

Midterm 1 Review

EECS 16A – Fall 2021

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October 2, 2021

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① Vectors – Basic Properties

Definition A *vector* is an *ordered* list of numbers $\vec{v} \in \mathbb{R}^n$.

For example $\vec{x} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$ is a vector in \mathbb{R}^3 , and $\vec{b} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$ is a vector in \mathbb{R}^2 .

Addition Adding together vectors \vec{a} and \vec{b} is only possible if they're in the same space \mathbb{R}^n , and then follows by adding element-wise.

scalar Multiplication We can multiply any vector \vec{a} by a number $\alpha \in \mathbb{R}$ and the vector will carry that multiple element-wise.

$$\vec{a} + \vec{b} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{bmatrix} \quad \alpha \vec{a} = \begin{bmatrix} \alpha a_1 \\ \alpha a_2 \\ \vdots \\ \alpha a_n \end{bmatrix}$$

Distributive The two properties above provide us with the distributive property for vectors

$$\alpha (\vec{a} + \vec{b}) = \alpha \vec{a} + \alpha \vec{b}.$$

Proving this would make for good practice with proofs!

① Vectors – Vector Subspaces

Definition A *vector space* V is a space inside \mathbb{R}^n where vectors can live AND must satisfy closure, meaning for any $\vec{a}, \vec{b} \in \mathbb{R}^n$ and any $\alpha, \beta \in \mathbb{R}$ we know $\alpha \vec{a} + \beta \vec{b} \in V$.

Take a moment, consider the cases $\alpha = \beta = 0$ and $\alpha = -1, \beta = 0$.

All vector spaces MUST contain $\vec{0}$.

Span Any vector space V in \mathbb{R}^n can be expressed by a *span* of vectors (in \mathbb{R}^n), which we write using the notation $V = \text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$. Formally the span is defined as

$$\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\} := \left\{ \vec{v} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_m \vec{v}_m \mid \alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{R} \right\}.$$

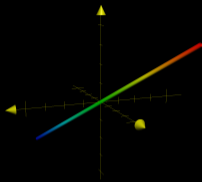
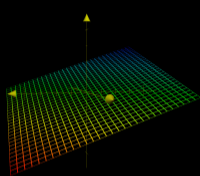
In english this means, “ V is the space containing any vector you can dream up by adding or scaling the vectors within the span”.

① Vectors – Vector Subspaces

Span Example Can you match these spans to their graphic renders?

$$\text{span}\left\{ \begin{bmatrix} 1/2 \\ -1 \\ -2/3 \end{bmatrix} \right\}$$

$$\text{span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1/3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1/3 \end{bmatrix} \right\}$$



Linear Dependence It is possible that a span contains redundant vectors, meaning that some vector $\vec{v} \in V$ can be expressed in multiple ways using the span vectors (for example: $\text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right\}$, since $\begin{bmatrix} 3 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix}$).

Formally, a set of vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ are said to be *linearly dependent* if it's possible to find constants $\alpha_1, \alpha_2, \dots, \alpha_m$ such that

$$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_m \vec{v}_m = \vec{0}.$$

Otherwise, the set $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ is *linearly independent* and forms a *basis* for V .

NOTE! The *dimension* of V is the least number of vectors needed to span it, ie. number of basis vectors.

② Matrices – as transformations

Dimensions A matrix $\mathbf{A}^{m \times n} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ is used (via matrix multiplication)

to map vectors between spaces $\mathbf{A}\vec{x} = \vec{y}$, where $\vec{x} \in \mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^m$.

NOTE: m goes down, n goes across (most common notation).

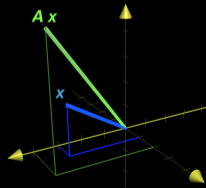
Matrix $\mathbf{A}^{m \times n}$ maps $\mathbb{R}^n \rightarrow \mathbb{R}^m$.

Transformations Square matrices $\mathbf{A}^{n \times n}$ (where $m = n$) can be imagined as transformations which manipulate a vector $\vec{x} \in \mathbb{R}^n$. In the past we've seen matrices induce rotations, mirrors, and scalings.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1/2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\mathbf{A}\vec{x} = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$$



② Matrices – Multiplication

Matrix-Vector Multiplication

$$\mathbf{A} \vec{x} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n \\ a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n \\ \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n \end{bmatrix}$$

Matrix-Matrix Multiplication We can multiply matrices $\mathbf{A}^{m \times n}$ and $\mathbf{B}^{p \times \ell}$ (in the order \mathbf{AB}) only if $n = p$. The resulting matrix of $\mathbf{C} = \mathbf{AB}$ has dimensions $\mathbf{C}^{m \times \ell}$.

Think about this: \mathbf{B} maps vectors $\mathbb{R}^\ell \rightarrow \mathbb{R}^p$ while \mathbf{A} maps vectors $\mathbb{R}^n \rightarrow \mathbb{R}^m$, so the mapping of $\vec{x} \in \mathbb{R}^\ell$ by $\mathbf{AB}\vec{x} = \mathbf{A}(\mathbf{B}\vec{x})$ is only possible when the output from \mathbf{B} agrees dimensionally with the input for \mathbf{A} .

$$\mathbf{AB} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1\ell} \\ b_{21} & b_{22} & \dots & b_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{n\ell} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1\ell} \\ c_{21} & c_{22} & \dots & c_{2\ell} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \dots & c_{m\ell} \end{bmatrix}$$

NOTE! Matrix-matrix multiplication is the general form of matrix-vector multiplication, which you can see if you consider $\ell = 1$, in which $\mathbf{B}^{n \times 1} \equiv \vec{b} \rightarrow \mathbf{AB} \equiv \mathbf{A}\vec{b}$.

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NOTE! Matrix-matrix multiplication is the general form of matrix-vector multiplication, which you can see if you consider $\ell = 1$, in which $\mathbf{B}^{n \times 1} \equiv \vec{b} \rightarrow \mathbf{AB} \equiv \mathbf{A}\vec{b}$.

② Matrices – multiplication properties

Associativity Matrix products are always associative, meaning for multiple products we can freely place the parentheses $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$.

