EECS 16A Designing Information Devices and Systems I
Spring 2016 Elad Alon, Babak Ayazifar **Final**

Exam location: Solutions

PRINT your student ID: ______________________________________

PRINT AND SIGN your name: ________________, ________________ ____________________
(last) (first) (signature)

PRINT your Unix account login: ee16a-____________

PRINT your discussion section and GSI (the one you attend): ______________________________________

PRINT your lab GSI (the one you attend): ______________________________________

Name of the person to your left: ______________________________________

Name of the person to your right: ______________________________________

Name of the person in front of you: ______________________________________

Name of the person behind you: ______________________________________

Section 0: Pre-exam questions (3 points)

1. **What are you looking forward to the most this summer? (1 pt)**

2. **What was your favorite lab/homework problem/discussion WS in 16A? What did you like the most about it? (2 pts)**

Do not turn this page until the proctor tells you to do so. You can work on Section 0 above before time starts.

Section 1: Context-free questions

Unless told otherwise, you must show work to get credit. There will be very little partial credit given in this section.

3. DeterminAnts (5 pts)

Compute the determinant of $\mathbf{A} = \begin{bmatrix} 2 & 6 & 2 \\ 1 & 2 & 1 \\ 2 & 6 & 4 \end{bmatrix}$.

Solutions:

We row reduce the following matrix into upper triangular form:

$$\begin{bmatrix} 2 & 6 & 2 \\ 1 & 2 & 1 \\ 2 & 6 & 4 \end{bmatrix}$$

$$\xrightarrow[-0.5\times(1)+(2)]{} \begin{bmatrix} 2 & 6 & 2 \\ 0 & -1 & 0 \\ 2 & 6 & 4 \end{bmatrix}. \text{ Elementary row operation matrix: } E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[-1\times(1)+(3)]{} \begin{bmatrix} 2 & 6 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}. \text{ Elementary row operation matrix: } E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Hence, we have

$$E_2 \times E_1 \times \begin{bmatrix} 2 & 6 & 2 \\ 1 & 2 & 1 \\ 2 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

giving us

$$det(E_2) \times det(E_1) \times det(\begin{bmatrix} 2 & 6 & 2 \\ 1 & 2 & 1 \\ 2 & 6 & 4 \end{bmatrix}) = det(\begin{bmatrix} 2 & 6 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix})$$

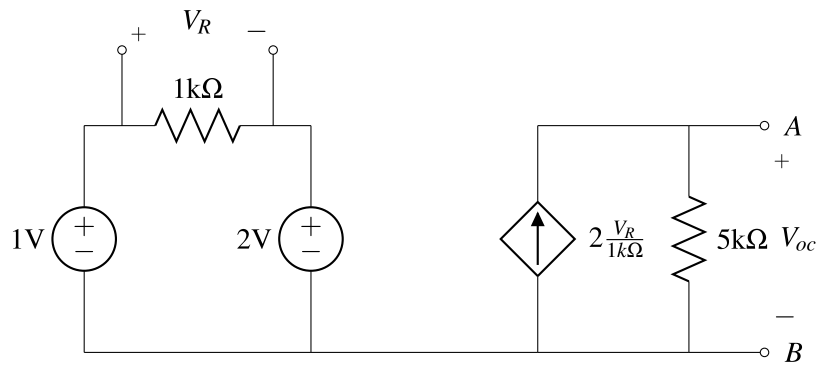
$$1 \times 1 \times det(\begin{bmatrix} 2 & 6 & 2 \\ 1 & 2 & 1 \\ 2 & 6 & 4 \end{bmatrix}) = -4$$

We have

$$det(\begin{bmatrix} 2 & 6 & 2 \\ 1 & 2 & 1 \\ 2 & 6 & 4 \end{bmatrix}) = -4.$$

4. Revenge of the Boxes (10 pts)

Consider the following circuit:



(a) Find the open-circuit voltage V_{oc} between nodes A and B.

Solutions: For convenience, we choose B to be "0V" or ground.
Using KVL on the left side:

$$0 = 1V - V_r - 2V$$
$$V_r = -1V$$

With this, we use KCL at A:

$$2\frac{V_r}{1k\Omega} - \frac{V_{oc}}{5k\Omega} = 0$$
$$\frac{V_{oc}}{5k\Omega} = 2\frac{-1V}{1k\Omega}$$
$$V_{oc} = -10V$$

$$\boxed{V_{oc} = -10V}$$

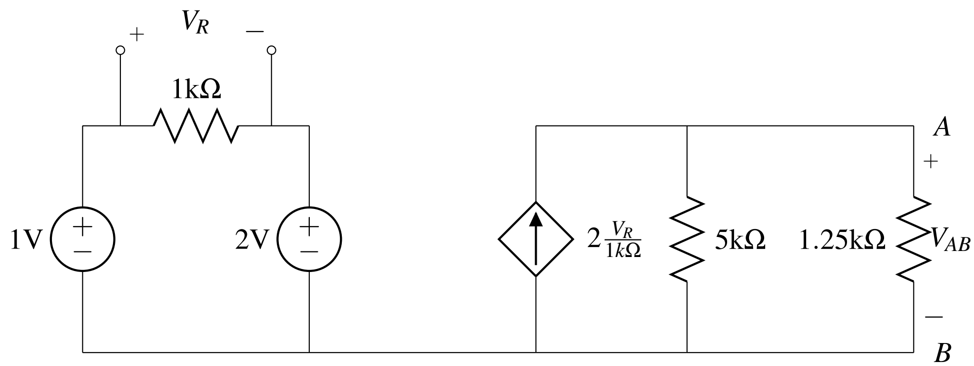
(b) Find the short-circuit current I_{sc} between the same two points.

Solutions: Shorting the bottom 5kΩ resistor out attaches the dependent current source to ground, and so all the current from the current source travels through the shorted out branch in this case.

$$\begin{aligned} I_{sc} &= 2\frac{V_r}{1k\Omega} \\ &= 2(-1) \text{ mA} \end{aligned}$$

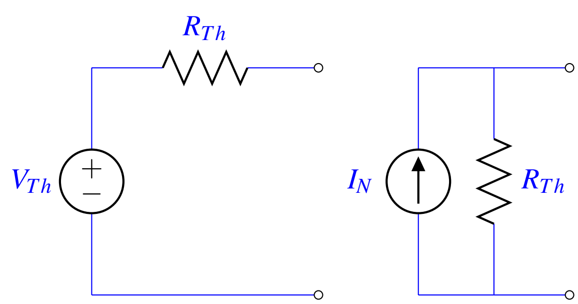
$\boxed{I_{sc} = -2\text{mA}}$ Mind your orders of magnitude!

(c) Now assume that V_{oc} and I_{sc} are as written in the boxes below. (Note that these may or may not be the correct answer to the previous parts.) If you attach a 1.25 kΩ resistor across nodes A and B (as shown below), what will be the value of the voltage V_{AB} (also defined below)?



V_{oc}	$-10V$
I_{sc}	$-2mA$

Solutions: For now, we'll work symbolically with V_{oc} and I_{sc}.
With V_{oc} and I_{sc}, you can find the Thevenin and Norton equivalents of the circuit:



where $R_{Th} = \frac{V_{oc}}{I_{sc}}$. Plugging in a resistor R at the appropriate location, you end up with a voltage or current divider for Thevenin and Norton equivalent circuits, respectively. For completeness, we'll solve for both:

Thevenin equivalent:

$$V_{AB} = V_{oc}\left(\frac{R_{AB}}{R_{AB}+R_{Th}}\right)$$

Norton equivalent:

$$V_{AB} = I_{sc}\left(\frac{1}{R_{AB}}+\frac{1}{R_{Th}}\right)^{-1}$$

Given $V_{oc} = I_{sc}R_{Th}$, these come out to be equivalent expressions. Plugging in the numbers from above:

$$R_{Th} = \frac{-10\text{V}}{-2\text{mA}} = 5\text{k}\Omega$$

$$V_{AB} = -10\text{V}\left(\frac{1.25k\Omega}{1.25k\Omega+5k\Omega}\right) = -2\text{V}$$

$\boxed{V_R = -2\text{V}}$

5. Full Bases (5 pts)

Let the orthonormal vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ be a basis $\mathscr{B}$. Provide a simple (symbolic) expression for the vector $\vec{x}$ (given below) as coordinates in the basis $\mathscr{B}$.

$$\vec{x} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

Solutions: Let the matrix U be:

$$U = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}$$

Since the columns are orthonormal, then $U^{-1} = U^{\top}$.

$$Ux_{[\mathscr{B}]} = x$$

$$x_{[\mathscr{B}]} = U^{-1}x = U^{\top}x$$

6. Bulletproof (10 pts)

(a) Let **A** and **B** be diagonalizable n x n matrices. Show that if **A** and **B** have the same set of eigenvectors, then **AB** = **BA**.

