# EECS 16A Designing Information Devices and Systems I
# Fall 2020 Final Exam – Instructions

**Read the following instructions before the exam.**

Good luck on the final exam! You've studied hard and we are rooting for you to do well!

**Our advice to you:** if you can't solve a particular problem, move on to another, or state and solve a simpler one that captures at least some of its essence. You will perhaps find yourself on a path to the solution. **We believe in you!**

**Format & How to Submit Answers**

**In this exam there are 10 problems (2 introductory questions and 8 exam questions containing subparts) with varying point number.** The problems are of varying difficulty, so pace yourself accordingly and avoid spending too much time on any one question until you have gotten all of the other points you can. If you are having trouble with one problem, there may be easier points available later in the exam!

Complete your exam using either the template provided or appropriately created sheets of paper. Either way, you should submit your answers to the *Gradescope* assignment that is marked Final for your specific exam group. Make sure you submit your assignment to the correct *Gradescope* assignment. You MUST select pages for each question. We cannot grade your exam if you do not select pages for each question. If you are having technical difficulties submitting your exam, you can email your answers to `eecs16a@berkeley.edu`.

In general, show all your work legibly to receive full credit; we cannot grade anything that we cannot read. For some problems, we may try to award partial credit for substantial progress on a problem, and showing your work clearly and legibly will help us do that.

**Timing & Academic Honesty**

You are expected to follow the rules provided in the Exam Proctoring Guidelines.
`https://docs.google.com/document/d/1EVb4Ca6FWSAykExY7X5ynFW4KdmHd0BI6KZ0ktM8ows/edit?usp=sharing`
The exam will be available to you at the link sent to you via email. The exam will start at 8am Pacific Time, Friday, December 18th, 2020, unless you have an exam accommodation. If you experience technical difficulties and cannot access your exam, let us know by making a private post on Piazza and we will try to help.

You have 180 minutes (3 hours) for the exam, with 40 minutes of extra time for scanning and submitting to *Gradescope*. Most of you will have to submit your exam by 11:40am unless you have another accommodation. Late submissions will be penalized exponentially. An exam that is submitted $N$ minutes after the end of the submission period will lose $2^N$ points. This means that if you are 1 minute late you will lose 2 points; if you are 5 minutes late you will lose 32 points and so on.

This is a closed-note, closed-book, closed-internet, and closed-collaboration exam. Calculators are not allowed. You may consult **three** handwritten 8.5" by 11" cheat sheets (front and back of three pieces of paper). Do not attempt to cheat in any way. We have a zero tolerance policy for violations of the Berkeley Honor Code.

# EECS 16A Designing Information Devices and Systems I
# Fall 2020 Final Exam

**1. HONOR CODE**

If you have not already done so, please copy the following statements into the box provided for the honor code on your answer sheet, and sign your name.

*I will respect my classmates and the integrity of this exam by following this honor code. I affirm:*

- *I have read the instructions for this exam. I understand them and will follow them.*
- *All of the work submitted here is my original work.*
- *I did not reference any sources other than my allocated reference cheat sheet(s).*
- *I did not collaborate with any other human being on this exam.*

**2.** (a) **(2 Points) Name someone who makes you feel happy.**
*All answers will be awarded full credit.*

(b) **(2 Points) What are you looking forward to over winter break?**
*All answers will be awarded full credit.*

**3. Save Baby Yoda! (8 points)**

Despite our best efforts, we have lost Baby Yoda to former agents of the Galactic Empire. Luckily we were able to conceal a receiver in his locket, so now it's time to save Baby Yoda using our 16A knowledge!

Baby Yoda has been delivered to an Imperial Star Destroyer. Rebel intel has provided us with access to their internal communication beacons. The ship's layout is 2-dimensional with 3 beacon locations specified in Table 1.

| Beacon | Coordinates | Distance to Baby Yoda |
|---|---|---|
| A | (5,5) | $\sqrt{20}$ |
| B | (2,3) | 1 |
| C | (1,1) | 2 |

Table 1: Data from Destroyer Beacons and their coordinates.

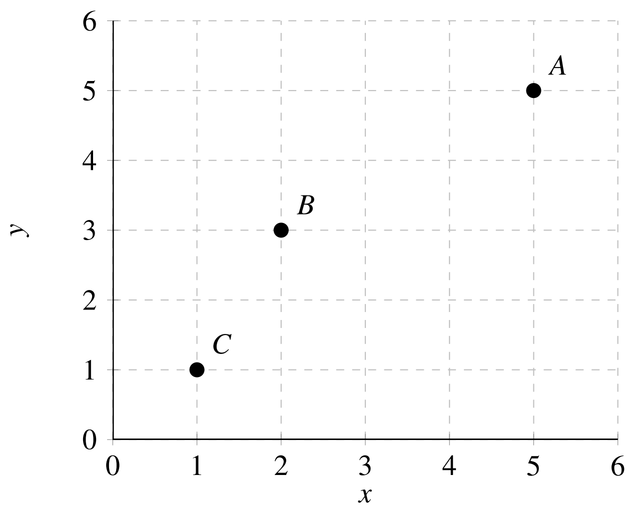



Figure 1: Diagram of the Destroyer's floor-plan with Beacon coordinates marked accordingly.

**Explicitly write out a *linear* system of equations (in matrix-vector form) using the data above for finding Baby Yoda's location $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. Draw a box around your final linear system, then solve for Baby Yoda's location**. Nonlinear terms are not permitted in your final system of equations. You must provide both the system and the location for full credit.

