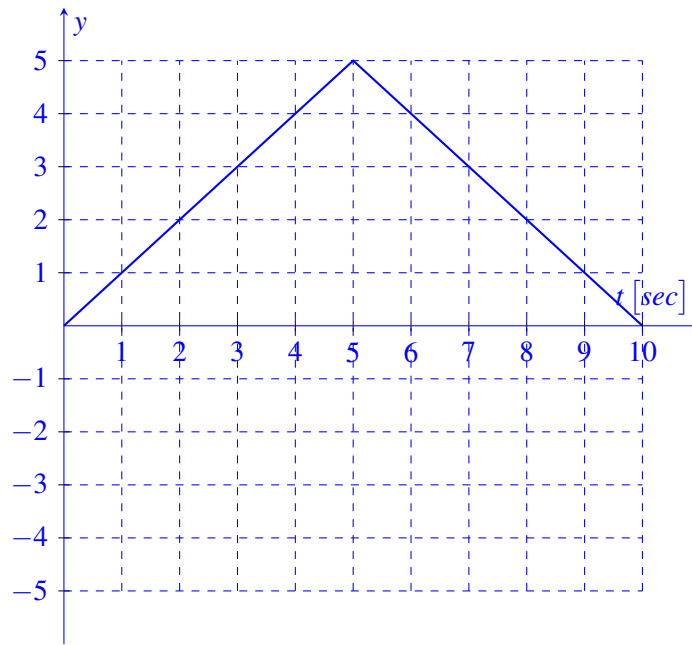
(d) (7 points) The *Multiply and Integrate* circuit has to integrate the product of the two signals, $g_2(t)$ and $r(t)$. Your friend has already designed the multiplier circuit. A Thévenin model of her multiplier circuit is shown below:
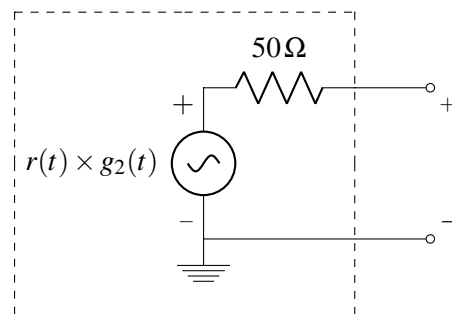
Design a circuit to complete the *Multiply and Integrate* block where the output of your circuit as a function of time, t, is:

$$v_{out}(t) = \int_0^t r(\tau) \times g_2(\tau)d\tau.$$

You may use **the multiplier circuit (Thevenin equivalent provided in the answer box), up to two op-amps, a capacitor, and two resistors** to implement your circuit. **Clearly define v_{out} and show numerical values of your circuit components.** As long as your circuits are in negative feedback, you may assume the voltage rails to your op-amps are large enough to not affect the output.

Solution:
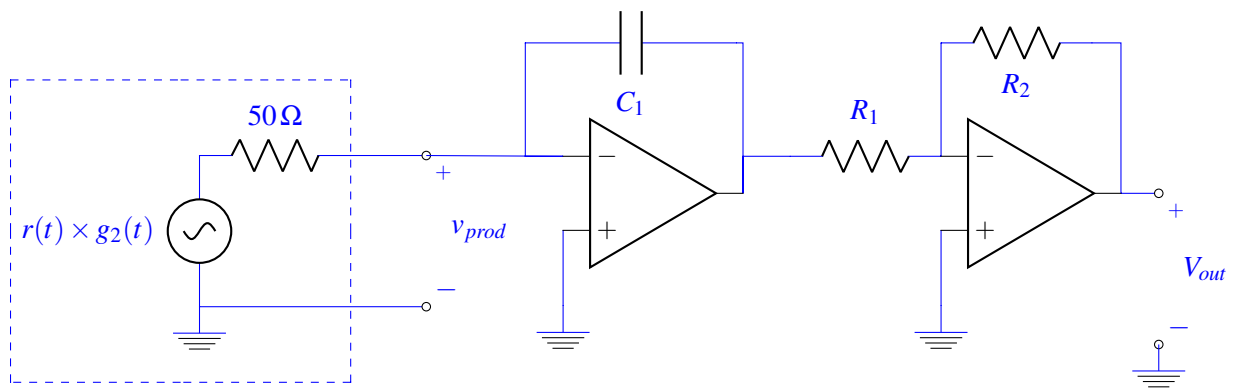
The next task is to size the resistors and capacitors. Notice the output of this circuit is:

$$\frac{R_2}{R_1} \times \frac{1}{50\Omega \times C_1} \int_0^t r(\tau) \times g_2(\tau)d\tau$$

We will chose $R_2 = R_1 = 1k\Omega$ and $C_1 = 0.02F$, such that the response is as desired.

(e) (2 points) Receivers also need to use the received data to calculate the position of the satellite. Each receiver will receive data from k satellites. Each satellite transmits the time, S_i, at which it started sending the message, where i is the index of the satellite, and $1 \leq i \leq k$. The receiver knows the time, T_i, at which each message arrives. You may assume the receiver and transmitter clocks are synchronized perfectly. Let c represent the speed of the signal.

Find an expression for d_i, the distance between the receiver and the ith satellite, in terms of S_i, T_i, and other relevant parameters.

Solution:
$$d_i = c \times (T_i - S_i)$$

(f) (2 points) Each satellite's position in 3D space is (u_i, v_i, w_i), where $1 \leq i \leq k$. The receiver position is given by (x, y, z). We need a linear system of equations the receiver can use to solve for its position, $\vec{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Due to limitations of the hardware, the receiver can only handle **linear** systems of equations.

How many satellites must the receiver get data from to solve for its position?

Solution: Here we need to solve for 3 variables, but the equations are non-linear. We will need one extra equation to make the equations linear, so 4 satellites total.

PRINT your name and student ID: _____

6. Besto Pesto (23 points)

Your TA Laura is struggling to keep her basil plant alive! She needs your help to determine how much water and sunlight her plant needs.

Let $x_h[k]$ be the plant's height on day k and $x_\ell[k]$ be the number of leaves on the plant on day k. The vector $\vec{x}[k] = \begin{bmatrix} x_h[k] \\ x_\ell[k] \end{bmatrix}$ defines the state of the plant. The evolution of the basil plant from one day to the next is defined by the **approximate** mathematical model:

$$\vec{x}[k+1] = \mathbf{A}\vec{x}[k] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_h[k] \\ x_\ell[k] \end{bmatrix}. \tag{1}$$

(a) (3 points) Our first goal is to estimate the elements of state transition matrix, $\mathbf{A}$: $a_{11}, a_{12}, a_{21}, a_{22}$. To do this we count the leaves and measure the height for the first N time steps, i.e. we know $\{\vec{x}[0], \vec{x}[1], \ldots, \vec{x}[N]\}$. Setup a least squares problem to estimate $\vec{a} = \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \end{bmatrix}$:

$$\hat{\vec{a}} = \underset{\vec{a}}{\operatorname{argmin}} \|\mathbf{M}\vec{a} - \vec{b}\|^2. \tag{2}$$

Write the matrix, M, and vector, $\vec{b}$, that would be used in the above least squares problem for $N = 3$.