Solutions: Use the eigendecomposition. Since **A** and **B** have the same eigenvectors, their cooresponding V matrices will be the same.

$$A = V\Lambda_a V^{-1} \qquad B = V\Lambda_b V^{-1}$$

Then through expansion:

$$\begin{aligned} AB &= V\Lambda_a V^{-1} V\Lambda_b V^{-1} \\ &= V\Lambda_a \Lambda_b V^{-1} \\ &= V\Lambda_b \Lambda_a V^{-1} \\ &= V\Lambda_b V^{-1} V\Lambda_a V^{-1} \\ &= BA \end{aligned}$$

$\Lambda_a \Lambda_b$ and $\Lambda_b \Lambda_a$ give the same result and $VV^{-1} = V^{-1}V = I$.
Therefore, the multiplication is commutative.

(b) Consider a symmetric n x n matrix **A**. Show that if $\vec{x}^\top \mathbf{A}\vec{x} > 0$ for all $\vec{x} \neq \vec{0}$, then all eigenvalues of **A** are positive. Fill in the blanks to complete the proof:

Let $\vec{x}$ be a/an ____________________ of **A**.

$$\mathbf{A}\vec{x} = \_\_\_\_\_$$

$$x^\top \mathbf{A}x = \_\_\_\_\_\_\_\_\_\_ = \_\_\_\_\_\_\_\_\_\_ (1)$$

We can rewrite __________ (1) with a summation: _______________ (2).
Expression (2) is equivalent to $\vec{x}^\top \mathbf{A}\vec{x}$, and thus must be stricly positive. Explain in a single sentence why this means that any eigenvalue of **A** must be positive.

Solutions:

Let $\vec{x}$ be an eigenvector of **A**.

$$\mathbf{A}\vec{x} = \lambda\vec{x}$$

$$\vec{x}^\top \mathbf{A}\vec{x} = \vec{x}^\top \lambda \vec{x} = \lambda \vec{x}^\top \vec{x}$$
$$\lambda \vec{x}^\top \vec{x} = \lambda \sum x_i^2$$

The right hand side is strictly positive, i.e. always positive. Since the summation is always positive, then λ must always be positive. This holds true for all eigenpairs of $\mathbf{A}$.

Section 2: Free-form Problems

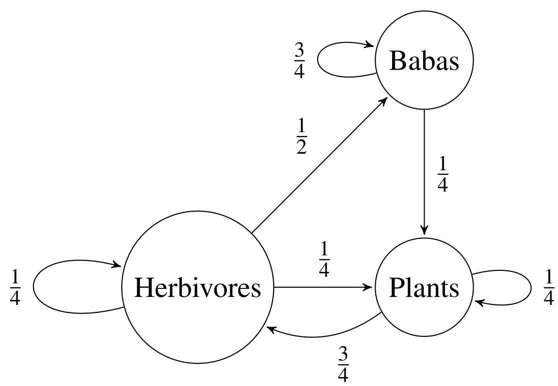
7. EEcology (22 pts)

In this problem we will study the evolution of a hypothetical ecosystem. Within an ecosystem, one can have all sorts of classifications of life forms. For example, your model could initially include only Herbivores and Plants. We will use what's called a "replacement model" to represent the dynamics of the ecosystem. Specifically, at every time step, some fraction of Plants is "eaten" by Herbivores, and the number of Herbivores increases by the number of Plants that were eaten. In addition, some fraction of Herbivores die, and, in the context of this problem, they become Plants. Finally, for both types of organisms, some fraction of the organisms remain in their current state. These same rules apply to new organisms introduced in the problem later on.

Note that throughout this problem, the total number of organisms remains the same. In other words, no organism can leave or enter the system.

(a) A new species of killer snake, the Baba Konstrictor, slithers its way into this peaceful environment. Babas are incredibly intelligent and hunt Herbivores. The new ecology network is depicted below.

What is the state transition matrix for this new ecosystem where $\vec{x}[k] = \begin{bmatrix} \text{\# of Herbivores}[k] \\ \text{\# of Plants}[k] \\ \text{\# of Babas}[k] \end{bmatrix}$?



Solutions: $\begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{3}{4} \end{bmatrix}$

(b) If we start with 100 of each organism in the new ecosystem, how many organisms of each type will there be after an infinite number of time steps?

Solutions: The total number of organisms in the system will remain unchanged. We will always have 300 organisms in the system.

After infinite steps, we should be in steady state. Remember that, for a steady state to occur, $A\vec{x} = \vec{x}$, and $\vec{x}$ is the steady state we are looking for. This corresponds to the eigenvector when $\lambda = 1$ (i.e. the importance scores).

Consider the previously found state transition matrix A and the steady state $\vec{x}$.

$$(A - I)x = \begin{bmatrix} \frac{-3}{4} & \frac{3}{4} & 0 \\ \frac{1}{4} & \frac{-3}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{-1}{4} \end{bmatrix} x = 0$$

Solving for the nullspace:

$$\begin{bmatrix} \frac{-3}{4} & \frac{3}{4} & 0 \\ \frac{1}{4} & \frac{-3}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{-1}{4} \end{bmatrix} \sim \begin{bmatrix} -3 & 3 & 0 \\ 1 & -3 & 1 \\ 2 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 1 & -3 & 1 \\ 2 & 0 & -1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

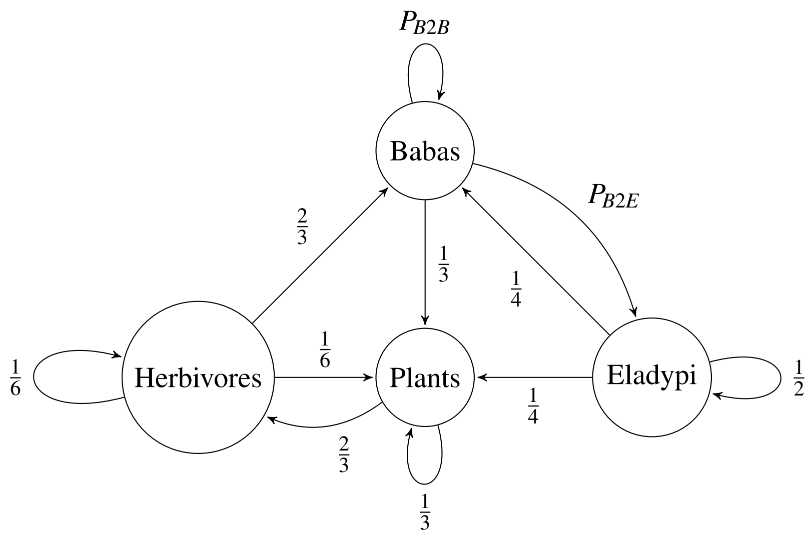
$$x_3 \text{ is free. } \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Since we are solving for steady state fractions within the system, we have to normalize this eigenvector. Normalizing in this case means making the sum of the vector be 1, so we divide each element in the vector by the sum of the vector. The normalized steady-state fractions are $\vec{x} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{2} \end{bmatrix}$

Note that the total number of organisms in the system is 300, so the final number of each type of organism is a fraction of 300, not a fraction of the original 100. Therefore, the number of each organism at steady state will be:

$$\lim_{n \to \inf} \vec{x}[n] = \begin{bmatrix} 75 \\ 75 \\ 150 \end{bmatrix}$$

(c) After a few million years, an aquatic-based lifeform, the Eladypus, evolves a set of legs and enters the ecosystem. Having spent many years feasting on eels and sea snakes, the Eladypus has a diet restricted to Babas in this new environment. The new ecosystem can be modeled by the diagram shown below. Note that we don't know yet know what fraction of the Babas is eaten by the Eladypi, but we will define that fraction as having a value of p_{B2E} (as indicated in the diagram). Given the rest of the constants indicated in the diagram below, as a function of p_{B2E}, what must be the value of p_{B2B} (i.e., at each time step, the fraction of Babas that remain in their current state)?