**4. Ultrasound Sensing with Op-Amps (14 points)**

The transresistance amplifier is often used to convert a current from a sensor to a voltage. In this problem we will use it to build an ultrasound sensor! When an ultrasonic wave hits our sensor, it generates a current, $i_{\text{ultra}}$. Whenever no ultrasonic wave hits our sensor zero current is generated, so $i_{\text{ultra}} = 0$.
*Note: An **ideal op-amp** is used in all subparts of this question. You can also assume that $V_{DD} = -V_{SS}$.*

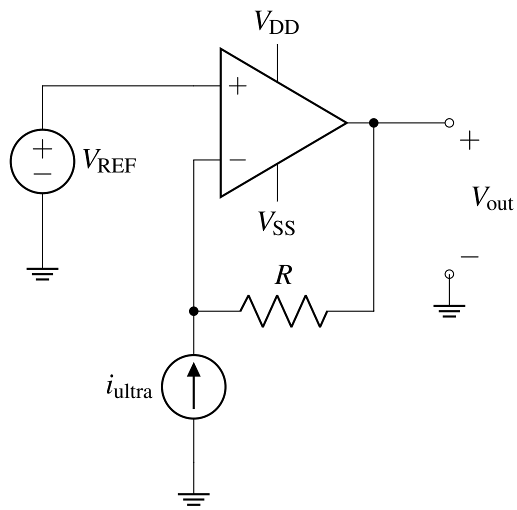



Figure 2: Transresistance sensor circuit

(a) (4 points) **Calculate the output voltage, $V_{\text{out}}$, of the transresistance sensor circuit shown in Fig. 2, as a function of the reference voltage, $V_{\text{REF}}$, the sensor input current, $i_{\text{ultra}}$, and the resistor, $R$, when an ultrasonic wave hits the sensor.** Clearly show all your work and justify your answer. Writing only the final expression will not be given full credit.

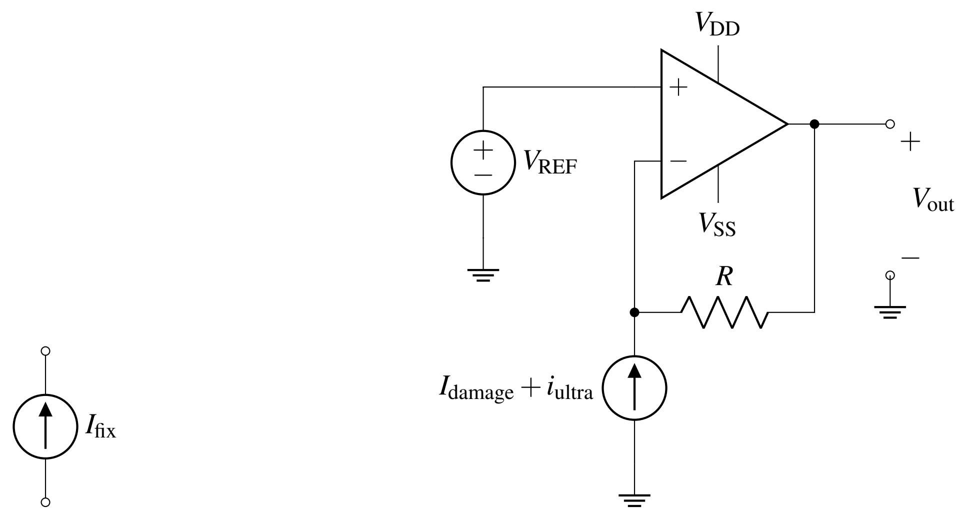
(b) (5 points) Assume that the amplitude of the ultrasonic wave hitting the sensor is such that the current $i_{\text{ultra}}$ fluctuates from a minimum value of $i_{\min} = 1 \cdot 10^{-6}$A, to a maximum value of $i_{\max} = 2 \cdot 10^{-6}$A. Also assume that the reference voltage is set to $V_{\text{REF}} = 1$V. In this case, calculate the following:

i. **The maximum value of the resistor, $R$, so that the output voltage, $V_{\text{out}}$, does not drop below 0V.** Clearly show all your work.

ii. Assuming you picked $R = 250 \cdot 10^3 \Omega$ (which may or may not be the correct answer to part (i)), **calculate the maximum value of the output voltage, $V_{\text{out}}$.** Clearly show all your work.

(c) (5 points) Unfortunately, after a few hours of successful ultrasound sensing, our sensor got damaged. It now constantly generates a huge background current, $I_{\text{damage}}$. So when an ultrasonic wave hits it, the sensor produces $I_{\text{damage}} + i_{\text{ultra}}$, as shown in Fig 3(b). When no ultrasonic wave hits it, the sensor produces just $I_{\text{damage}}$. However, the huge background current causes our output to constantly be $V_{\text{out}} = V_{SS}$, so we are not able to tell whether an ultrasonic wave is present or not.

We would like to fix this in our circuit by canceling the background current and retaining only the useful signal. For this purpose we are going to use a current source, $I_{\text{fix}}$, shown in Fig. 3(a), whose value we can choose. **How would you connect this current source in your circuit and what value would you pick for it?** Redraw the entire circuit with the new current source, $I_{\text{fix}}$, added and give the value of $I_{\text{fix}}$ in terms of $I_{\text{damage}}, i_{\text{ultra}}, R, V_{\text{REF}}$. **Explain how your design works.**




(a) Constant current source $I_{\text{fix}}$.

(b) Damaged transresistance amplifier circuit.

Figure 3: Circuits detailing the transresistance amplifier design, including the background signal $I_{\text{damage}}$.

**5. Saving Lives with Op-Amps (19 points)**

An electrocardiogram, or ECG, is a medical device used to detect electrical signals in your heart. Typically, the voltage signal from the human heart is only $1 \times 10^{-3}$ V at maximum. However, in order for healthcare professionals to properly interpret ECGs, these signals need to be amplified so that abnormalities are more obvious. In this problem we will do so by using **ideal op-amps**.
*Note: Assume that $V_{DD} = -V_{SS}$ in all subparts.*

(a) (3 points) We need to amplify the voltage signal recorded by the electrodes $V_{in}$ by a factor of 1000. Using the op-amp in Figure 4 below and 2 resistors, **draw a circuit that achieves $V_{out} = 1000 \cdot V_{in}$. Write an equation for $V_{out}$ in terms of $V_{in}$ and the resistor(s), label the resistors you use (i.e. $R_1, R_2$), and choose their values**. You should redraw the entire circuit in your answer sheet, but there is no need to draw the human as long as you label $V_{in}$. Clearly explain and show your work.