Solution:

$$\mathbf{M} = \begin{bmatrix} x_h[0] & x_\ell[0] & 0 & 0 \\ 0 & 0 & x_h[0] & x_\ell[0] \\ x_h[1] & x_\ell[1] & 0 & 0 \\ 0 & 0 & x_h[1] & x_\ell[1] \\ x_h[2] & x_\ell[2] & 0 & 0 \\ 0 & 0 & x_h[2] & x_\ell[2] \end{bmatrix} \tag{3}$$

$$\vec{b} = \begin{bmatrix} x_h[1] \\ x_\ell[1] \\ x_h[2] \\ x_\ell[2] \\ x_h[3] \\ x_\ell[3] \end{bmatrix} \tag{4}$$

(b) (4 points) Now we would like to use the observations of the current state of the plant to determine the initial state: $\vec{x}[0] = \begin{bmatrix} x_h[0] \\ x_\ell[0] \end{bmatrix}$. For convenience of measurement, we use a Launchpad with a camera attached that periodically takes pictures and gives us a measurement vector, $\vec{y}[k] = \begin{bmatrix} y_h[k] \\ y_\ell[k] \end{bmatrix} \in \mathbb{R}^2$, where $y_h[k]$ is a noisy measurement of the plant's height, $y_\ell[k]$ is a noisy count of leaves. Since the

measurement might not be accurate, we will apply least squares to estimate $\vec{x}[0]$.

We have the dynamics and sensor models,

$$\vec{x}[k+1] = \mathbf{A}\vec{x}[k], \tag{5}$$
$$\vec{y}[k] = \mathbf{C}\vec{x}[k] + \vec{n}[k], \tag{6}$$

where $\vec{x}[k]$ is as before, $\vec{y}[k]$ is the sensor reading at time k, and $\vec{n}[k] \in \mathbb{R}^2$ is sensor noise/ error. The matrices $\mathbf{C}$ and $\mathbf{A}$ are given by

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{7}$$

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.5 \end{bmatrix}. \tag{8}$$

We took two actual readings from the Launchpad: $\vec{y}[0] = \begin{bmatrix} y_h[0] \\ y_\ell[0] \end{bmatrix}$ and $\vec{y}[1] = \begin{bmatrix} y_h[1] \\ y_\ell[1] \end{bmatrix}$. Now we need to use **both readings** to setup a least squares problem **to estimate the initial state of the plant, $\vec{x}[0]$.**

$$\hat{\vec{x}}[0] = \underset{\vec{x}[0]}{\operatorname{argmin}} \|\mathbf{M}\vec{x}[0] - \vec{b}\|^2. \tag{9}$$

Write the matrix, M, and vector, $\vec{b}$, that would be used in the above least squares problem to estimate $\vec{x}[0]$.

Solution: Applying the sensor and dynamic models, we have the following:

$$\vec{y}[0] = \mathbf{C}\vec{x}[0] + \vec{n}[0]$$
$$\vec{y}[1] = \mathbf{CA}\vec{x}[0] + \vec{n}[1]$$

We cannot solve for $\vec{x}[0]$ directly due to the presence of the noise ($\vec{n}[0]$ and $\vec{n}[1]$), so we apply least squares to the following:

$$\begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} \vec{x}[0] \approx \begin{bmatrix} \vec{y}[0] \\ \vec{y}[1] \end{bmatrix}$$

So we have that:

$$\mathbf{M} = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.8 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} \vec{y}[0] \\ \vec{y}[1] \end{bmatrix} = \begin{bmatrix} y_h[0] \\ y_\ell[0] \\ y_h[1] \\ y_\ell[1] \end{bmatrix}$$

(c) (2 points) The evolution of the basil plant from one day to the next is defined by:

$$\vec{x}[k+1] = \mathbf{A}\vec{x}[k],$$

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.5 \end{bmatrix},$$

$$\vec{x}[0] \neq \vec{0}.$$

What will happen to the number of leaves and the height of the plant as $k \to \infty$?

Solution: The state evolution matrix, $\mathbf{A}$, is in diagonal form. The eigenvalues of the matrix are on the diagonal; $\lambda = 0.8, 0.5$. The plant height and number of leaves will go to zero over time. This is regardless of $\vec{x}[0]$.