Solutions: The sum of the transitions leaving Babas have to sum to 1.

$$\frac{1}{3} + P_{B2B} + P_{B2E} = 1$$

$$P_{B2B} = \frac{2}{3} - P_{B2E}$$

(d) It's "Eladypi Week" on the Discovery Channel and we're hooked to the television trying to learn as much as we can about this unique creature. We learn that an ideal ecosystem has 350 Eladypi. Currently, we observe 600 Plants, 600 Herbivores, 300 Babas, and 400 Eladypi in our ecosystem.
What is the fraction of Babas that the Eladypi can eat in order for the ecosystem to reach the ideal state after one transition? That is, what is the value of p_{B2E} that enables the ecosystem to reach the ideal state (in terms of the number of Eladypi) from its current state in one time step? With this value of p_{B2E}, how many Herbivores, Plants, and Babas are there after this single time step?

Solutions: Supposed our state is now $\vec{x}[\text{k}] = \begin{bmatrix} \#\text{ of Herbivores}[k] \\ \#\text{ of Plants}[k] \\ \#\text{ of Babas}[k] \\ \#\text{ of Eladypi}[k] \end{bmatrix}$. Setting the state transition matrix, we get:

$$\begin{bmatrix} \frac{1}{6} & \frac{2}{3} & 0 & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{4} \\ \frac{2}{3} & 0 & P_{B2B} & \frac{1}{4} \\ 0 & 0 & P_{B2E} & \frac{1}{2} \end{bmatrix}$$

We also have our initial state $\vec{x}[0] = \begin{bmatrix} 600 \\ 600 \\ 300 \\ 400 \end{bmatrix}$. We can isolate the last row to get the desired P_{B2E}.

$$300\, P_{B2E} + \frac{1}{2}400 = 350$$

$$P_{B2E} = \frac{1}{2}$$

Using the equation previously found, we can solve for P_{B2B}

$$\begin{aligned} P_{B2B} &= \frac{2}{3} - P_{B2E} \\ &= \frac{2}{3} - \frac{1}{2} \\ &= \frac{1}{6} \end{aligned}$$

We can now solve for the next step

$$\begin{bmatrix} \frac{1}{6} & \frac{2}{3} & 0 & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{4} \\ \frac{2}{3} & 0 & \frac{1}{6} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 600 \\ 600 \\ 300 \\ 400 \end{bmatrix} = \begin{bmatrix} 500 \\ 500 \\ 550 \\ 350 \end{bmatrix}$$

$P_{B2E} = \frac{1}{2}$. In this state, there are 500 Herbivores, 500 Plants, 550 Babas

(e) Given the value of p_{B2E} and the state (i.e., the number of Herbivores, Plants, Babas, and Eladypis after the single time step) you computed in part (d), will the number of Eladypi in the ecosystem remain constant after all future timesteps? You must mathematically justify your answer in order to receive credit for this problem.

Solutions: Mathematically, the question is asking whether the new state is an eigenvector.[1] We could solve for the steady state of this system and compare it to the state we found. However, it will be easier just to test the state that we have. Let's look at one more iteration in time.

$$\begin{bmatrix} \frac{1}{6} & \frac{2}{3} & 0 & 0 \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{4} \\ \frac{2}{3} & 0 & \frac{1}{6} & \frac{1}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 500 \\ 500 \\ 550 \\ 350 \end{bmatrix} = \begin{bmatrix} 416 \\ 521 \\ 512 \\ 450 \end{bmatrix}$$

This is an entirely different state, so the previous state was not stable. This should not be terribly surprising. Since we were only given one of the state values, we did not know enough information about the entire system to arrive stably at equilibrium.

[1] More specifically, a real eigenvector.

8. Correlated Circuitry (20 pts)

In labs and homeworks you have been implementing `cross-correlation` in iPython and using helpful numpy functions such as `numpy.correlate` to calculate cross-correlation. This problem instead will guide you through designing a circuit that implements cross-correlation of an unknown input signal with a known signal.

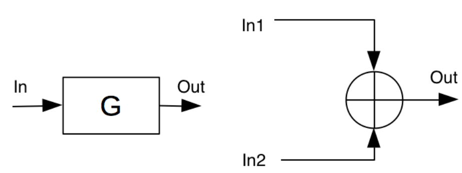
(a) Compute the cross correlation of $\vec{s}_1$ = [3 7] with respect to $\vec{s}_2$ = [2 -3], where $\vec{s}_1$ and $\vec{s}_2$ are both periodic. Your output should be $\vec{y} = [y_0 \; y_1]$ where y_k corresponds to delaying $\vec{s}_2$ by k timesteps.

Solutions: (-15,5)

(b) Now let's start designing a circuit that outputs the cross-correlation of an input signal $\vec{v}_{in}$ with a known signal $\vec{s}$. Once again, the input signal $\vec{v}_{in}^T = [v_0 \; v_1]$ and the given signal $\vec{s}^T = [a \; b]$ are discrete signals with periods of **2**. The values of a and b are known, fixed and given to you, while the values of the input voltages v_0 and v_1 are unknown. Note that v_0 and v_1 can be treated as indpenedent inputs to your design.

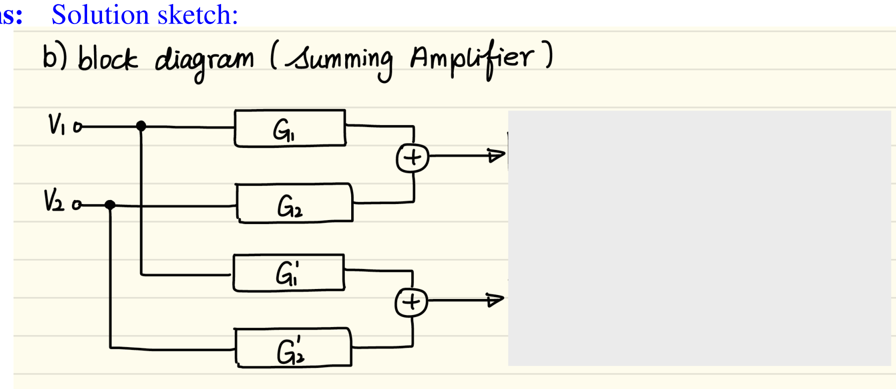
Draw a block diagram of a circuit that cross-correlates the input signal v_{in} with the given signal s and outputs the cross-correlation $\vec{y}^T = [y_0, y_1]$, with the outputs being defined the same way as they are in part (a).

Example diagrams for multipliers and summers are shown below to clarify what we mean by a block diagram for this problem.