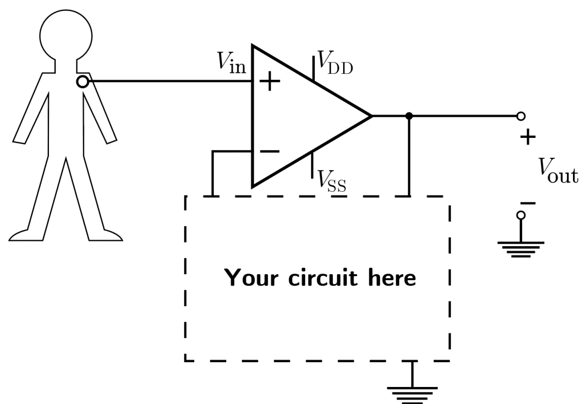



Figure 4: Unfinished ECG amplification circuit.

(b) (4 points) A friend of yours is also working on an ECG amplification circuit, and shows you their design in Figure 5. Their design uses $R_{\text{electrode}} = 1 \cdot 10^3\,\Omega$, $R_1 = 1 \cdot 10^3\,\Omega$, and $R_2 = 1 \cdot 10^6\,\Omega$. They claim their circuit gives, $V_{\text{out}} = -1000 \cdot V_{\text{in}}$. **Is their claim true?**

- **If yes, justify why.**
- **If no, how would you choose the value of $R_2$ to achieve $V_{\text{out}} = -1000 \cdot V_{\text{in}}$, assuming that both $R_{\text{electrode}}$ and $R_1$ are fixed at $R_{\text{electrode}} = R_1 = 1 \times 10^3\,\Omega$?** Clearly show your work, and justify your answers.

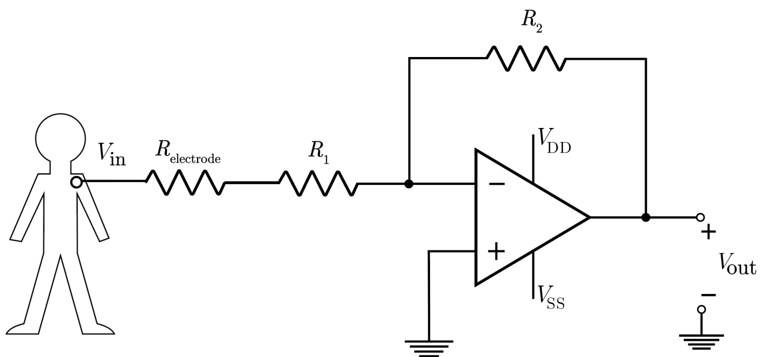



Figure 5: An alternative ECG op-amp circuit.

(c) (4 points) Another configuration often used by healthcare professionals is to attach one electrode to the heart (recording its electrical signal, $V_{\text{in}}$) and another electrode to the right leg to serve as a reference voltage, as shown in Figure 6. **What is the output voltage, $V_{\text{out}}$, as a function of $V_{\text{in}}$, $V_{\text{RL}}$, $R_{\text{bottom}}$, and $R_{\text{top}}$?** Clearly show your work.

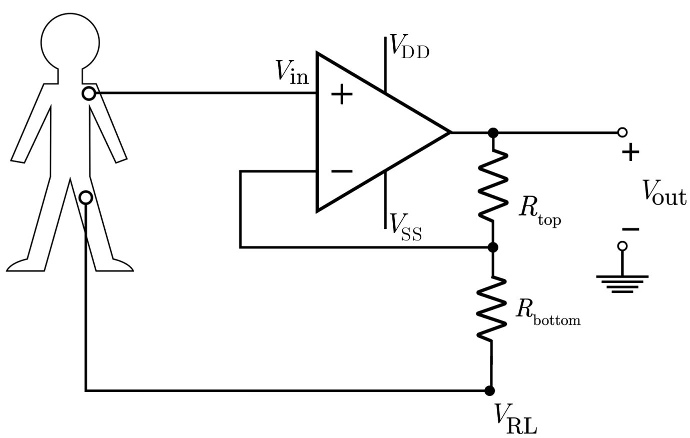



Figure 6: Alternative op-amp ECG topology.

(d) (8 points) Even after amplification, certain peaks of your ECG signal are too low to be discerned. You want to sample them and amplify them a bit more. To this end, you use the circuit in Figure 7. The circuit cycles through two phases: in phase 1, switches labeled $\phi_1$ are ON and $\phi_2$ are OFF, while in phase 2, switches labeled $\phi_2$ are ON and $\phi_1$ are OFF. **Calculate the output voltage, $V_{\text{out}}$, during phase 2, after steady state has been reached, in terms of $C_1$, $C_2$ and $V_{\text{in}}$.** Clearly show your work.

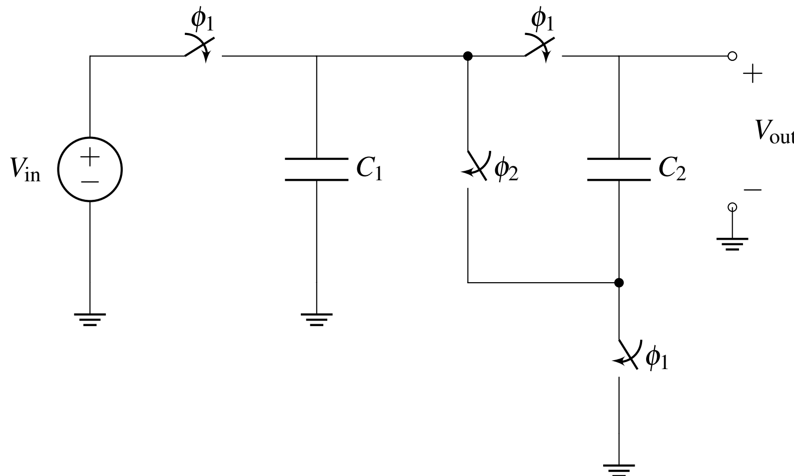
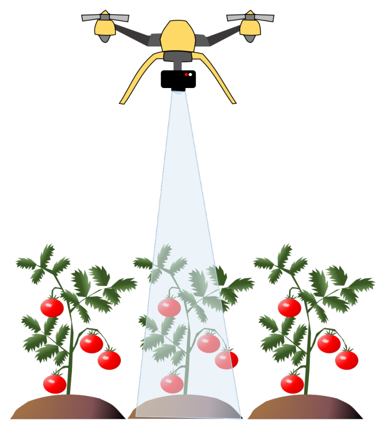



Figure 7: Switch capacitor voltage boosting circuit.