(d) (6 points) Now suppose the system evolves according to a new state transition matrix $\mathbf{A}$, where

$$\vec{x}[k+1] = \mathbf{A}\vec{x}[k],$$

$$\mathbf{A} = \begin{bmatrix} -1 & -3 \\ 4 & 6 \end{bmatrix}.$$

We want to diagonalize $\mathbf{A}$ to calculate the system evolution easily.

Find matrices $\mathbf{V}$ and Λ such that $\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^{-1}$. Use this diagonalization to write out the value of $x[N]$ in terms of $x[0]$.

Solution: To find the eigenvalues, we solve the equation:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$
$$(-1 - \lambda)(6 - \lambda) + 12 = 0$$
$$\lambda^2 - 5\lambda - 6 + 12 = 0$$
$$\lambda^2 - 5\lambda + 6 = 0$$
$$(\lambda - 2)(\lambda - 3) = 0$$
$$\lambda = 2, 3$$

If $\lambda = 2$, we need to find $\vec{a}$ such that:

$$(\mathbf{A} - 2\mathbf{I})\vec{a} = \vec{0}$$
$$\begin{bmatrix} -3 & -3 \\ 4 & 4 \end{bmatrix}\vec{a} = 0$$

Thus the dimension of the nullspace of $\mathbf{A} - 2\mathbf{I}$ is 1, and we can find one linearly independent vector in this basis:

$$\vec{a}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

If $\lambda = 3$, we need to find $\vec{a}$ such that:

$$(\mathbf{A} - 3\mathbf{I})\vec{a} = \vec{0}$$
$$\begin{bmatrix} -4 & -3 \\ 4 & 3 \end{bmatrix}\vec{a} = 0$$

Thus the dimension of the nullspace of $\mathbf{A} - 3\mathbf{I}$ is 1, and we can find one linearly independent vector in this basis:

$$\vec{a}_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

Now we define:

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\mathbf{V} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$$

Now we find $\mathbf{V}^{-1}$. We solve for the inverse using:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\mathbf{V}^{-1} = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$$

Then $\mathbf{A}$ can be diagonalized as $\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^{-1}$.
Applying this diagonalization,

$$\vec{x}[N] = \mathbf{A}^N \vec{x}[0] = \mathbf{V}\Lambda^N\mathbf{V}^{-1}\vec{x}[0]$$

$$\vec{x}[N] = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^N \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} \vec{x}[0]$$

(e) (2 points) Assuming that

$$\vec{x}[k+1] = \mathbf{A}\vec{x}[k],$$

$$\mathbf{A} = \begin{bmatrix} -1 & -3 \\ 4 & 6 \end{bmatrix},$$

$$\vec{x}[0] \neq \vec{0},$$

what will happen to the number of leaves and the height of the plant as $k \to \infty$?

Solution: To find the eigenvalues, we solved the equation in the previous part:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$
$$(-1 - \lambda)(6 - \lambda) + 12 = 0$$
$$\lambda^2 - 5\lambda - 6 + 12 = 0$$
$$\lambda^2 - 5\lambda + 6 = 0$$
$$(\lambda - 2)(\lambda - 3) = 0$$
$$\lambda = 2, 3$$

The plant height and leaf count will go to positive infinity or negative infinity as $k \to \infty$.

(f) (3 points) Laura decides to enlist the help of her friend Vijay to design an automatic watering and shading system. Let $u_w[k] \in \mathbb{R}$ be the amount of water added or removed from the soil each day and let $u_s[k] \in \mathbb{R}$ be the amount of time the plant spends in the sun each day. Vijay decides to design the automatic control system as a negative feedback controller (inspired by op-amps)!