- For instance, this implies $\mathbf{A}(\mathbf{A}(\mathbf{A}\vec{x})) = (\mathbf{A}(\mathbf{AA}))\vec{x} = (\mathbf{A})^3\vec{x}$.
- The usual matrix dimension requirements must hold: $\mathbf{A}^{m \times n}$, $\mathbf{B}^{n \times \ell}$, and $\mathbf{C}^{\ell \times p}$.

Commutativity Matrix products are generally *NOT* commutative, meaning $\mathbf{AB} \neq \mathbf{BA}$.

- In fact, it's possible that the product can only exist in one order. For $\mathbf{A}^{m \times n}$ and $\mathbf{B}^{p \times \ell}$, \mathbf{AB} is only defined when $n = p$ whereas \mathbf{BA} is only defined when $\ell = m$.
- The usual matrix dimension requirements must hold $\mathbf{A}^{m \times n}$, $\mathbf{B}^{n \times \ell}$, and $\mathbf{C}^{\ell \times p}$.

Square Matrices Only for $\mathbf{A}^{n \times n}$ (note $m = n$) can we meaningfully define an *inverse* $(\mathbf{A})^{-1}$ (and only for certain matrices...) for \mathbf{A} , or the *eigenvalue* problem $\mathbf{A}\vec{v} = \lambda\vec{v}$.

② Matrices – Extra tid-bits

Inverse properties Consider matrices $\mathbf{A}^{n \times n}$ and $\mathbf{B}^{n \times n}$:

- If either \mathbf{A} or \mathbf{B} has no inverse, then neither \mathbf{AB} nor \mathbf{BA} will have an inverse.
- Otherwise $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ and $(\mathbf{BA})^{-1} = \mathbf{A}^{-1}\mathbf{B}^{-1}$.
- Generally, $(\mathbf{A} + \mathbf{B})^{-1} \neq \mathbf{A}^{-1} + \mathbf{B}^{-1}$.

Column Space The columns of a matrix $\mathbf{A}^{m \times n}$ span a vector-space we call the *column space* of \mathbf{A} . It is sometimes called the *range* of \mathbf{A} since we can think of $\mathbf{A}\vec{x}$ as

$$\mathbf{A}\vec{x} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} \uparrow \\ \vec{a}_1 \\ \downarrow \end{bmatrix} + x_2 \begin{bmatrix} \uparrow \\ \vec{a}_2 \\ \downarrow \end{bmatrix} + \dots + x_n \begin{bmatrix} \uparrow \\ \vec{a}_n \\ \downarrow \end{bmatrix},$$

which effectively defines \vec{x} as coefficients for summing vectors \vec{a}_j in the column space. If there is no solution to $\mathbf{A}\vec{x} = \vec{b}$, we could say that \vec{b} lies outside of the column space.

③ Gaussian Elimination – Systems of equations

Systems of equations As we had seen in the beginning, a matrix-vector product lets us easily write down a system of equations into the form $\mathbf{A}\vec{x} = \vec{b}$. For example:

$$\begin{array}{rcl} x - 2y + 3z + w & = & 20 \\ 5x + y - 2w & = & 0 \\ 6y - z & = & -7 \\ x + y + z + w & = & -2 \end{array} \quad \longrightarrow \quad \begin{bmatrix} 1 & -2 & 3 & 1 \\ 5 & 1 & 0 & -2 \\ 0 & 6 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \\ -7 \\ -2 \end{bmatrix}$$

This prescription lets us systematically add/subtract/scale these equations until we can identify the solution for x, y, z , and w .

- If there are equal unknowns and *unique* equations, then there is a *unique solution* \vec{x} for $\mathbf{A}\vec{x} = \vec{b}$.
- If there are fewer unique equations than unknowns **and** the equations do not contradict each other, then there are *infinite solutions*.
- If there are more unique equations than unknowns **or** the equations contradict each other, then there is *no solutions*.

③ Gaussian Elimination – Systems of equations

Conditional cases –

- 1 – If there are equal unknowns and *unique* equations, then there is a *unique solution* \vec{x} for $\mathbf{A}\vec{x} = \vec{b}$.

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

- 2 – If there are fewer unique equations than unknowns **and** the equations do not contradict each other, then there are *infinite solutions*.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

- 3 – If there are more unique equations than unknowns **or** the equations contradict each other, then there is *no solutions*.

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

③ Gaussian Elimination – Method

Formal Method –

- 0 – Note: A *pivot* is an element equal to 1 where every term to its left or down is zero.
- 1 – Start with the top left term (the assumed pivot), then scale the row so this left-most term equals one.
 - If the left-most term is zero, then swap with the next-highest row holding a nonzero left-most term.
 - If the entire column has only zeros, this column has no pivots. Restart step 1 assuming the pivot is over right one column.
- 2 – Subtract (a scaling of) this row from all the others below it in a way that cancels any nonzero terms below the pivot to zero.
- 3 – Now we have created a pivot. The next candidate pivot is down one and right one. Repeat from step 1 treating this new pivot as the left-most term.

Example –

$$\begin{array}{rcl} \frac{1}{2}x + y - \frac{1}{4}z - w & = & 1 \\ 2x + 4y - z - w & = & 1 \\ -3x - 6y + z - 4w & = & 1 \end{array} \longrightarrow \begin{bmatrix} 1/2 & 1 & -1/4 & -1 \\ 2 & 4 & -1 & -1 \\ -3 & -6 & 1 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \longrightarrow \left[\begin{array}{cccc|c} 1/2 & 1 & -1/4 & -1 & 1 \\ 2 & 4 & -1 & -1 & 1 \\ -3 & -6 & 1 & -4 & 1 \end{array} \right]$$