**6. Hyperspectral Classification of Tomatoes (14 points)**

You're a high-tech farmer who just bought a new hyperspectral sensor to monitor your crops.
*NOTE: You do not need to understand how a hyperspectral sensor works to solve this problem.*

You attach the sensor to a drone and fly it over your crops, taking measurements of the hyperspectral signature for different points along the field. You want to use these measurements to identify which crops are healthy and which crops are getting sick. Your sensor gives you a spectral signature for each plant as a length 5 vector, where each entry of the vector represents a different frequency. Scientists have determined that healthy versus sick tomato plants will have the following spectral signatures as shown in Figure 8.



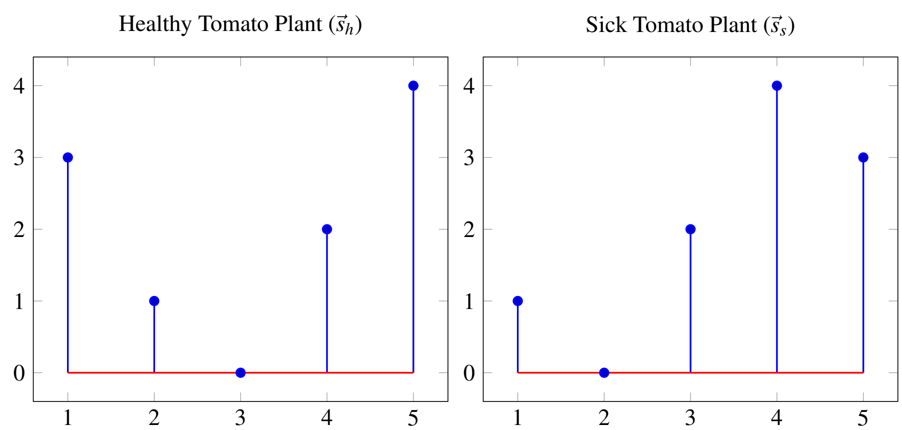



Figure 8: Spectral signature for healthy tomato plant ($\vec{s}_h$) and sick tomato plant ($\vec{s}_s$).

They can also be represented in vector form:

$$\vec{s}_h = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \qquad \vec{s}_s = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \\ 3 \end{bmatrix}.$$

(a) (6 points) Using your spectral sensor, you measure the following spectral signature for one of your tomato plants as shown in Figure 9. This measurement has some noise in it.

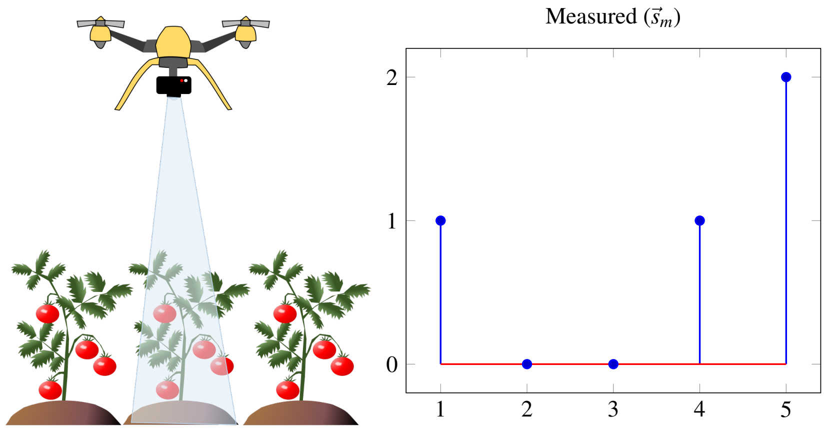



Figure 9: Spectral signature for the measurement, $\vec{s}_m$.

The spectral signatures for healthy, sick, and your measured tomato plants can also be represented in vector form as

$$\vec{s}_h = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \qquad \vec{s}_s = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \\ 3 \end{bmatrix}, \qquad \vec{s}_m = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

Since spectral signatures never exactly match, the standard procedure is to calculate the angle between signature vectors to determine how close they are. **Compute the angle between $\vec{s}_m$ and $\vec{s}_h$ and the angle between $\vec{s}_m$ and $\vec{s}_s$. Is your measured vector closer to the sick plants or the healthy plants?** Classify your plant's health based on the angle between your measured spectral signature ($\vec{s}_m$) and the known spectral signatures, ($\vec{s}_h$, $\vec{s}_s$). Show your work and justify your answer.
*NOTE: Table 2 can be helpful for finding the angles.*

Table 2: Cosine Table

| $\cos(\theta)$ | $\theta(^\circ)$ |
|---|---|
| $\frac{9}{\sqrt{180}}$ | 47.87 |
| $\frac{10}{\sqrt{180}}$ | 41.81 |
| $\frac{11}{\sqrt{180}}$ | 34.93 |
| $\frac{12}{\sqrt{180}}$ | 26.57 |
| $\frac{13}{\sqrt{180}}$ | 14.31 |

(b) (4 points) It's a windy day and the drone got pushed as it was taking a measurement, so now the measurement has a linear combination of measurements for several different tomato plants (some of which are healthy and some of which are sick). So your measurement is

$$\vec{s}_m = \alpha\vec{s}_h + \beta\vec{s}_s + \vec{e}, \tag{1}$$

where $\vec{e}$ represents an error vector that is unknown.
The values you get for your measurement are:

$$\vec{s}_m = \begin{bmatrix} 5 \\ 1 \\ 4 \\ 10 \\ 10 \end{bmatrix}.$$

The measurement is also shown in Figure 10.