The new evolution of the basil plant from one day to the next is defined by:

$$\vec{x}[k+1] = \mathbf{A}\vec{x}[k] + \vec{u}[k] = \mathbf{A}\begin{bmatrix} x_h[k] \\ x_\ell[k] \end{bmatrix} + \begin{bmatrix} u_w[k] \\ u_s[k] \end{bmatrix},$$

$$\mathbf{A} = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.5 \end{bmatrix},$$

$$\vec{u}[k] = -\mathbf{F}\vec{x}[k] = -\begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix}\begin{bmatrix} x_h[k] \\ x_\ell[k] \end{bmatrix}.$$

We want to express the new system as $\vec{x}[k+1] = \mathbf{R}\vec{x}[k]$, where $\mathbf{R}$ is the new state transition matrix. **Find R in terms of f_1, f_2, f_3, f_4, and numerical constants.**

Solution: Plugging in the negative feedback form of $u[k]$:

$$\vec{x}[k+1] = \mathbf{A}\vec{x}[k] - \mathbf{F}\vec{x}[k]$$

$$\vec{x}[k+1] = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.5 \end{bmatrix}\vec{x}[k] - \begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix}\vec{x}[k]$$

$$\vec{x}[k+1] = (\begin{bmatrix} 0.8 & 0 \\ 0 & 0.5 \end{bmatrix} - \begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix})\vec{x}[k]$$

$$\vec{x}[k+1] = \begin{bmatrix} 0.8 - f_1 & -f_2 \\ -f_3 & 0.5 - f_4 \end{bmatrix}\vec{x}[k]$$

$$\mathbf{R} = \begin{bmatrix} 0.8 - f_1 & -f_2 \\ -f_3 & 0.5 - f_4 \end{bmatrix}$$

(g) (3 points) Laura wants her basil plant to be at steady state; she wants the number of leaves and the height of the plant to stay the same over time. Vijay needs to choose values for the elements of matrix **F** from part (f) and asks for your help. Assume that $f_2 = 0$ and $f_3 = 0$. **Choose values for f_1 and f_4 that will keep the height of the plant and number of leaves constant over time.**

Solution: We can use our representation of the system from the previous part:

$$\vec{x}[k+1] = \begin{bmatrix} 0.8 - f_1 & -f_2 \\ -f_3 & 0.5 - f_3 \end{bmatrix}\vec{x}[k]$$

We want to place the eigenvalues of the matrix **R** at exactly 1. We choose

$$f_1 = -0.2$$
$$f_4 = -0.5$$

This will place the eigenvalues of **R** to be exactly 1.

PRINT your name and student ID: _____

7. OMP for Matrices (27 points)

In the real world, it is not computationally practical to directly solve for the eigenbasis for large matrices as you might do for small matrices on paper. You need to build an algorithm that sequentially computes the eigenbasis for a square symmetric matrix $\mathbf{Q} = \mathbf{A}^T\mathbf{A}$ (Note: Any matrix that can be written with $\mathbf{A} \in \mathbb{R}^{N \times N}$ in this form is symmetric).

To accomplish this we are given access to a function,

$$(\vec{v}_1, \lambda_1) = f(\mathbf{Q}), \tag{10}$$

that returns the **largest** eigenvalue of the matrix Q and the corresponding eigenvector. We will use this function to build a sequential greedy algorithm (similar to Orthogonal Matching Pursuit) that computes the eigenvectors and eigenvalues in descending order.

IMPORTANT: Throughout the problem, you should assume that the magnitude of each eigenvector is 1 (i.e. $\|\vec{v}_i\| = 1$), that the eigenvalues are real, unique, and distinct, and that $\lambda_1 > \lambda_2 > \cdots > \lambda_N > 0$.

(a) (2 points) Show that the matrix $\mathbf{Q} = \mathbf{A}^T\mathbf{A}$ is symmetric (i.e. $\mathbf{Q} = \mathbf{Q}^T$) when,

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}. \tag{11}$$

Solution:

$$\mathbf{Q} = \mathbf{A}^T\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix} \tag{12}$$

$$\mathbf{Q}^T = (\mathbf{A}^T\mathbf{A})^T = \left(\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \right)^T = \left(\begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix} \right)^T = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix} \tag{13}$$

They are equal and therefore $\mathbf{Q}$ is symmetric.