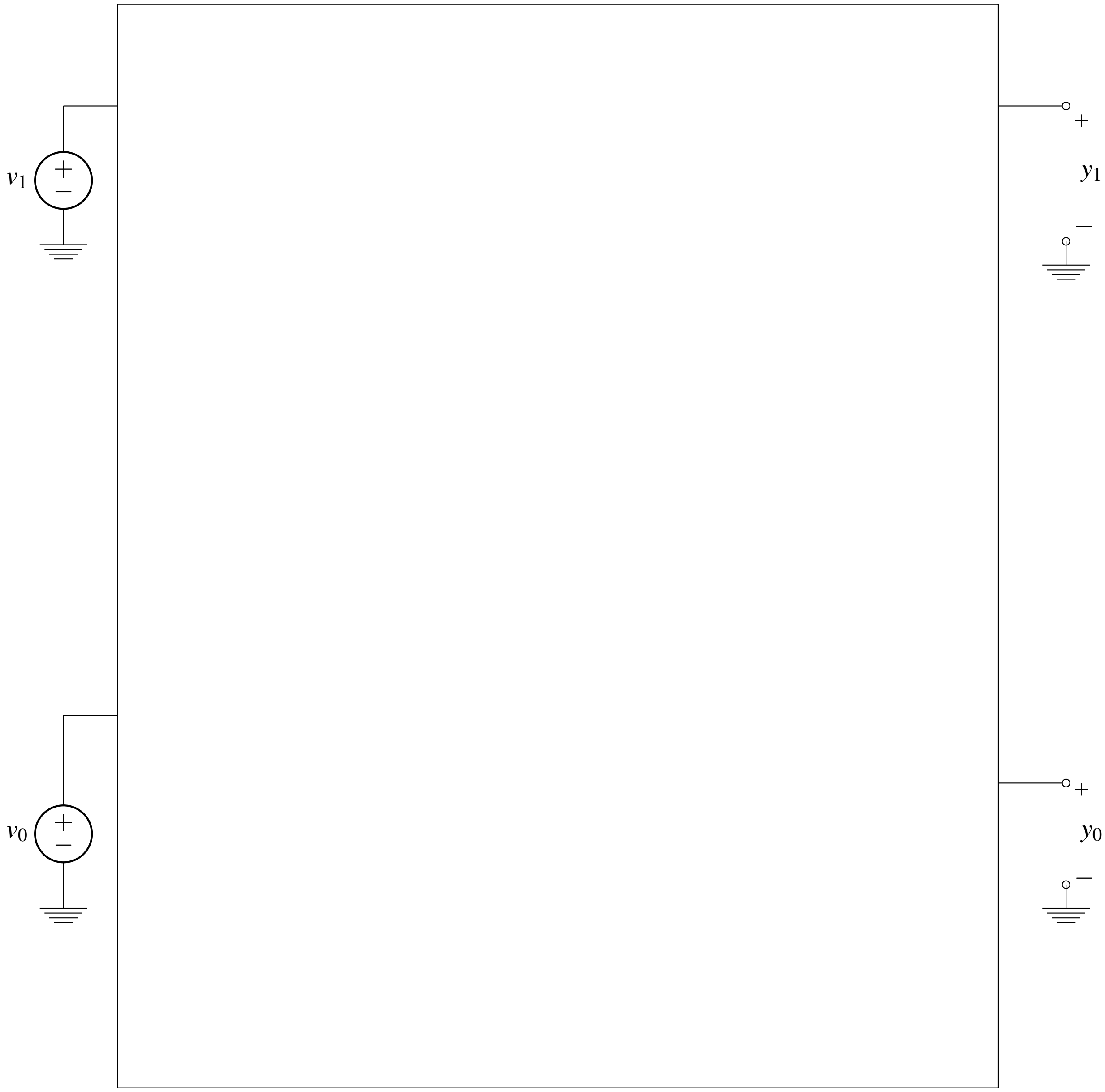


Multiplier Summer

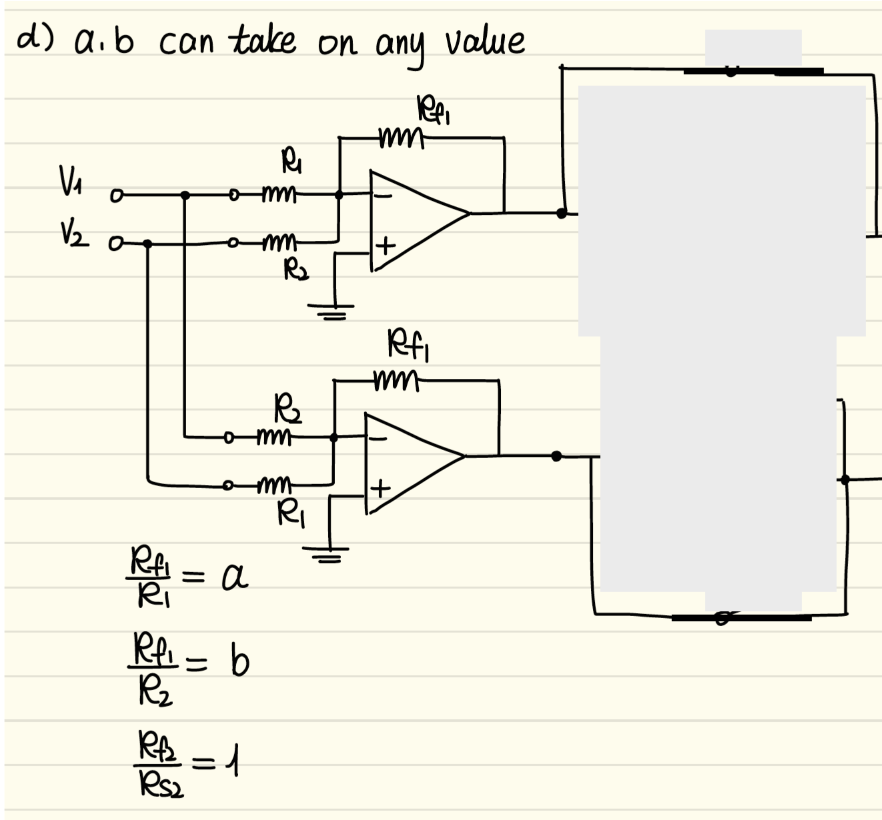
Solutions: Solution sketch:



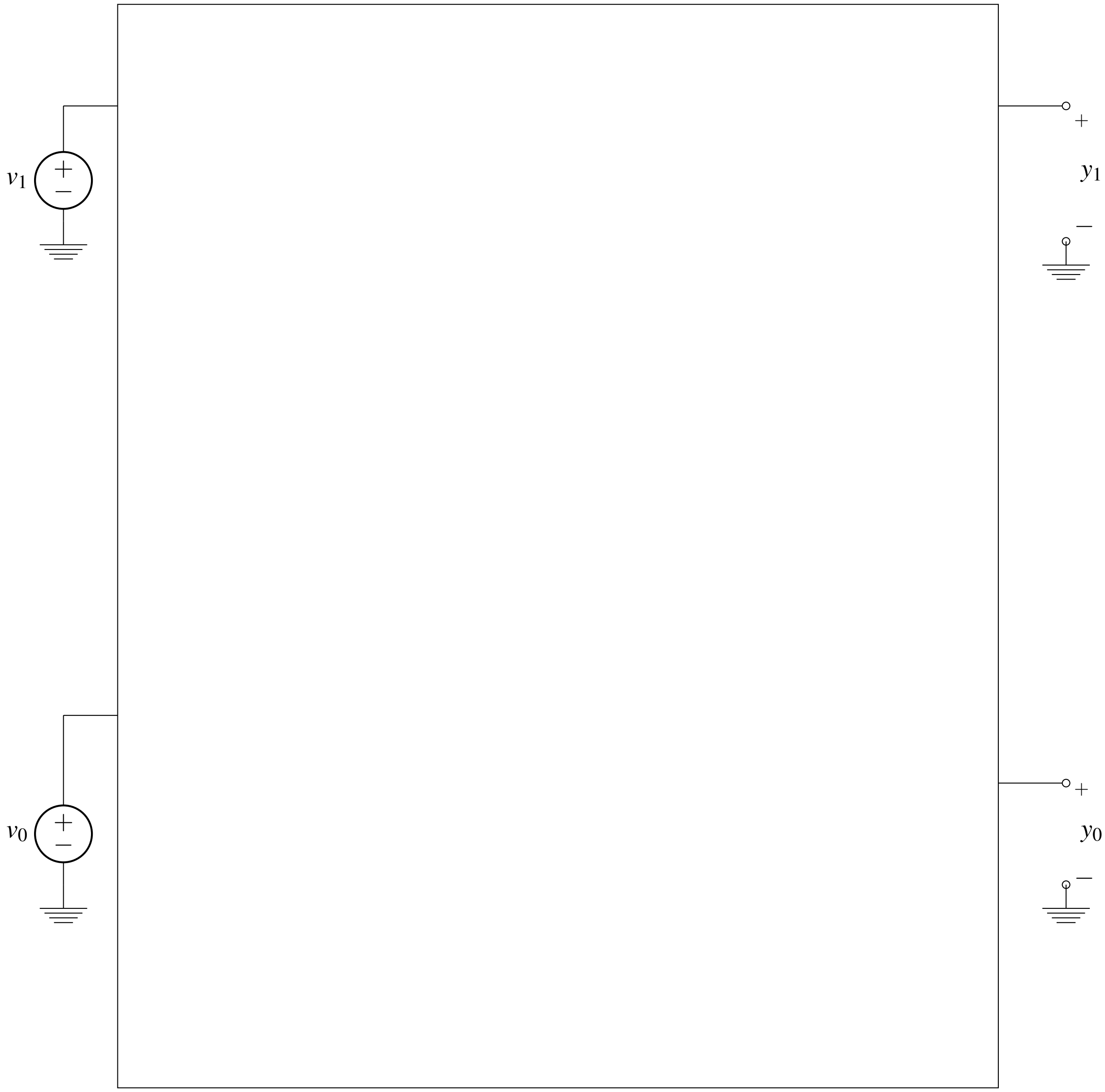
(c) Design a circuit that outputs the cross-correlation of $\vec{v}_{in}^T = [v_0\ v_1]$ and $\vec{s}^T = [a\ b]$ for a = -2 and b = -4 using only **resistors** and **op amps**. Be sure to label your components and provide their values; your design must be drawn within the box below and use v_0 and v_1 as indicated.



Solutions: Solution sketch

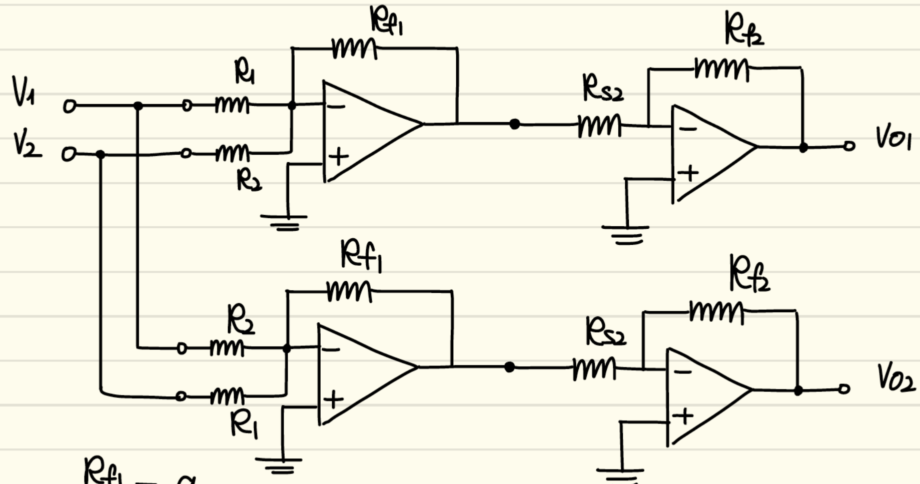


(d) Design a circuit that outputs the cross-correlation of $\vec{v}_{in}^T = [v_0 \ v_1]$ and $\vec{s}^T = [a \ b]$ for $a = 2$ and $b = 4$ using only **resistors** and **op amps**. Be sure to label your components and provide their values; your design must be drawn within the box below and use v_0 and v_1 as indicated.



Solutions:

c) Assume $a, b > 0$



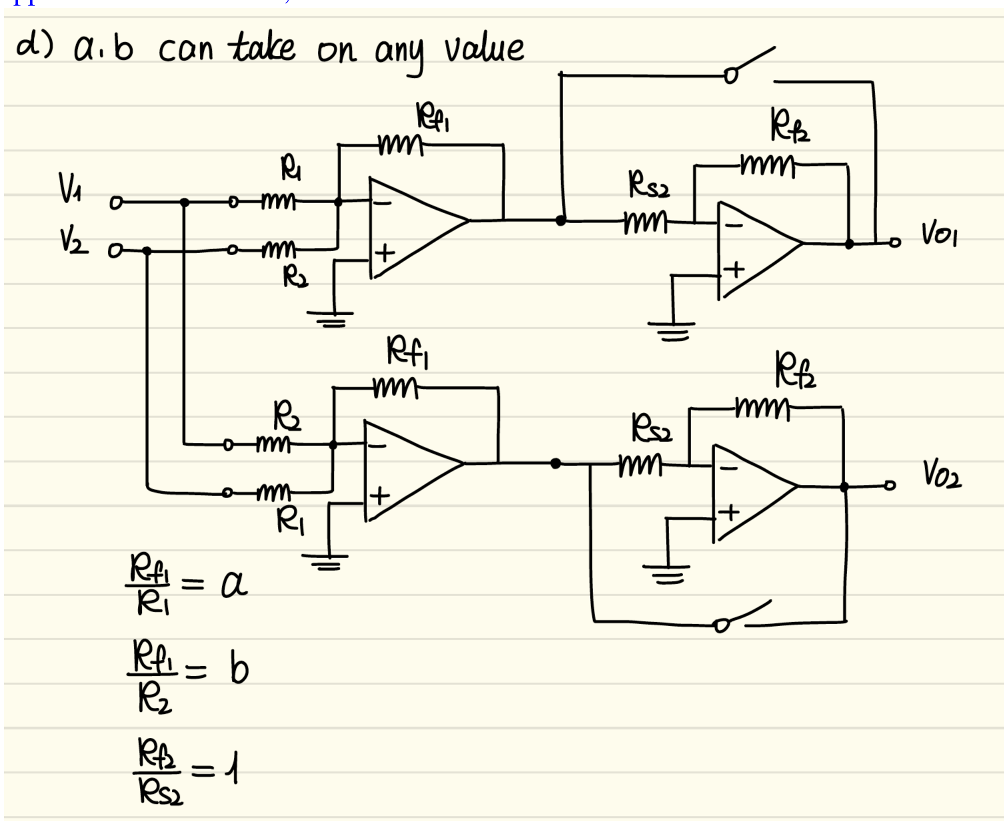
$$\frac{R_{f1}}{R_1} = a$$

$$\frac{R_{f1}}{R_2} = b$$

$$\frac{R_{f2}}{R_{s2}} = 1$$

(e) **(BONUS: 5 pts)** Now let's assume that we want to be able to correlate v_{in} with respect to either $\vec{s_1}^T$ = [2 4] or $\vec{s_2}^T$ = [-2 -4], where the choice between the two options is made by some external control signal(s). Using switches, blocks representing correct implementations of parts (c) and (d), and any additional resistors, sources, and op amps, design a circuit that would achieve this functionality. Be sure to label your components and provide their values.

Solutions: use swiches to switch between your solution for part (a) and (b), below is a general solution that applies to all values of a, and b.



9. Track This (20 pts)

In this question we will be working on a robot that must track an object moving along a straight path at an unknown but constant velocity. Specifically, the trajectory of the object is $y(t) = \alpha + \beta t$, where α is the unknown initial position of the object, and β is the unknown velocity of the object.