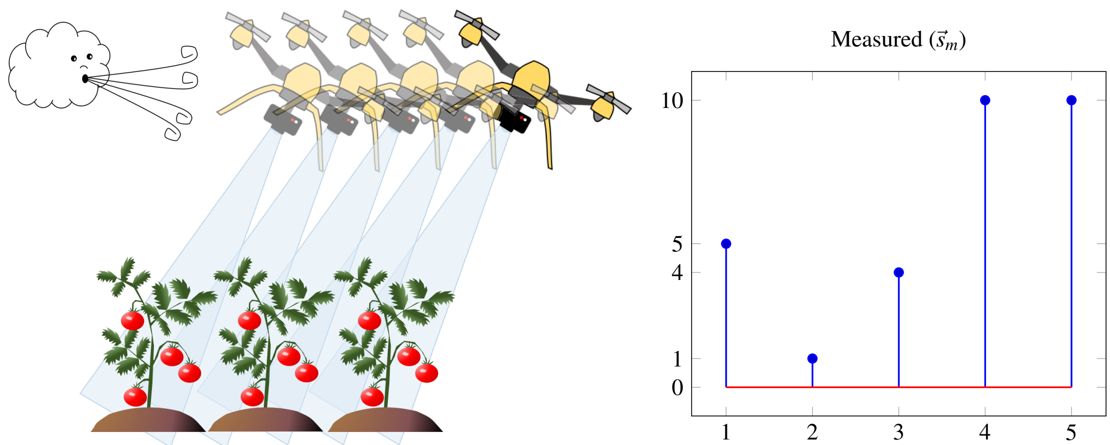


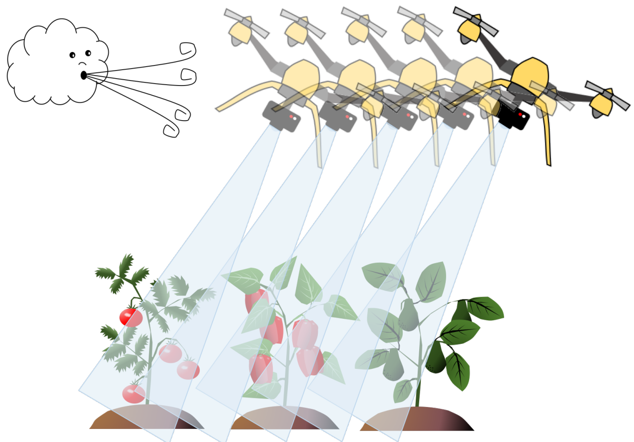

Figure 10: Spectral signature for your measurement, $\vec{s}_m$.

Recall that

$$\vec{s}_h = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \qquad \vec{s}_s = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \\ 3 \end{bmatrix}.$$

You want to identify the unknowns $\alpha$ and $\beta$. **Write a least squares problem in the format $\mathbf{A}\vec{x} = \vec{b}$ to identify the unknowns $\alpha$ and $\beta$. Show your work.** You do not have to solve for $\alpha$ and $\beta$.

(c) (4 points) Your drone got pushed by the wind again, but this time it was while it was taking a measurement on the border of three adjacent fields - your tomato, pepper, and avocado fields.



Tomato, pepper, and avocado plants have unique spectral signatures with a length of 5. The notations are described as the following:

- $\vec{s}_h$ and $\vec{s}_s$ represent the spectral signatures of healthy and sick tomato plants
- $\vec{s}_{ph}$ and $\vec{s}_{ps}$ represent the spectral signatures of healthy and sick pepper plants
- $\vec{s}_{ah}$ and $\vec{s}_{as}$ represent the spectral signatures of healthy and sick avocado plants

Your measurement is now a linear combination of 6 possible spectral signatures:

$$\vec{s}_m = \alpha_1\vec{s}_h + \beta_1\vec{s}_s + \alpha_2\vec{s}_{ph} + \beta_2\vec{s}_{ps} + \alpha_3\vec{s}_{ah} + \beta_3\vec{s}_{as}. \tag{2}$$

Here $\alpha_1, \alpha_2, \alpha_3$ are the unknown weights of healthy tomato, pepper, and avocado plants respectively. $\beta_1, \beta_2, \beta_3$ are the unknown weights of sick tomato, pepper, and avocado plants respectively. **Is it possible to uniquely determine the weights of healthy/sick tomatoes, peppers, and avocados from your measurement in equation 2? Why or why not? Show your work and justify your answer.**

**7. Cross-correlation (28 points)**

We are building our own Acoustic Positioning System.
*NOTE: The signatures $\vec{s}_1$, $\vec{s}_2$ in each sub-part are different; each prompt is independent from the others.*

(a) (6 points) We have two signatures/gold codes of length-5, given by $\vec{s}_1$ and $\vec{s}_2$ as in Figure 11. So far we have numerically computed their linear cross-correlation $\text{Corr}_{\vec{s}_1}(\vec{s}_2)$, yet a few entries have been tragically lost! Fortunately we can compute these omitted terms by hand. **Please compute the missing cross-correlation values at shifts $k = -1$ and $k = +2$. Show your work and justify your answer.**

$$\vec{s}_1 = \begin{bmatrix} +1 \\ 0 \\ -1 \\ 0 \\ +1 \end{bmatrix} \qquad \vec{s}_2 = \begin{bmatrix} +1 \\ +1 \\ 0 \\ -1 \\ +1 \end{bmatrix}.$$

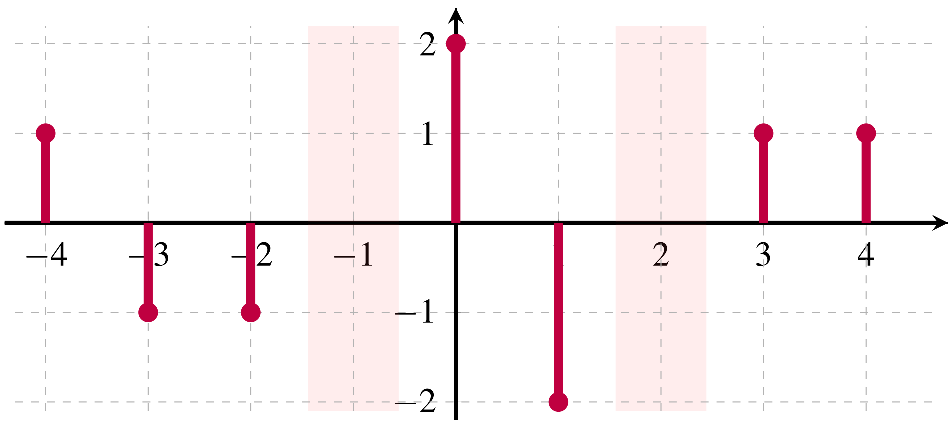



Figure 11: Linear cross-correlation plot of the two signals $\text{Corr}_{\vec{s}_1}(\vec{s}_2)$. The x-axis represents the shift.