(b) (4 points) Given two distinct eigenvalue/eigenvector pairs, $(\lambda_k, \vec{v}_k)$ and $(\lambda_\ell, \vec{v}_\ell)$, show that for the symmetric matrix $\mathbf{Q} = \mathbf{A}^T\mathbf{A}$, if $\lambda_k \neq \lambda_\ell$ then $\langle \vec{v}_k, \vec{v}_\ell \rangle = 0$ (i.e. any pair of eigenvectors with distinct eigenvalues is orthogonal). Hint: *Consider:*

$$\mathbf{Q}\vec{v}_k = \lambda_k \vec{v}_k, \tag{14}$$

$$\vec{v}_\ell^T \mathbf{Q} = \lambda_\ell \vec{v}_\ell^T. \tag{15}$$

Solution: Start by considering,

$$\vec{v}_l^T \mathbf{Q} \vec{v}_k = \vec{v}_l^T \mathbf{A}^T \mathbf{A} \vec{v}_k = \vec{v}_l^T \lambda_k \vec{v}_k \tag{16}$$

$$\vec{v}_l^T \mathbf{Q} \vec{v}_k = \vec{v}_l^T \mathbf{A}^T \mathbf{A} \vec{v}_k = \lambda_l \vec{v}_l^T \vec{v}_k \tag{17}$$

We see the two expressions on the left are equal, so set the two expressions on the right equal to each other:

$$\lambda_k \vec{v}_l^T \vec{v}_k = \lambda_l \vec{v}_l^T \vec{v}_k \tag{18}$$

If $\lambda_k \neq \lambda_l$, then the only possible solution is that $\vec{v}_l^T \vec{v}_k = 0$, which means $\vec{v}_l$ and $\vec{v}_k$ are orthogonal.

(c) (3 points) Let us consider $\mathbf{V}$ to be an orthonormal matrix, where

$$\mathbf{V} = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_N \\ | & | & & | \end{bmatrix}. \tag{19}$$

Show that if $\mathbf{V}$ is an orthonormal matrix, then $\mathbf{V}^T\mathbf{V} = \mathbf{I}$.
Solution:

$$\mathbf{V}^T\mathbf{V} = \begin{bmatrix} \vec{v}_1^T\vec{v}_1 & \vec{v}_2^T\vec{v}_1 & \dots & \vec{v}_n^T\vec{v}_1 \\ \vec{v}_2^T\vec{v}_1 & \vec{v}_2^T\vec{v}_2 & \dots & \vec{v}_n^T\vec{v}_2 \\ \vdots & \vdots & & \vdots \end{bmatrix} = \mathbf{I} \tag{20}$$

All $\vec{v}_i^T\vec{v}_i = \|\vec{v}_i\|^2 = 1$ and when $i \neq j$ all $\vec{v}_i^T\vec{v}_j = 0$ because the eigenvectors are orthogonal.

(d) (3 points) Recall that the columns of orthonormal matrix $\mathbf{V}$ form a basis for $\mathbb{R}^N$, as you proved in the discussion sections. Assume $\vec{\alpha} = [\alpha_1 \dots \alpha_n]^T$ represents $\vec{x}$ in the basis of $\mathbf{V}$. Find $\langle \vec{v}_i, \vec{x} \rangle$.
Hint: *Write $\vec{x}$ as a linear combination of the column vectors $\vec{v}_1, \cdots, \vec{v}_N$.*
Solution:

$$\langle \vec{v}_i, \vec{x} \rangle = \vec{v}_i^T\vec{x} = \vec{v}_i^T \sum_{j=1}^{N} \alpha_j\vec{v}_j = 0 + \cdots + \alpha_i\vec{v}_i^T\vec{v}_i + \cdots + 0 = \alpha_i \tag{21}$$