③ Gaussian Elimination – Method

Example –

$$\left[\begin{array}{cccc|c} 1/2 & 1 & -1/4 & -1 & 1 \\ 2 & 4 & -1 & -1 & 1 \\ -3 & -6 & 1 & -4 & 1 \end{array} \right]$$

$$\xrightarrow{R_1 \rightarrow 2R_1}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1/2 & -2 & 2 \\ 2 & 4 & -1 & -1 & 1 \\ -3 & -6 & 1 & -4 & 1 \end{array} \right]$$

$$\xrightarrow{R_2 \rightarrow R_2 - 2R_1}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1/2 & -2 & 2 \\ 0 & 0 & 0 & 3 & -3 \\ -3 & -6 & 1 & -4 & 1 \end{array} \right]$$

$$\xrightarrow{R_3 \rightarrow R_3 + 3R_1}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1/2 & -2 & 2 \\ 0 & 0 & 0 & 3 & -3 \\ 0 & 0 & -1/2 & -10 & 7 \end{array} \right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1/2 & -2 & 2 \\ 0 & 0 & -1/2 & -10 & 7 \\ 0 & 0 & 0 & 3 & -3 \end{array} \right]$$

$$\xrightarrow{R_2 \rightarrow -2R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1/2 & -2 & 2 \\ 0 & 0 & 1 & 20 & -14 \\ 0 & 0 & 0 & 3 & -3 \end{array} \right]$$

$$\xrightarrow{R_3 \rightarrow (1/3) R_3}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1/2 & -2 & 2 \\ 0 & 0 & 1 & 20 & -14 \\ 0 & 0 & 0 & 1 & -1 \end{array} \right]$$

□

③ Gaussian Elimination – Applications

3 key applications of Gaussian elimination (row-reduction procedure):

1 – Solving systems of linear equations –

As in prior slides, we can find \vec{x} for which $\mathbf{A}\vec{x} = \vec{b}$ or identify whether there are infinite or no solutions. To do so we row reduce the augmented form $\left[\mathbf{A} \mid \vec{b} \right]$.

2 – Computing the null-space (check linear independence) –

The *null space* of a matrix \mathbf{A} is the vector subspace V with $\vec{x} \in V$ for which $\mathbf{A}\vec{x} = \vec{0}$. This is a particular case where $\vec{b} = \vec{0}$, so we row-reduce $\left[\mathbf{A} \mid \vec{0} \right]$.

NOTE: You can drop the right side of the augmented form here, since row operations will never alter $\vec{0}$.

- If we find no zero row from \mathbf{A} , the null space $V = \vec{0}$ only contains the origin point. In this case we have proven \mathbf{A} columns (and rows) constitute *linearly independent* vectors.
- Else the zero row(s) will motivate choosing the free parameter(s) within \vec{x} , leading us to a span of vectors defining the null space.

3 – Identifying inverses –

Provided \mathbf{A} 's null space is $V = \vec{0}$ **AND** \mathbf{A} is square ($m = n$), we can compute an inverse matrix $(\mathbf{A})^{-1}$ for which $(\mathbf{A})^{-1}\mathbf{A} = \mathbf{I}$. We use row reduction on the augmented form $\left[\mathbf{A} \mid \mathbf{I} \right]$, but go further after getting pivots until $\left[\mathbf{I} \mid \mathbf{A}^{-1} \right]$.

③ Gaussian Elimination – Finding a null space

Get the null space for $\mathbf{A}^{3 \times 5} = \begin{bmatrix} 1 & 3 & 0 & 2 & 0 \\ 2 & 6 & 1 & 8 & -1 \\ 0 & 0 & 1 & 4 & 4 \end{bmatrix}$, solving $\vec{x} \in \text{Null}(\mathbf{A})$ such that $\mathbf{A}\vec{x} = \vec{0}$.

– **1 – Row reduction** –

$$\left[\begin{array}{ccccc|c} 1 & 3 & 0 & 2 & 0 & 0 \\ 2 & 6 & 1 & 8 & -1 & 0 \\ 0 & 0 & 1 & 4 & 4 & 0 \end{array} \right] \xrightarrow{R_2 \rightarrow R_2 - 2R_1} \left[\begin{array}{ccccc|c} 1 & 3 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 4 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 & 0 \end{array} \right] \xrightarrow{R_3 \rightarrow R_3 - R_2} \left[\begin{array}{ccccc|c} 1 & 3 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array} \right]$$

– **2 – choosing free parameters** –

Work from the bottom row up!

- The bottom row declares $x_5 = 0$.
- The middle row still has 2 unknowns! We can set either x_3 or x_4 to a *free parameter*, here we'll choose $x_4 = \alpha$ for $\alpha \in \mathbb{R}$. Then by plugging in we find $x_3 + 4\alpha - 0 = 0 \rightarrow x_3 = -4\alpha$.
- The top row also has 2 unknowns x_1 and x_2 ! Here we choose $x_2 = \beta$ for $\beta \in \mathbb{R}$. This lets us solve for x_1 from the top $x_1 + 3\beta + 0 + 2\alpha + 0 = 0 \rightarrow x_1 = -2\alpha - 3\beta$.

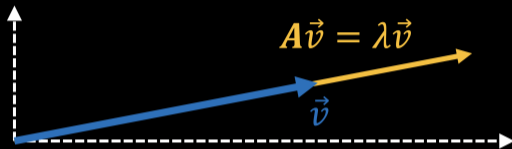
– **3 – formalizing span describing the space** –

Lastly we factor \vec{x} by α and β to identify the span:

$$\vec{x} = \begin{bmatrix} -2\alpha - 3\beta \\ \beta \\ -4\alpha \\ \alpha \\ 0 \end{bmatrix} = \begin{bmatrix} -2\alpha \\ 0 \\ -4\alpha \\ \alpha \\ 0 \end{bmatrix} + \begin{bmatrix} -3\beta \\ \beta \\ 0 \\ 0 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \text{Null}(\mathbf{A}) = \text{span} \left\{ \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

④ Eigenvalue Problems – Conceptual basics

Definition An eigenvalue λ and corresponding eigenvector $\vec{v} \in \mathbb{R}^n$ (for $\vec{v} \neq \vec{0}$) for the square matrix $\mathbf{A}^{n \times n}$ satisfy the *eigenvalue problem*: $\mathbf{A}\vec{v} = \lambda\vec{v}$.



Casually stated... “For some matrix \mathbf{A} , there are some vectors \vec{v} for which $\mathbf{A}\vec{v}$ looks just like a scaling of \vec{v} by some number λ ”.

Ground rules –

- $\mathbf{A}^{n \times n}$ has between 1 and n eigenvalues λ .
Algebraically there are ALWAYS n eigenvalues, but some can repeat, so like $\lambda_1 = \lambda_2$.
- Each eigenvalue λ has a nonempty *eigenspace* of vectors satisfying $\mathbf{A}\vec{v} = \lambda\vec{v}$, which is ultimately the null space of $(\mathbf{A} - \lambda\mathbf{I})$.