(b) (4 points) We are trying out some new codes $\vec{s}_1$ and $\vec{s}_2$. We only know that the codes are normalized ($\langle \vec{s}_1, \vec{s}_1 \rangle = 1$, $\langle \vec{s}_2, \vec{s}_2 \rangle = 1$) and their inner-product is $\langle \vec{s}_1, \vec{s}_2 \rangle = 0.3$. During our test we have received the signal $\vec{r} = \frac{1}{2}\vec{s}_1 + \frac{1}{3}\vec{s}_2$. Without knowing any more information about our codes, **compute $\text{Corr}_{\vec{r}}(\vec{s}_1)$ at the shift $k = 0$. Show your work and justify your answer.**

(c) (4 points) We again have two new signals $\vec{s}_1$ and $\vec{s}_2$, and are now given the plot of $\mathrm{Corr}_{\vec{s}_1}(\vec{s}_2)$ as shown in Figure 12. Our receiver identified a signal $\vec{r}$ which we know to be related to the code $\vec{s}_2$ by some scaling, shifting, and/or reflection. However, we only know the linear cross-correlation $\mathrm{Corr}_{\vec{s}_1}(\vec{r})$ as shown in Figure 13. **Can you express $\vec{r}$ in terms of $\vec{s}_2$? Show your work and justify your answer.**

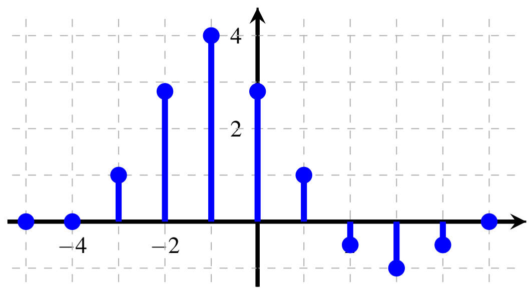



Figure 12: Linear cross-correlation plots for $\mathrm{Corr}_{\vec{s}_1}(\vec{s}_2)$.

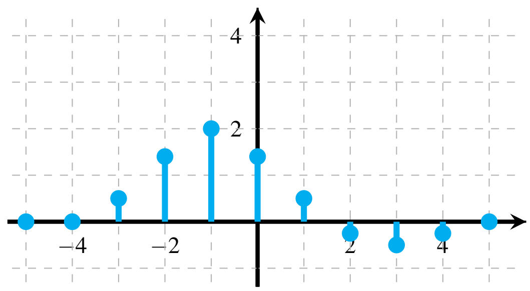



Figure 13: Linear cross-correlation plots for $\mathrm{Corr}_{\vec{s}_1}(\vec{r})$.

(d) (4 points) With a little effort we managed to create two *good* gold codes of length 100, $\vec{s}_1$ and $\vec{s}_2$. The linear cross-correlation of $\vec{s}_1$ and $\vec{s}_2$ is small at all shifts while the autocorrelation of each signal is also small, except at shift $k = 0$. We receive our first signal $\vec{r}$ which we know to be a combination of both codes

$$\vec{r}[n] \;=\; \vec{s}_1[n-k_1] \;+\; \vec{s}_2[n-k_2]. \tag{3}$$

The linear cross-correlation $\mathrm{Corr}_{\vec{r}}(\vec{s}_1)$ has been computed and plotted in Figure 14, and similarly $\mathrm{Corr}_{\vec{r}}(\vec{s}_2)$ is plotted in Figure 15. **Determine the shifts for $\vec{s}_1$ and $\vec{s}_2$ in the received signal $\vec{r}$, i.e. solve for $k_1$ and $k_2$ in equation (3). Explain your answer.**

*Note: Don't worry too much about identifying the exact value for $k_1$ and $k_2$. As long as your answer is reasonable, you will receive full credit.*

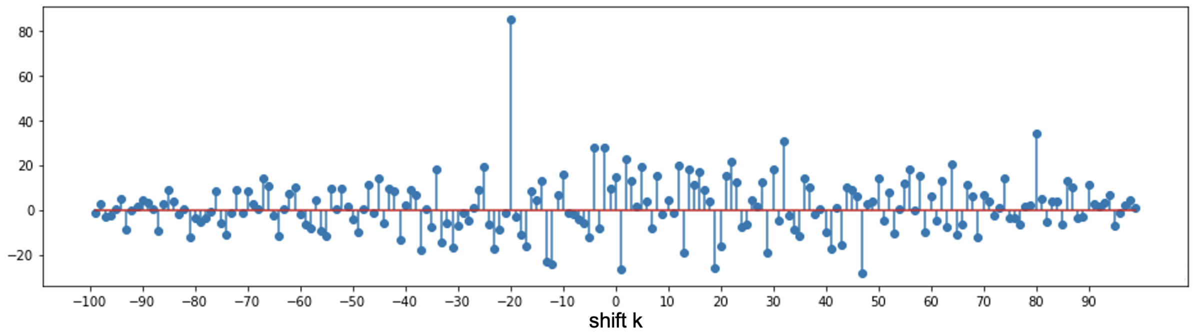



Figure 14: Linear cross-correlation plots for $\mathrm{Corr}_{\vec{r}}(\vec{s}_1)$.