(e) (4 points) Let us define,

$$\mathbf{V}^{(2)} = \begin{bmatrix} | & & | \\ \vec{v}_2 & \dots & \vec{v}_N \\ | & & | \end{bmatrix}. \tag{22}$$

Assume $\vec{\alpha} = [\alpha_1 \dots \alpha_n]^T$ represents $\vec{x}$ in the basis of $\mathbf{V}$. Find $\hat{\vec{x}}$, the projection of a vector $\vec{x} \in \mathbb{R}^N$ onto the columns of the orthonormal matrix $\mathbf{V}^{(2)}$:

$$\hat{\vec{x}} = \text{proj}_{\text{Col}(\mathbf{V}^{(2)})}(\vec{x}). \tag{23}$$

What is the magnitude of the error vector, $\vec{e} = \hat{\vec{x}} - \vec{x}$?
Solution:

$$\hat{\vec{x}} = \mathbf{V}^{(2)} \begin{bmatrix} \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \tag{24}$$

$$\|\vec{e}\| = \|\hat{\vec{x}} - \vec{x}\| = \|\alpha_1\vec{v}_1\| = \alpha_1 \tag{25}$$

The order which you subtract does not matter as all we care about is the magnitude of the error vector.

(f) (4 points) We know that because $\mathbf{Q}$ is symmetric, $\mathbf{V}$ is an orthonormal matrix. We use the idea of diagonalization and part (c) of the problem to express $\mathbf{Q}$ as:

$$\mathbf{Q} = \mathbf{A}^T\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^{-1} = \mathbf{V}\Lambda\mathbf{V}^T = \sum_{i=1}^{N}\lambda_i\vec{v}_i\vec{v}_i^T \tag{26}$$

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & \dots & 0 \\ 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_N \end{bmatrix} \tag{27}$$

Let $\mathbf{Q}^{(2)} = \mathbf{Q} - \lambda_1\vec{v}_1\vec{v}_1^T$. Thus, $\mathbf{Q}^{(2)}$ represents $\mathbf{Q}$ after the component associated with direction $\vec{v}_1$ is removed. Show that $\vec{v}_1$ is in the null space of $\mathbf{Q}^{(2)}$.

Hint: *Can you write $\mathbf{Q}^{(2)}$ using Eq. (26)?*

Solution: To show that $\vec{v}_1$ is in the nullspace of $\mathbf{Q}^{(2)}$, we want to show that $\mathbf{Q}^{(2)}\vec{v}_1 = \vec{0}$.

Method 1:

$$\mathbf{Q}^{(2)} = \mathbf{Q} - \lambda_1\vec{v}_1\vec{v}_1^T$$
$$= \sum_{i=1}^{N}\lambda_i\vec{v}_i\vec{v}_i^T - \lambda_1\vec{v}_1\vec{v}_1^T$$
$$= \sum_{i=2}^{N}\lambda_i\vec{v}_i\vec{v}_i^T$$
$$\mathbf{Q}^{(2)}\vec{v}_1 = \sum_{i=2}^{N}\lambda_i\vec{v}_i\vec{v}_i^T\vec{v}_1$$

Since $\vec{v}_1$ is orthogonal to all of the other eigenvectors, $\vec{v}_i^T\vec{v}_1 = 0$ for all $i \neq 1$.

$$\mathbf{Q}^{(2)}\vec{v}_1 = \sum_{i=2}^{N}\vec{v}_i\lambda_i 0$$
$$= \vec{0}$$

We've shown that $\vec{v}_1$ is in the null space of $\mathbf{Q}^{(2)}$.