NOTE! This is because $\mathbf{A}\vec{v} = \lambda\vec{v} \equiv \lambda\mathbf{I}\vec{v} \rightarrow \mathbf{A}\vec{v} - \lambda\mathbf{I}\vec{v} = \vec{0} \rightarrow (\mathbf{A} - \lambda\mathbf{I})\vec{v} = \vec{0}$.

NOTE! To get eigenvectors, we must FIRST identify the eigenvalues.

④ Eigenvalue Problems – Machinery

Solving for eigenvalues The key lies in rearranging $\mathbf{A}\vec{v} = \lambda\vec{v} \rightarrow (\mathbf{A} - \lambda\mathbf{I})\vec{v} = \vec{0}$.

- This tells us we've found a λ once $\mathbf{A} - \lambda\mathbf{I}$ has a nonzero null-space.
- Randomly trying different λ 's and row-reducing would be *brutal*.
- So far, we can only compute $\mathbf{A}^{2 \times 2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ eigenvalues λ using this theorem:

$(\mathbf{A} - \lambda\mathbf{I})$ has a nonzero null space **only if** $\det(\mathbf{A} - \lambda\mathbf{I}) := (a - \lambda)(d - \lambda) - bc \equiv 0$.

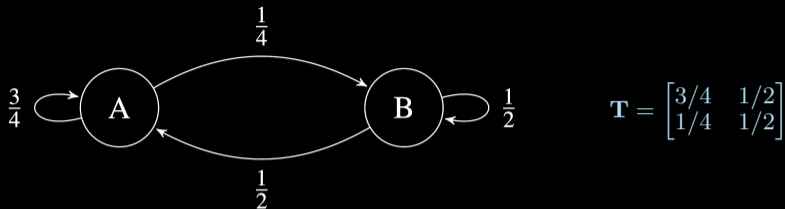
- $\det(\mathbf{A} - \lambda\mathbf{I})$ is called the *characteristic* polynomial of \mathbf{A} , and one can expand this expression into a form $\lambda^2 + \alpha\lambda + \beta = 0$ which is solved using the quadratic formula, $\lambda = -\frac{1}{2}\alpha \pm \frac{1}{2}\sqrt{\alpha^2 - 4\beta}$. Note there are always two roots!

Eigenvalues can be complex if $\alpha^2 - 4\beta < 0$ (recall $z = x + iy \in \mathbb{C}$, and $i^2 = -1$).

Just notice $\sqrt{\alpha^2 - 4\beta} = \sqrt{(-1)(4\beta - \alpha^2)} = i\sqrt{4\beta - \alpha^2}$.

Solving for eigenvectors Once the eigenvalues are known (or given), the eigenspace for each λ can be identified as the null space of $\mathbf{A} - \lambda\mathbf{I}$, which is something we've solved often using Gaussian elimination!

⑤ Transition Matrices



Definition Flow problems involve some initial state vector $\vec{x}[1]$, in the case above

$\vec{x}[1] = \begin{bmatrix} x_A[1] \\ x_B[1] \end{bmatrix}$, and the state after one iteration becomes $\vec{x}[2] = \mathbf{T}\vec{x}[1]$ where \mathbf{T} is the *transition matrix*.

- If the columns of \mathbf{T} each sum to 1, the system is *conservative* (eg. $x_a[n] + x_b[n] = x_a[n+1] + x_b[n+1]$).
- The inverse of \mathbf{T} (noted as $\mathbf{T}^{-1}\vec{x}[n+1] = \vec{x}[n]$) does *not* have the same network diagram with flipped arrows, and it may not correspond to a physical process.
- In the limit as $n \rightarrow \infty$, we approach the *steady state* solution \vec{x}_f of the system $\vec{x}_f = \lim_{n \rightarrow \infty} \mathbf{T}^n \vec{x}[1]$. This steady state should satisfy $\mathbf{T}\vec{x}_f = \vec{x}_f$.

⑤ Transition Matrices

Let us focus on the transition matrix problem now for a matrix \mathbf{M} (which may not be conservative) with eigenvalues in mind...

$$\mathbf{M}(\mathbf{M}\vec{v}) = \mathbf{M}(\lambda\vec{v}) = \lambda\mathbf{M}\vec{v} = \lambda^2\vec{v}$$

Eigenvalue relations –

Suppose $\mathbf{M}^{3 \times 3}$ has 3 eigenvalues $|\lambda_1| < 1$, $\lambda_2 = 1$, and $|\lambda_3| > 1$, and consider an initial state $\vec{x}[1] = \alpha\vec{v}_1 + \beta\vec{v}_2 + \gamma\vec{v}_3$. Expanding a vector into eigenvectors is *very powerful*...

$$\begin{aligned}\vec{x}[2] &= \mathbf{M}\vec{x}[1] = \alpha(\mathbf{M}\vec{v}_1) + \beta(\mathbf{M}\vec{v}_2) + \gamma(\mathbf{M}\vec{v}_3) \\ &= (\lambda_1\alpha)\vec{v}_1 + \beta\vec{v}_2 + (\lambda_3\gamma)\vec{v}_3\end{aligned}$$

$$\vec{x}[n+1] = \mathbf{M}(\mathbf{M}(\dots(\mathbf{M}\vec{x}[1])\dots)) = \mathbf{M}^n\vec{x}[1] = \alpha(\lambda_1)^n\vec{v}_1 + \beta\vec{v}_2 + \gamma(\lambda_3)^n\vec{v}_3$$

· We find our steady-state vector \vec{x}_f depends on our initial state!

$$\vec{x}_f = \lim_{n \rightarrow \infty} \vec{x}[n+1] = \beta\vec{v}_2 + \gamma(\infty)\vec{v}_3$$

⑥ Proofs

Proofs for vectors, matrices, or eigenvalue properties can be extremely helpful in organizing concepts in your head, and being able to formalize a proof is a valuable skill to learn early!

Claim: Given a matrix $\mathbf{A}^{n \times n}$ with unequal eigenvalues $\lambda_1 \neq \lambda_2$ and corresponding eigenspaces V_1, V_2 , the bases of V_1 and V_2 are linearly independent from each other.