In general, our robot measures the position of the object at M time instants $t_1, t_2, \ldots, t_M$. We denote our measurements by $y_1, \ldots, y_M$. Unfortunately, our measurements are prone to error, caused by imperfections in our measurement instruments. What we therefore get is the following set of M expressions in two unknowns:

$$\begin{aligned} \alpha + t_1 \beta &\approx y_1 \\ \alpha + t_2 \beta &\approx y_2 \\ &\vdots \\ \alpha + t_M \beta &\approx y_M. \end{aligned}$$

(a) Rewrite the measurement expressions in matrix-vector form $\mathbf{A}\vec{x} \approx \vec{y}$, where $\vec{x}^T = [\alpha\ \beta]$.

Solutions: Rewrite the original system equations:

$$\underbrace{\begin{bmatrix} 1 & t_1 \\ \vdots & \vdots \\ 1 & t_m \\ \vdots & \vdots \\ 1 & t_M \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_m \\ \vdots \\ y_M \end{bmatrix}}_{\vec{y}} \tag{1}$$

(b) We know the least-squares solution to part (a) for $\widehat{\vec{x}}$ is given by

$$\widehat{\vec{x}} = \left(\mathbf{A}^\top \mathbf{A}\right)^{-1} \mathbf{A}^\top \vec{y}.$$

Determine a reasonably simple form for each of $\mathbf{A}^\top \mathbf{A}$ and $\mathbf{A}^\top \vec{y}$. Each of your expressions should be in terms of an appropriate subset of M, t_m, and y_m, where $m = 1, \ldots, M$.
Note: Do NOT carry out the multiplication

$$\left(\mathbf{A}^\top \mathbf{A}\right)^{-1} \mathbf{A}^\top \vec{y}.$$

Solutions: $A^T A = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ t_1 & t_2 & \ldots & t_m \end{bmatrix} \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_m \end{bmatrix} = \begin{bmatrix} \sum_{m=1}^{M} 1 & \sum_{m=1}^{M} 1 \cdot t_m \\ \sum_{m=1}^{M} t_m \cdot 1 & \sum_{m=1}^{M} t_m \cdot t_m \end{bmatrix} = \begin{bmatrix} M & \sum_{m=1}^{M} t_m \\ \sum_{m=1}^{M} t_m & \sum_{m=1}^{M} t_m^2 \end{bmatrix}$

Note that $A^T A$ is symmetric as expected

$A^T \vec{y} = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ t_1 & t_2 & \ldots & t_m \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} \sum_{m=1}^{M} y_m \\ \sum_{m=1}^{M} t_m y_m \end{bmatrix}$

(c) Next we'll apply the Gram-Schmidt method to orthogonalize (but not orthonormalize) the columns of $\mathbf{A}$. Take as your first orthogonal vector $\vec{z}_1 = \vec{1}$ — that is, a vector of size M whose entries are all equal to 1. (Note that $\vec{z}_1$ does not have unit length, and we won't bother normalizing its length.)
Prove that each entry $z_{2,i}$ of the second orthogonal vector $\vec{z}_2$ is given by $z_{2,i} = t_i - \bar{t}$, where

$$\bar{t} = \frac{1}{M} \sum_{m=1}^{M} t_m.$$

Specifically, you must be sure to prove that $\vec{z}_2$ is in fact orthogonal to $\vec{z}_1$.

Solutions: $\vec{z}_1 = \vec{1} \implies$ The projection matrix onto spanz_1 is $P_1 = \frac{\vec{z}_1\vec{z}_1^T}{\vec{z}_1^T\vec{z}_1}$

This, together with the fact that $\vec{z}_1^T\vec{z}_1 = M \implies P_1 = \frac{1}{M}\vec{1}\vec{1}^T = \frac{1}{M}\underbrace{\begin{bmatrix}1 & \dots & 1\\ \vdots & & \vdots \\ 1 & \dots & 1\end{bmatrix}}_{All-ones-matrix}$

The second orthogonal vector is

$$\vec{z}_2 = \vec{a}_2 - P_1\vec{a}_2 = \begin{bmatrix}t_1\\ \vdots \\ t_m\end{bmatrix} - \frac{1}{M}\begin{bmatrix}1 & \dots & 1\\ \vdots & & \vdots \\ 1 & \dots & 1\end{bmatrix}\begin{bmatrix}t_1\\ \vdots \\ t_m\end{bmatrix} = \begin{bmatrix}t_1\\ \vdots \\ t_m\end{bmatrix} - \frac{\sum_{m=1}^{M} t_m}{M}\begin{bmatrix}1\\ \vdots \\ 1\end{bmatrix} = \begin{bmatrix}t_1 - \bar{t}\\ \vdots \\ t_m - \bar{t}\end{bmatrix}$$

To prove $\vec{z}_1$ is orthogonal to $\vec{z}_2$:

$$\vec{z}_1^T\vec{z}_2 = \vec{1}^T(\vec{a}_2 - P_1\vec{a}_2) = \vec{1}^T\vec{a}_2 - \vec{1}^T P_1\vec{a}_2 = \vec{1}^T\vec{a}_2 - \vec{1}^T\frac{\vec{1}\vec{1}^T}{M}\vec{a}_2 = \vec{1}^T\vec{a}_2 - \vec{1}^T\vec{a}_2 = 0$$

Note that in the previous step, $\frac{\vec{1}^T\vec{1}}{M} = 1$

Interpretation: The second orthogonal vector $\vec{z}_2$ is obtained by subtracting the arithmetic mean of the measurement from each measurement time. That is, we shift the origin of time to $\bar{t}$. This corresponds to looking at $y(\tau) = \hat{\alpha} + \hat{\beta}\tau$, where $\tau = t - \bar{t} \implies y(t) = (\hat{\alpha} - \hat{\beta}\bar{t}) + \hat{\beta}t \implies \alpha = \hat{\alpha} - \hat{\beta}\bar{t}$ and $\beta = \hat{\beta}$

(d) Suppose we measure using a noisy apparatus the positions of the moving object at time instants $t_1 = 0$, $t_2 = 1$, and $t_3 = 5$ seconds, and that we've registered the following position values:

$$y(0) = 1, \qquad y(1) = 3, \qquad y(5) = 11.$$

Using the orthognalization we developed in part (c), find the values of α and β that represent the line that best fits the measured data (in the least squared error sense).

Solutions: First, given the 3 measurement times, we figure out $\bar{t} = \frac{t_1+t_2+t_3}{3} = 2$
Let $\tau = t - \bar{t} = t - 2 \implies y(\tau) = \hat{\alpha} + \hat{\beta}\tau$ where $\tau = -2, -1, 3$
For the least square expression $\hat{A}\hat{\vec{x}} \approx \vec{y}$:

$$\hat{A} = \begin{bmatrix} 1 & \tau_1 \\ 1 & \tau_2 \\ 1 & \tau_3 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 3 \end{bmatrix}$$

Solve for $\hat{\vec{x}}$: $\vec{A}^T\vec{A} = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 14 \end{bmatrix} \implies (\hat{A}^T\hat{A})^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{14} \end{bmatrix}$

$$\hat{A}^T\hat{y} = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 11 \end{bmatrix} = \begin{bmatrix} 15 \\ 28 \end{bmatrix}$$

$$\hat{x}^T = (\hat{A}^T\hat{A})^{-1}\hat{A}^T\vec{y} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{14} \end{bmatrix} \begin{bmatrix} 15 \\ 28 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

From part (c), $\beta = \hat{\beta} = 2$, $\alpha = \hat{\alpha} - \hat{\beta}\bar{t} = 5 - 2 \cdot 2 = 1$
$y(t) = 1 + 2t$

10. Jumpbot Jumpbot Jumpbot (24 pts)

In this problem you will be designing circuits allowing a robot named Jumpbot to execute a set of commands that will be described below. Specifically, the output voltages produced by your circuits are interpretted by Jumpbot as setting its vertical position in meters.

Note that throughout this problem you must label all circuit component values, and that if you use any switches in your circuit, you must explain/note what voltage in your circuit would set the state of the switch to be open or closed. (In other words, you can't assume that switches will just be set to the appropriate state by some external signal at the appropriate time. Instead, your circuit must produce the control voltage for the switches.)