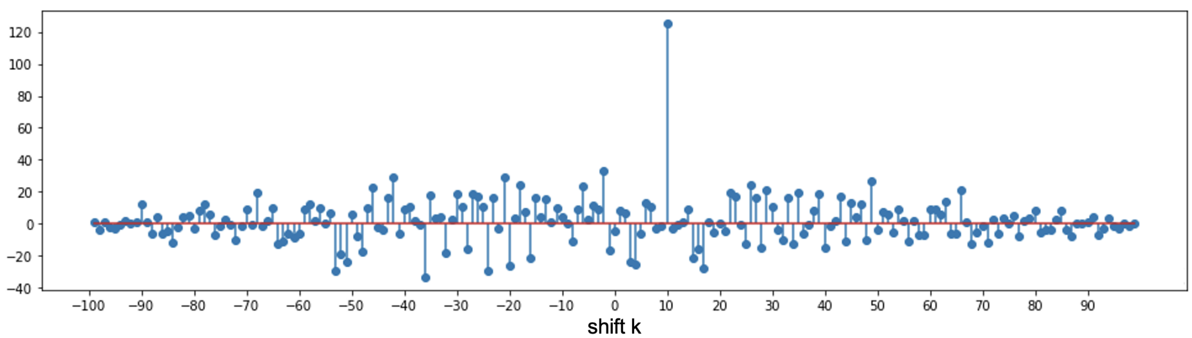



Figure 15: Linear cross-correlation plots for $\mathrm{Corr}_{\vec{r}}(\vec{s}_2)$.

(e) (4 points) It appears that making codes orthogonal to each other improves the robustness of our Acoustic Positioning System. Knowing this, we want to use our knowledge of projections to write our first code as $\vec{s}_1 = \vec{a} + \vec{b}$, where $\langle \vec{b}, \vec{s}_2 \rangle = 0$ and $\vec{a} = \alpha \, \vec{s}_2$ (for some constant $\alpha$) as illustrated in Figure 16. **Compute $\alpha$ and $\vec{b}$ in terms of $\vec{s}_1$ and $\vec{s}_2$. Show your work and justify your answer.**

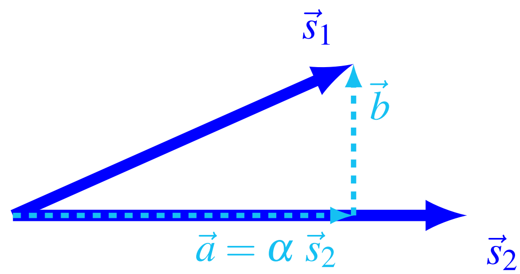



Figure 16: 2D figure of $\vec{s}_1 = \vec{a} + \vec{b}$.

(f) (6 points) After optimizing two orthogonal codes $\vec{s}_1$ and $\vec{s}_2$ (i.e. $\langle \vec{s}_1, \vec{s}_2 \rangle = 0$), we would next like to include another code $\vec{s}_3$ and make it orthogonal to $\vec{s}_1$ and $\vec{s}_2$. We can start by writing $\vec{s}_3$ as $\vec{s}_3 = \vec{a} + \vec{b}$, such that $\vec{a}$ belongs to the span$\{\vec{s}_1, \vec{s}_2\}$ and $\vec{b}$ is orthogonal to span$\{\vec{s}_1, \vec{s}_2\}$, i.e. $\langle \vec{b}, \vec{s}_1 \rangle = 0$ and $\langle \vec{b}, \vec{s}_2 \rangle = 0$. **Use the idea of projections to write both $\vec{a}$ and $\vec{b}$ in terms of $\vec{s}_1$, $\vec{s}_2$, and $\vec{s}_3$, and inner-products thereof. (For full credit your final answer may not contain matrices nor matrix-vector products).** Show your work and justify your answer.

**8. Warm for the Holidays (14 points)**

Winter is coming, and both you and your roommate are in desperate need of electric heating eye pads to avoid overly dry eyes this holiday. Tragically the circuit for your eye pads broke, yet fortunately you've taken EECS16A and have come up with a clever fix by designing a voltage divider and a comparator circuit!

(a) (4 points) First you build a circuit that converts temperature change to voltage change. Your design is shown in Fig. 17. In your design you use two temperature dependent resistors, whose values are given by $R_0 + \alpha T$, $R_0 - \alpha T$, where $R_0$ is the resistor value at 0 degrees centigrade, $\alpha$ is a thermal coefficient, and $T$ is the temperature of your eye pads.
**What is the temperature dependent output voltage, $V_\mathrm{T}$, of this circuit, as a function of $V_s$, $R_0$, $\alpha$, and $T$? Is $V_\mathrm{T}$ a linear function of T?** Clearly show all your work.

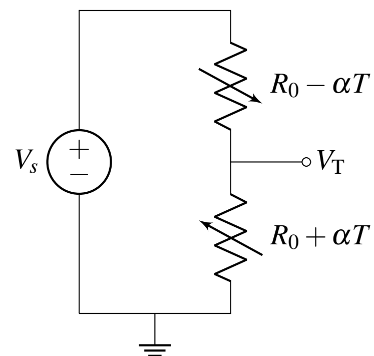



Figure 17: Temperature Sensing Circuit

(b) (4 points) We want to use a comparator to turn the heat ON and OFF, and you set up the circuit in Fig. 18. You process the $V_T$ to make $V_\mathrm{in} = (1 - \frac{T}{T_0})$[Volts], where $T_0 = 30°\mathrm{C}$. The heat will turn on when $V_\mathrm{out} = V_\mathrm{DD}$. **For what range of temperatures, $T$, is $V_\mathrm{out} = V_\mathrm{DD}$? Give your answer in terms of °C.** Clearly show all your work.

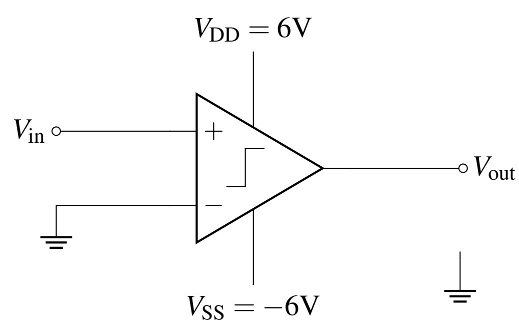



Figure 18: First attempt eye-pad control circuit.

(c) (6 points) Your TA, Moses, points out that just using the circuit in Figure 18 will cause your heat to turn ON and OFF due to very small fluctuations. Instead, he suggests analyzing the following circuit in Figure 19. **Find the voltage $u_+$ at the positive terminal of the comparator, as a function of $V_{\text{out}}$, $R_1$, $R_2$, and $V_{\text{ref}}$.** Clearly show all your work.