Proof by Contradiction:

- Suppose the bases are linearly dependent on each other, in which case one could select $\vec{v}_1 \in V_1$ and $\vec{v}_2 \in V_2$ such that $\vec{v}_1 = \alpha \vec{v}_2$ for some $\alpha \in \mathbb{R}$.
- **Note:** Every vector space contains the zero vector $\vec{0}$ by definition, thus we may exclude the cases where $\vec{v}_2 = \vec{0}$ and where $\alpha = 0$, since either claim would imply $\vec{v}_1 = \vec{0}$ which is in V_1 trivially.

$$\begin{aligned}\vec{0} &= \mathbf{M} \vec{v}_1 - \mathbf{M} \vec{v}_1 \\ &= \lambda_1 \vec{v}_1 - \mathbf{M}(\alpha \vec{v}_2) \\ &= \lambda_1(\alpha \vec{v}_2) - \alpha(\mathbf{M}\vec{v}_2) \\ &= \alpha \lambda_1 \vec{v}_2 - \alpha(\lambda_2 \vec{v}_2) = \alpha(\lambda_1 - \lambda_2) \vec{v}_2\end{aligned}$$

- The only possible way we can reconcile our result is if $\lambda_1 = \lambda_2$, but that contradicts our setup! Thus there is no way to select a $\vec{v}_1 \in V_1$ and $\vec{v}_2 \in V_2$ so that they're linearly dependent; we've proven the claim!! \square

⑦ General Tips – For a smooth exam experience

Keep Moving –

If you get stuck on a question, move on and then come back. Try to spend your time answering things you know well, and feel free to skip parts if you get stuck. You can always go back to the tougher parts if you have time.

Show your work –

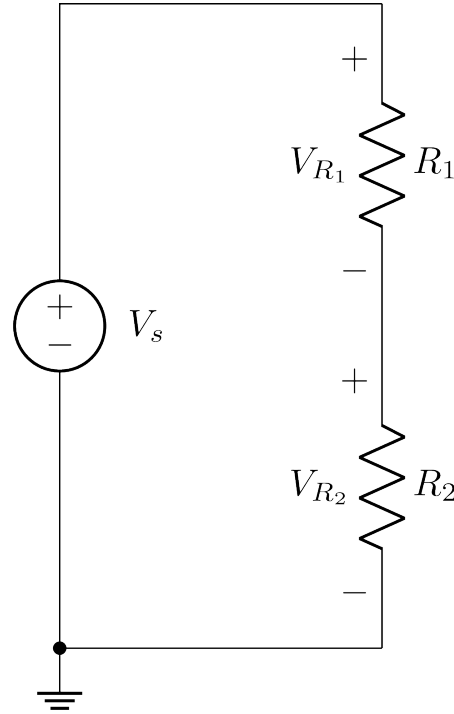
Be aware that you can only assume things that were shown on homework and in class. You cannot reference some theorem that was not covered.

Good Luck :)

Voltage Divider

$$V = IR$$

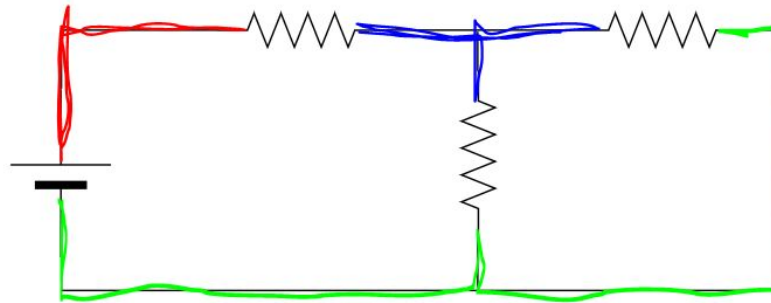
Note, the resistors MUST be in SERIES to use this equation



$$V_{R_2} = V_s \frac{R_2}{R_1 + R_2}$$

Identifying nodes

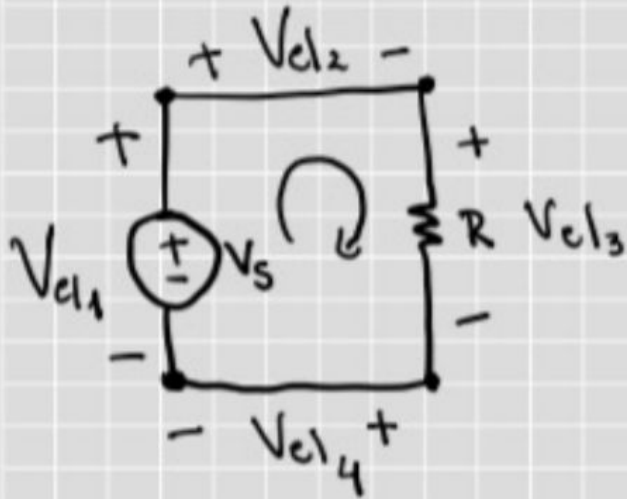
- A node is anywhere in a circuit that is at the same potential.
- Wires don't change your potential, only crossing components do that
- Start somewhere and start coloring until you hit a component. Turn around and color in a different direction. When you've colored everywhere, start somewhere else with a different color



Red = 1 node
Blue = 1 node
Green = 1 node
3 nodes total!

KVL/KCL

Sum of Voltages across the elements in a loop equal zero

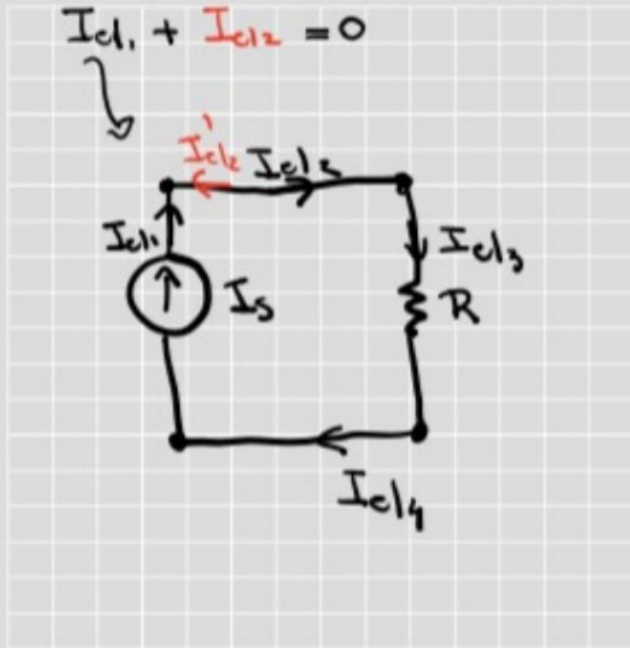


$$V_{e1} - V_{e2} - V_{e3} - V_{e4} = 0$$

$$\begin{array}{l} + \\ V_{e1} \\ - \end{array} \quad \begin{array}{l} + \\ - \\ V_{e2} \\ + \\ - \\ V_{e3} \\ + \\ - \\ V_{e4} \end{array}$$