(a) The first command we will design a circuit for is to enable Jumpbot to climb a ramp at a constant vertical rate of $20\frac{m}{s}$. Design a circuit that outputs a ramp voltage of slope $20\frac{V}{s}$. Label the output terminal V_{ramp}.

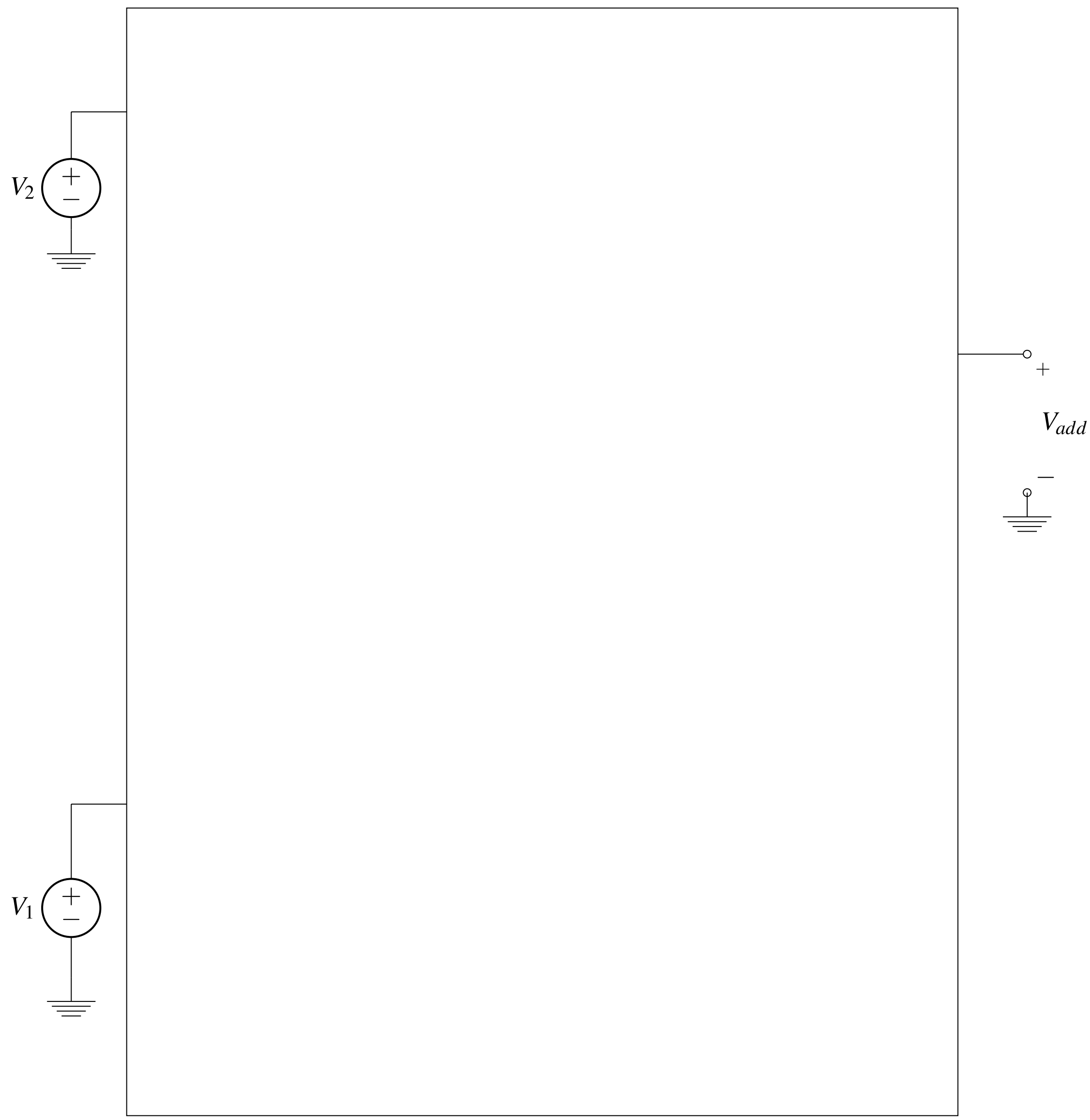
You can use only the following components in your design:

- Current Sources
- Voltage Sources
- Capacitors
- Resistors

Solutions: Form a loop with just a current source I and capacitor C. Measurement terminal should be across C, with the $+$ to $-$ of measurement terminal in the same direction as flow of current. The relationship between I and C is as follows: $\frac{I}{C} = 20\frac{V}{s}$

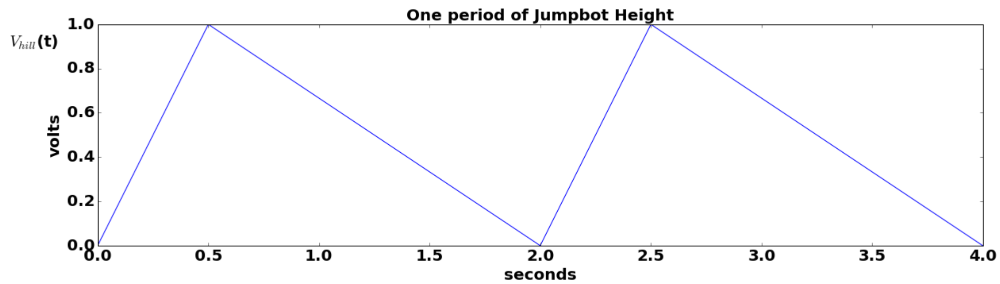
(b) Now we need to augment Jumpbot to perform the combination of two commands at once. Design a circuit that takes in two voltages (V_1 and V_2) and outputs $V_{add} = V_1 + V_2$. Your design must be drawn within the box below, and use V_1 and V_2 as indicated.

Your design for this circuit can use only op-amps and resistors.



Solutions: a non-inverting summing amplifier with gain 1 (or a circuit with equivalent behavior)

(c) Due to an issue with Jumpbot's mechanical design, he must take three times as long to go down a ramp as he does to go up one. Design a circuit that outputs the periodic V_{hill}(t) shown below (note that two periods are shown in the figure) and that reflects this constraint.



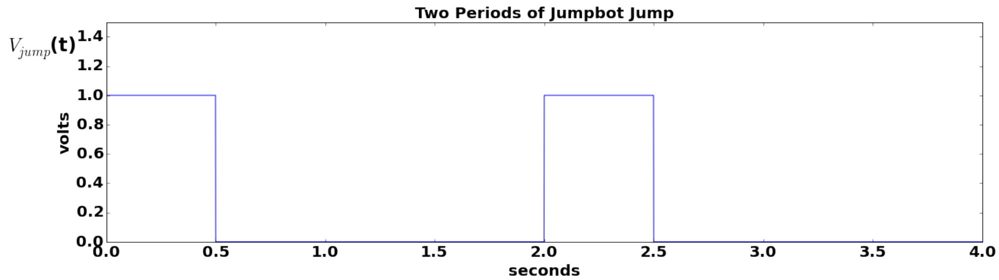
For this design, you may use only the following components:

- Current Sources
- Voltage Sources
- Op-Amps
- Capacitors
- Resistors
- Switches

Note: You can complete the rest of this problem without having a correct design for this sub-part. **Solutions:** see solution to part d

(d) Now let's allow Jumpbot to actually jump. His jumps occur with a periodicity of 2 seconds, and when he jumps, he immediately reaches a height of 1m for 0.5 seconds, after which he immediately returns to ground level.