Method 2:

$$\mathbf{Q}^{(2)} = \mathbf{Q} - \lambda_1 \vec{v}_1 \vec{v}_1^T$$
$$= \mathbf{V}\mathbf{\Lambda}\mathbf{V}^T - \lambda_1 \vec{v}_1 \vec{v}_1^T$$

$$= \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & .. & \vec{v}_N \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_N \end{bmatrix} \begin{bmatrix} - & \vec{v}_1 & - \\ - & \vec{v}_1 & - \\ & \vdots & \\ - & \vec{v}_N & - \end{bmatrix} - \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & .. & \vec{v}_N \\ | & | & & | \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} - & \vec{v}_1 & - \\ - & \vec{v}_1 & - \\ & \vdots & \\ - & \vec{v}_N & - \end{bmatrix}$$

$$= \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & .. & \vec{v}_N \\ | & | & & | \end{bmatrix} \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_N \end{bmatrix} \begin{bmatrix} - & \vec{v}_1 & - \\ - & \vec{v}_1 & - \\ & \vdots & \\ - & \vec{v}_N & - \end{bmatrix}$$

From the diagonalization above, the eigenvalue associated with $\vec{v}_1$ is 0 in the matrix $\mathbf{Q}^{(2)}$. Therefore, by the definition of eigenvalues and eigenvectors, $\mathbf{Q}^{(2)}\vec{v}_1 = 0\vec{v}_1 = \vec{0}$. This means $\vec{v}_1$ is in the nullspace of $\mathbf{Q}^{(2)}$

Common Mistakes:

- Stating that "$\vec{v}_1$ has been removed from $\mathbf{Q}$, so $\vec{v}_1$ must be part of the nullspace":
  This is too vague to receive credit because we have no definition of what it means to "remove" a vector from a matrix.
- Stating that "$\vec{v}_1$ cannot be represented by the other vectors $\vec{v}_2...\vec{v}_N$ so it is not in the column space of $\mathbf{Q}^{(2)}$ so it must be in the nullspace":
  If we remove a *dimension* of the column space, it must appear in the nullspace, but removing a *vector* from the column space does not mean it will be in the nullspace. In fact, vectors can appear in both the column space and nullspace at the same time. For example, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is in both the column space and nullspace of $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. The vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is in neither the column space nor the nullspace.
- Stating that $\vec{v}_i \vec{v}_i^T = \|\vec{v}_i\|^2$:
  The outer product is a matrix, not a scalar. $\vec{v}_i^T \vec{v}_i = \|\vec{v}_i\|^2$.
- Forgetting that matrices and vectors don't commute. For example, stating "$\mathbf{Q}^{(2)}\vec{v}_1 = \vec{v}_1(\mathbf{Q} - \lambda_1 \vec{v}_1 \vec{v}_1^T)$":
  If you right-multiply by a vector or matrix, you can't move it to the left side on the next line! Also, left-multiplying a matrix by a vector (without taking the transpose) usually is not a valid operation since the dimensions are wrong.
- Stating that $\mathbf{Q}$ or $\mathbf{Q}^{(2)}$ is an orthogonal matrix. This is not necessarily true, only V is orthogonal.

(g) (7 points) Recall the function that returns the **largest** eigenvalue and corresponding eigenvector of a matrix,

$$(\vec{v}_1, \lambda_1) = f(\mathbf{Q}). \tag{28}$$

Design a sequential greedy algorithm that returns a list of eigenvalues of matrix $\mathbf{Q}$ in descending order of values. You may assume that all the eigenvalues of $\mathbf{Q}$ are positive (> 0). You are allowed to use

the function defined in Eq. (28) that returns the largest eigenvalue and corresponding eigenvector and what you know from part (f).

Solution:

- Let $\mathbf{Q}^{(1)} = \mathbf{Q}$
- For i in range(1,N-1):
- $\qquad (\vec{v}_i, \lambda_i) = f(\mathbf{Q}^{(i)})$
- $\qquad \mathbf{Q}^{(i+1)} = \mathbf{Q}^{(i)} - \lambda_i \vec{v}_i \vec{v}_i^T$
- return $(\vec{v}_i, \lambda_i)_{i=1}^N$