KVL/KCL

The current flowing into any junction must equal the current flowing out



$$I_{c11} = I_{c12}$$

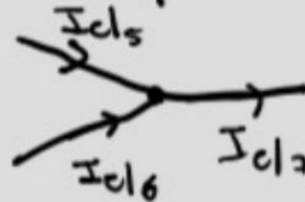
$$I_{c12} = I_{c13}$$

$$I_{c13} = I_{c14}$$

~~$$I_{c14} = I_{c11}$$~~

$$I_{c12} - I_{c11} = 0$$

Example 2:



$$I_{c15} + I_{c16} = I_{c17}$$

Circuit Analysis Steps

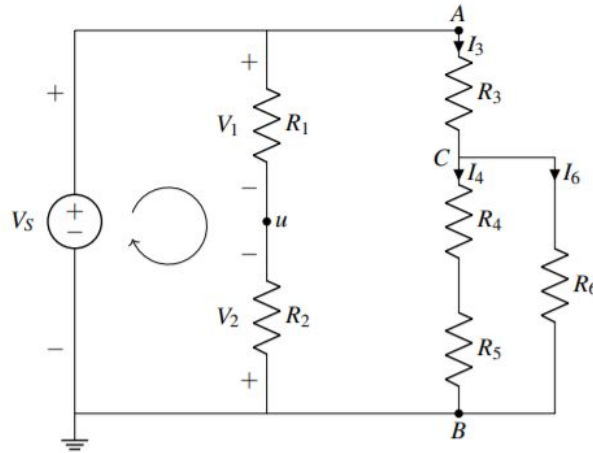
1. Label ground node
2. Label other nodes
3. Label currents through non-wire elements
4. Add +/- labels for elements
5. Set up unknowns and system of equations. Unknowns are your element currents and node potentials.
6. Use KCL to fill in equations
7. Use I-V relationships/voltage potentials of non-wire elements to generate more equations
8. Solve the system!

Circuit Analysis Example

3. Circuit Analysis (18 points)

For the circuit in the following diagram, answer parts (a) – (e).

You should **not** change the labels that are already given in the diagram. If you add any additional labels for your analysis, you should show your labels in the answer sheet for the corresponding part(s).



- (a) (3 points) Redraw the circuit diagram in your answer sheet. Following the passive sign convention, **label** (i) the current I_S through the voltage source V_S , (ii) the current I_1 through the resistor R_1 , and (iii) the voltage V_3 across the resistor R_3 .
- (b) (2 points) **Write the KVL expression** for the loop drawn in the circuit diagram in terms of voltages V_S , V_1 , and V_2 .
- (c) (2 points) **Write the KCL expression** at node C in terms of currents I_3 , I_4 , and I_6 as labeled in the circuit diagram.

Pumps Problem

6. Steady the Traffic (29 points)

- (a) (4 points) Your friend wants to study the flow of traffic around the Bay Area and asks for your help. From her observations, your friend finds that the number of cars in San Francisco, Berkeley, San Jose and Fremont can be represented in the following way:

$$\begin{bmatrix} x_{SF}[n+1] \\ x_B[n+1] \\ x_{SJ}[n+1] \\ x_F[n+1] \end{bmatrix} = \mathbf{A} \begin{bmatrix} x_{SF}[n] \\ x_B[n] \\ x_{SJ}[n] \\ x_F[n] \end{bmatrix} \quad (1)$$

The flow of traffic is represented in the diagram below. **Write the transition matrix \mathbf{A} corresponding to this diagram.**

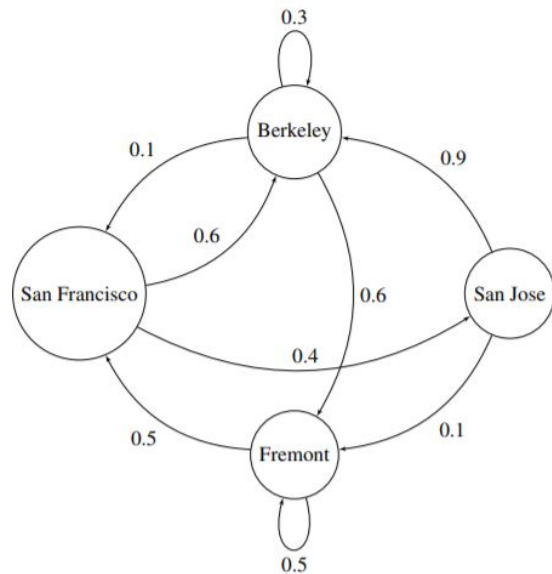


Figure 6.1: A flow diagram to represent how model \mathbf{A} transforms state vector $\vec{x}[n]$.

- (b) (4 points) Your friend takes measurements of the number of cars at the first 3 cities (San Francisco, Berkeley, San Jose) during Thanksgiving weekend and finds the following transition matrix:

$$\mathbf{T} = \begin{bmatrix} 0.25 & 0 & 0.3 \\ 0.75 & 1 & 0.4 \\ 0 & 0 & 0.3 \end{bmatrix}$$

The new state vector is:

$$\vec{x}[n] = \begin{bmatrix} x_{\text{SF}}[n] \\ x_{\text{B}}[n] \\ x_{\text{SJ}}[n] \end{bmatrix}.$$

You are performing some simulations to see how the traffic evolves at each time step. You start your simulation with 200 cars at San Francisco, 150 cars at Berkeley and 100 cars at San Jose. **Calculate the number of cars at each city in the next time step.**

- (c) (5 points) It would be helpful for your simulations to know the eigenvectors of this transition matrix. **Calculate the steady state eigenvector associated with the eigenvalue $\lambda = 1$ for the above matrix \mathbf{T} .**