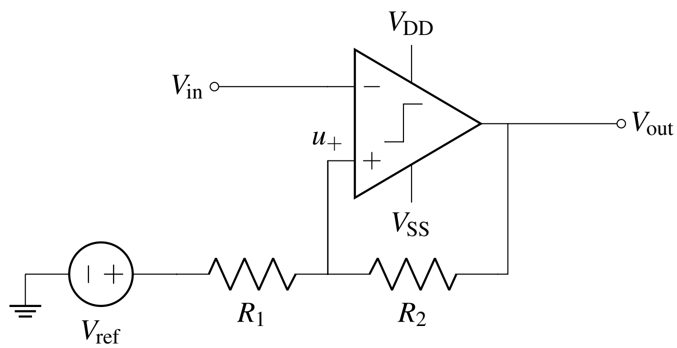



Figure 19: Proposed eye-pad control circuit.

## 9. Least Squares for Robotics (16 points)

Robots rely on sensors for understanding their environment and navigating in the real world. These sensors must be calibrated to ensure accurate measurements, which we explore in this problem.

(a) (3 points) Your robot is equipped with two forward-facing sensors – a radar and camera.
However, the sensors are placed with an offset (i.e. a gap) of $\ell$ in meters (m), as depicted in Fig. 20, and you want to find its value. The radar returns a range $\rho$ in meters (m) and heading angle $\theta$ in radians (rad) with respect to the object. In contrast, the camera only returns an angle, $\phi$ in radians (rad), with respect to the object.

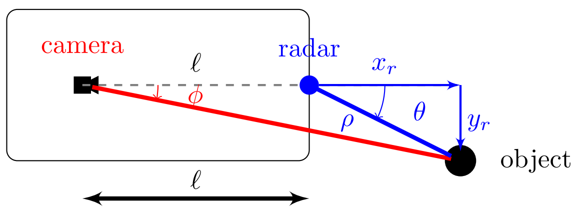



Figure 20: Sensor Placement and Offset $\ell$.

These relationships are summarized by the following sensor model, where $x_r$ and $y_r$ are the Cartesian coordinates of the object with respect to the radar:

$$x_r = \rho\cos(\theta)\,, \tag{4}$$

$$y_r = \rho\sin(\theta)\,, \tag{5}$$

$$\tan(\phi) = \frac{y_r}{x_r + \ell}\,. \tag{6}$$

**Assuming $\phi \neq 0$, use equations (4), (5), (6) to express $\ell$ in terms of $\rho$, $\theta$, and $\phi$.**

(b) (5 points) Often it is difficult to precisely identify the value of $\ell$. To learn the value of $\ell$ you decide to take a series of measurements. In particular, you take $N$ measurements and get the equations:

$$a\,\ell + e_i = b_i$$

for $1 \leq i \leq N$. Here $a \neq 0$ is a fixed and known constant. Each $b_i$ represents your $i^{\text{th}}$ measurement and $e_i$ represents the error in your measurement. While you know all of the $b_i$ values, you do not know the error values $e_i$.

We can write this equation in a vector format as:

$$\mathbf{A}\ell + \vec{e} = \vec{b},$$

where $\mathbf{A} = \begin{bmatrix} a \\ \vdots \\ a \end{bmatrix}$, $\vec{e} = \begin{bmatrix} e_1 \\ \vdots \\ e_N \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_N \end{bmatrix}$.

In this simple 1-D case, the least squares solution is a scaled version of the average of $\{b_i\}_{i=1}^{N}$.

**Find the best estimate for $\ell$, denoted as $\hat{\ell}$, using least squares. Simplify your expression and express $\hat{\ell}$ in terms of $a$, $b_i$, and $N$. Your answer may not include any vector notation.**

*Note: A is a vector and not a matrix.*

(c) (8 points) Now we turn to the task of controlling the robot's velocity and acceleration, which is a key requirement for navigation.

We use the following model for the robot, which describes how the velocity and acceleration of the robot changes with timestep k:

$$\begin{bmatrix} v[k+1] \\ a[k+1] \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v[k] \\ a[k] \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} j[k],$$

where

- $k$ is the timestep;
- $v[k]$ is the velocity state at timestep $k$;
- $a[k]$ is the acceleration state at timestep $k$;
- $j[k]$ is the jerk (derivative of acceleration) control input at timestep $k$.

We start at a known initial state $\begin{bmatrix} v[0] \\ a[0] \end{bmatrix}$, and we want to find $j[0]$ to set $\begin{bmatrix} v[1] \\ a[1] \end{bmatrix}$ as close to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as possible. For this, we minimize:

$$E = \left\| \begin{bmatrix} v[1] \\ a[1] \end{bmatrix} \right\|^2.$$

**Find the best estimate for the optimal choice of jerk, $\hat{j}[0]$, by using least squares method to minimize $E$. Express your solution in terms of $v[0]$ and $a[0]$. Show your work.**

*Hint: Rewrite E in terms of $j[0]$ and other relevant terms.*

**10. Proof (10 points)**

Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n \times n}$. The eigenvalues and eigenvectors of $\mathbf{A}$ are given by $(\alpha_1, \vec{v}_1)$, $(\alpha_2, \vec{v}_2)$, $\cdots$, $(\alpha_n, \vec{v}_n)$, where all the $\alpha_i,\ 1 \leq i \leq n$, are distinct. Similarly the eigenvalues and eigenvectors of $\mathbf{B}$ are given by $(\beta_1, \vec{v}_1)$, $(\beta_2, \vec{v}_2)$, $\cdots$, $(\beta_n, \vec{v}_n)$, where all the $\beta_i,\ 1 \leq i \leq n$, are distinct.
*NOTE:* $\mathbf{A}, \mathbf{B}$ *have identical eigenvectors.*

Prove that:

$$\mathbf{A}\,\mathbf{B}\,\vec{x} \;=\; \mathbf{B}\,\mathbf{A}\,\vec{x},$$

for any vector $\vec{x} \in \mathbb{R}^n$.