- (d) (6 points) Next you are interested in investigating the traffic flows during New Year's weekend. Your friend tells you the following information about the transition matrix for this period:

i. The transition matrix is conservative

ii. The eigenvector corresponding to $\lambda = 1$ is $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

iii. All other eigenvalues $|\lambda_i| < 1$

If the initial state vector is $\vec{x}[0] = \begin{bmatrix} 30 \\ 50 \\ 20 \end{bmatrix}$, **what steady state will this system converge to?**

- (e) (10 points) Next, for a new transition matrix \mathbf{S} , you investigate the traffic flow of commuters throughout the year. You calculate the eigenvalues $\lambda_1 = 1$, $\lambda_2 = 4$, and $\lambda_3 = 0.25$ with corresponding eigenvectors:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

For the given values of $\vec{x}[0]$, write down whether the system will converge to a non-zero steady state, decay to zero or keep growing infinitely.

i. If $\vec{x}[0] = \begin{bmatrix} 350 \\ 50 \\ 300 \end{bmatrix}$

ii. If $\vec{x}[0] = \begin{bmatrix} 15 \\ 10 \\ 25 \end{bmatrix}$

7. A Problem N(o)-body Can Solve (24 points)

An N-body simulation is a method of modeling the interactions between a set of particles, and it is commonly implemented in an astrophysics context to study the movements of celestial bodies and galactic formation under the constraints of gravitational forces. In each timestep, the core algorithm iterates through particle pairs to calculate the force on each particle and update its current position.

As part of your work in a research lab, you are developing an efficient N-body simulation for the solar system that exploits computationally fast operations on matrices to speed up runtime (good thing you're taking EECS16A!). You represent each body as a vector in 3D space:

$$\vec{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

You calculate that the position of one particular body – Earth – is updated in the following way during every timestep:

- $x[t + 1] = 0.5x[t] + 0.7y[t] + 0.3z[t]$
- $y[t + 1] = 0.6y[t] + 0.1z[t]$
- $z[t + 1] = 0.3x[t] + 0.2y[t] + z[t]$

- (a) (4 points) After one timestep, at time $t = 1$, Earth is located at $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$. You want to calculate the position

of Earth at $t = 0$. **Formulate this problem as a matrix-vector equation in the form $A\vec{x} = \vec{b}$.** You do not need to solve for Earth's position.

Solution: The system of equations representing the position of Earth between timesteps t and $t + 1$ can be substituted with $t = 0$ and $t + 1 = 1$ respectively. We can then rewrite the system so $\vec{x} = \vec{p}_{\text{Earth}}[0]$ (the unknown initial position of Earth), $\vec{b} = \vec{p}_{\text{Earth}}[1]$ (the known position of Earth at $t = 1$), and A represents the coefficients relating $\vec{p}_{\text{Earth}}[1]$ to $\vec{p}_{\text{Earth}}[0]$:

$$\mathbf{A}\vec{p}_{\text{Earth}}[0] = \vec{p}_{\text{Earth}}[1]$$
$$\begin{bmatrix} 0.5 & 0.7 & 0.3 \\ 0 & 0.6 & 0.1 \\ 0.3 & 0.2 & 1 \end{bmatrix} \begin{bmatrix} x[0] \\ y[0] \\ z[0] \end{bmatrix} = \begin{bmatrix} x[1] \\ y[1] \\ z[1] \end{bmatrix}$$
$$\begin{bmatrix} 0.5 & 0.7 & 0.3 \\ 0 & 0.6 & 0.1 \\ 0.3 & 0.2 & 1 \end{bmatrix} \begin{bmatrix} x[0] \\ y[0] \\ z[0] \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix},$$

substituting for the Earth's known position at $t = 1$.

- (b) (6 points) You have determined that Neptune's position in each timestep is updated according to the following matrix:

$$\mathbf{N} = \begin{bmatrix} 1 & 0.2 & 0 \\ 0 & -0.2 & 0.1 \\ -1 & 0 & 0.1 \end{bmatrix}$$

You let the simulation run in the background for a while, and at $t = n$, Neptune is located at $\begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix}$. You then realize that you've forgotten to record position data since you started! **Is it possible to recover Neptune's position uniquely at $t = n - 1$? If it is, use Gaussian elimination to find the inverse of \mathbf{N} , \mathbf{N}^{-1} .**

Solution: We can use Gaussian elimination to calculate N^{-1} by augmenting the N matrix with the identity matrix, and row-reducing N until it is transformed into the identity matrix:

$$\begin{aligned}
 & \left[\begin{array}{ccc|ccc} 1 & 0.2 & 0 & 1 & 0 & 0 \\ 0 & -0.2 & 0.1 & 0 & 1 & 0 \\ -1 & 0 & 0.1 & 0 & 0 & 1 \end{array} \right] \xrightarrow{R_3=R_3+R_1} \left[\begin{array}{ccc|ccc} 1 & 0.2 & 0 & 1 & 0 & 0 \\ 0 & -0.2 & 0.1 & 0 & 1 & 0 \\ 0 & 0.2 & 0.1 & 1 & 0 & 1 \end{array} \right] \xrightarrow{R_3=R_3+R_2} \\
 & \left[\begin{array}{ccc|ccc} 1 & 0.2 & 0 & 1 & 0 & 0 \\ 0 & -0.2 & 0.1 & 0 & 1 & 0 \\ 0 & 0 & 0.2 & 1 & 1 & 1 \end{array} \right] \xrightarrow{R_3=5R_3} \left[\begin{array}{ccc|ccc} 1 & 0.2 & 0 & 1 & 0 & 0 \\ 0 & -0.2 & 0.1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 5 & 5 & 5 \end{array} \right] \xrightarrow{R_1=R_1+R_2} \\
 & \left[\begin{array}{ccc|ccc} 1 & 0 & 0.1 & 1 & 1 & 0 \\ 0 & -0.2 & 0.1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 5 & 5 & 5 \end{array} \right] \xrightarrow{R_2=10R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0.1 & 1 & 1 & 0 \\ 0 & -2 & 1 & 0 & 10 & 0 \\ 0 & 0 & 1 & 5 & 5 & 5 \end{array} \right] \xrightarrow{R_2=R_2-R_1} \\
 & \left[\begin{array}{ccc|ccc} 1 & 0 & 0.1 & 1 & 1 & 0 \\ 0 & -2 & 0 & -5 & 5 & -5 \\ 0 & 0 & 1 & 5 & 5 & 5 \end{array} \right] \xrightarrow{R_2=-\frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0.1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2.5 & -2.5 & 2.5 \\ 0 & 0 & 1 & 5 & 5 & 5 \end{array} \right] \xrightarrow{R_1=R_1-\frac{1}{10}R_3} \\
 & \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.5 & 0.5 & -0.5 \\ 0 & 1 & 0 & 2.5 & -2.5 & 2.5 \\ 0 & 0 & 1 & 5 & 5 & 5 \end{array} \right]
 \end{aligned}$$

The N^{-1} matrix we can use to recover Neptune's previous positions is:

$$N^{-1} = \begin{bmatrix} 0.5 & 0.5 & -0.5 \\ 2.5 & -2.5 & 2.5 \\ 5 & 5 & 5 \end{bmatrix}$$

(c) (8 points) Additionally, you have the following matrix for Pluto:

$$\begin{bmatrix} 0.1 & 0 & 0.1 \\ 0.1 & 0.1 & 0.2 \\ 0.2 & 0.1 & 0.3 \end{bmatrix}$$

If Pluto is positioned at some unspecified $\begin{bmatrix} x[0] \\ y[0] \\ z[0] \end{bmatrix}$ at $t = 0$, **are there any points in \mathbb{R}^3 space that you cannot reach at $t = 1$? If so, what is the subspace that Pluto can be located in?**

Note: You do not have to provide rigorous justification.

Solution: Performing Gaussian elimination on Pluto's update matrix reveals that the rows (and columns) are linearly dependent, indicating that the columns cannot span \mathbb{R}^3 .

$$\begin{bmatrix} 0.1 & 0 & 0.1 \\ 0.1 & 0.1 & 0.2 \\ 0.2 & 0.1 & 0.3 \end{bmatrix} \xrightarrow{R_2=R_2-R_1} \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0 & 0.1 & 0.1 \\ 0.2 & 0.1 & 0.3 \end{bmatrix} \xrightarrow{R_3=R_3-2R_1} \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0 & 0.1 & 0.1 \\ 0 & 0.1 & 0.1 \end{bmatrix} \xrightarrow{R_3=R_3-R_2} \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0 & 0.1 & 0.1 \\ 0 & 0 & 0 \end{bmatrix}$$

Because one row was reduced to 0's during the process of row-reduction, the subspace Pluto's position can be in at $t = 1$ is two-dimensional. We can select any two column vectors from the matrix to represent the span of this subspace; a possible solution is:

$$\text{span} \left\{ \begin{bmatrix} 0.1 \\ 0.1 \\ 0.2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0.1 \\ 0.1 \end{bmatrix} \right\}$$

- (d) (6 points) After running your simulation repeatedly, you notice that with the current update matrices you have entered, Venus and Mars are all moving within the same 2D orbital plane. You refer back to your calculations, but you notice there is a smudge obscuring one element of matrix \mathbf{M} :

$$\mathbf{V} = \begin{bmatrix} 0.3 & 0.4 & 0.1 \\ 0 & 0.7 & 0.7 \\ 0.7 & 1.1 & 0.4 \end{bmatrix}, \mathbf{M} = \begin{bmatrix} 0.4 & 0.6 & 0.2 \\ -1.4 & 0 & 1.4 \\ 0.6 & m_{32} & 0.8 \end{bmatrix}$$

\mathbf{V} and \mathbf{M} are the update matrices for Venus and Mars respectively. **Fill in the missing matrix element (denoted by " m_{32} ") in a way that would explain the behavior of these 2 planets.**

Solution: $m_{32} = 1.4$ For both planets to be moving on the same 2D plane within \mathbb{R}^3 , all three update matrices must contain exactly two linearly independent column vectors that span the plane. These spans must be equivalent between the independent column vectors for V and M .

We denote C_i as the i th column of a mentioned matrix. By inspection, we see that $C_3 = C_2 - C_1$ for V , $C_1 = C_2 - C_3$ for M . This implies that $? = 0.8 + 0.6 = 1.4$ in M . Replacing the $?$ with this value would create a scenario in which Venus and Mars can reach any position in the 2D plane defined by the span below:

$$\text{span} \left\{ \begin{bmatrix} 0.3 \\ 0 \\ 0.7 \end{bmatrix}, \begin{bmatrix} 0.4 \\ 0.7 \\ 1.1 \end{bmatrix} \right\}$$

8. Proof (10 points)

You are told that a $\mathbf{A} \in \mathbb{R}^{2 \times 2}$ is a conservative transition matrix. **Prove that it has an eigenvalue of $\lambda = 1$.**

Solution: First, we must start with what we know, which is that $A \in \mathbb{R}^{2 \times 2}$ is a conservative transition matrix.

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

where $a + c = 1$ and $b + d = 1$.

Here are two possible ways you can prove this:

- i The more challenging way is to solve directly for the eigenvalues of A using the $\det(A - \lambda I) = 0$ formula and substituting in $b = 1 - d$ and $c = 1 - a$ where appropriate.

$$\begin{aligned} \det(A - \lambda I) &= (a - \lambda)(d - \lambda) - bc \\ &= ad - (a + d)\lambda + \lambda^2 - (1 - a)(1 - d) \\ &= ad - (a + d)\lambda + \lambda^2 - (1 - a - d + ad) \\ &= \lambda^2 - (a + d)\lambda + (-1 + a + d) \\ &= (\lambda - 1)(\lambda - (-1 + a + d)) \end{aligned}$$

From here we can see that $\lambda = 1$ is a root of this characteristic equation.

- ii A quicker method that we may notice is that we are only looking to show that $\lambda = 1$ is an eigenvalue of A , not find all eigenvalues. If we assume that A does have eigenvalue of 1, then we know that $\det(A - I) = 0$. Using our knowledge that $b = 1 - d$ and $c = 1 - a$ we can show this is true.

$$\begin{aligned} \det(A - I) &= (a - 1)(d - 1) - bc \\ &= (a - 1)(d - 1) - (1 - a)(1 - d) \\ &= 0 \end{aligned}$$