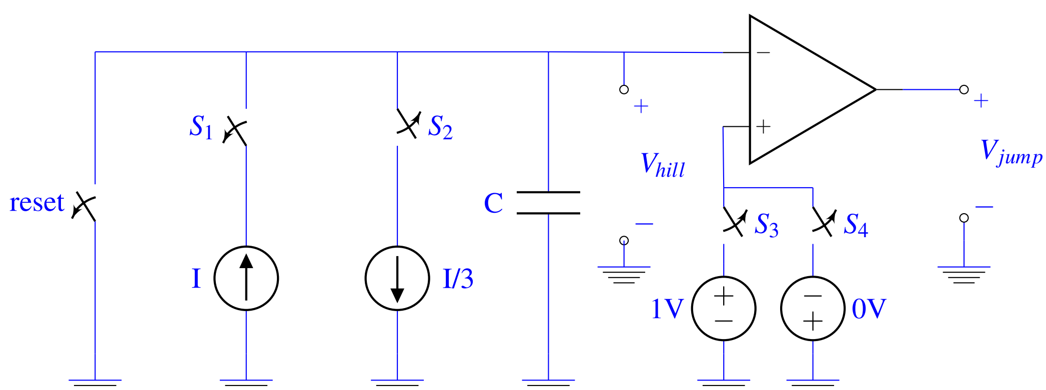
Design a circuit that implements this jump command (two periods of which are shown below). Label your output terminal V_{jump}.



You can use only the following components in your design:

- Current Sources
- Voltage Sources
- Op-Amps
- Capacitors
- Resistors
- Switches
- A voltage source producing V_{hill}(t) from part (c)

Solutions: Conceptually, if we could make a triangle wave, we can get a square wave out of it by using a comparator. Recalling that a constant current in to a capacitor gives a ramp voltage (i.e., one side of a triangle wave), we can start with the following initial implementation:



The idea in this circuit is that we use the current sources plus the capacitor to generate a triangle wave on V_{hill} that swings between 0 V and 1 V; when V_{hill} hits one of the two limits, we want the sign of both the current flowing in to the capacitor and the voltage at V_{jump} to flip. Thus, the switches S_1 and S_3 should be on when V_{jump} = 1 V (to drive V_{hill} towards 1V) and switches S_2 and S_4 should be on when V_{jump} = 0 V (to drive V_{hill} back towards 0 V).
The relationship between I and C is the following: $\frac{I}{C} = 2\frac{V}{s}$.

(e) **(BONUS: 6 pts)** Design a circuit that would allow Jumpbot to climb a hill with a slope of $20\frac{m}{s}$ while every 2 seconds also jumping over rocks that are 0.5m high, and that require him to be in the air for 0.5 seconds to clear the rocks.

You can use the following components for this design:

- Current Sources
- Voltage Sources
- Op-Amps
- Capacitors
- Resistors
- Switches
- A voltage source producing V_{ramp}(t)
- A voltage source producing V_{jump}(t)
- A correct implementation of the box from part (b)

Solutions: Using a block diagram: sum V_{ramp}(t) and V_{jump}(t) with part(b)

11. Of Greedy Pursuits and Multipath Signals (22 pts) In wireless communications, multipath propagation is the phenomenon in which a transmitted signal arrives at a receiver through two or more transmission paths—typically due to reflections from objects along the way.

Imagine a beacon that transmits a periodic discrete-time signal x. Each period of the signal is given by the vector

$$\vec{x} = \begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

In other words, the transmitted signal has period 7, so $x[n+7] = x[n]$ for all integers n.

Recall that the autocorrelation function for a periodic signal $\vec{x}$ is given by

$$R_{xx}[k] = \langle \vec{x}, \mathbf{S}^k \vec{x} \rangle = \vec{x}^{\mathsf{T}} \mathbf{S}^k \vec{x},$$

where $\mathbf{S}$ is the circular shift matrix:

$$\mathbf{S} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

(a) Determine the numerical value of $R_{xx}[0]$ and $R_{xx}[1]$.

Solutions: $R_{xx}[0] = \langle \vec{x}, \vec{x} \rangle = ||\vec{x}||^2$ because $\vec{x}$ is in $\mathbb{R}^7$ and it has only 1 or -1 entries

$$R_{xx}[1] = \langle \vec{x}^T, \mathbf{S}\vec{x} \rangle = \begin{bmatrix} 1 & 1 & 1 & -1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = -1$$

In fact, it turns out $R_{xx}[k] = -1$ for all k such that k mod 7 $\neq 0$.

For the remainder of this problem assume that the received signal y is a linear combination of delayed and scaled versions of the transmitted signal x. In particular, the received signal in vector form $\vec{y}$ is given by:

$$\vec{y} = \alpha_L \mathbf{S}^L \vec{x} + \alpha_M \mathbf{S}^M \vec{x} = \alpha_L \vec{z}_L + \alpha_M \vec{z}_M, \tag{2}$$

where $\vec{z}_k = \mathbf{S}^k \vec{x}$ for $k = L, M$. Note that $\vec{z}_k$ is the vector that represents one period of the k-sample delayed version of the transmitted signal x.

Let's now explore how to use OMP to determine the values of L and M as well as α_L and α_M from the received vector $\vec{y}$.

(b) Assuming that $|R_{xx}[0]| \gg |R_{xx}[k]|$ for all $k \neq 0$, suggest an appropriate dictionary of vectors to use for OMP that will allow us to directly find L, M, α_L, and α_M. Explain in one to two sentences why you picked this dictionary.

Solutions: The appropriate dictionary consists of $\vec{x}$ and all possible shifted versions of it. That is, $\mathbb{D} = \{\vec{x}, \mathbf{S}\vec{x}, \mathbf{S}^2\vec{x}, \ldots, \mathbf{S}^6\vec{x}\}$

Regardless of your answer to part (b), let's assume that you picked a dictionary of vectors consisting of the columns $\vec{z}_i$ of the matrix $\mathbf{Z} = \begin{bmatrix} \vec{z}_0 & \vec{z}_1 & \vec{z}_2 & \vec{z}_3 & \vec{z}_4 & \vec{z}_5 & \vec{z}_6 \end{bmatrix}$ below:

$$\mathbf{Z} = \begin{bmatrix} 1 & -1 & 1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 \\ -1 & 1 & -1 & -1 & 1 & 1 & 1 \end{bmatrix}$$

To save you some time in doing mechanical calculations, it is worth noting that the inner product of $\vec{z}_i$ with $\vec{z}_j$ is equal to 7 for $i = j$, and equal to -1 for $i \neq j$.

Finally, the received signal in vector form $\vec{y}$ is:

$$\vec{y} = \begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ y[4] \\ y[5] \\ y[6] \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3/2 \\ 3/2 \\ -1/2 \\ -3/2 \\ 1/2 \\ -1/2 \end{bmatrix}.$$

(c) Now let's examine what happens during the first iteration of OMP. Assuming a Grunge-Fighting Oracle provides you with the information below, which column of $\mathbf{Z}$ (i.e., which $\vec{z}_i$) will be selected? Given this choice, provide the expressions you would use to compute the residue signal at the end of the first iteration.

$$\vec{y}^{\mathsf{T}}\mathbf{Z} = \begin{bmatrix} \langle \vec{y}, \vec{z}_0 \rangle & \langle \vec{y}, \vec{z}_1 \rangle & \cdots & \langle \vec{y}, \vec{z}_6 \rangle \end{bmatrix} = \begin{bmatrix} \frac{13}{2} & \frac{5}{2} & -\frac{3}{2} & -\frac{3}{2} & -\frac{3}{2} & -\frac{3}{2} & -\frac{3}{2} \end{bmatrix}$$

Solutions: $\vec{r}^{[0]} = \vec{y} \rightarrow$ look at $|\langle \vec{r}^{[0]}, \vec{z}_k \rangle|$ and select the vecotr $\vec{z}_k$ for this this quantity is the largest. Clearly, $\vec{z}_0$ yields the largest quantity. $\vec{r}^{[1]} = \vec{r}^{[0]} - \frac{\vec{z}_0 \vec{z}_0^T}{\vec{z}_0^T \vec{z}_0} \vec{r}^{[0]} = \vec{y} - \frac{\vec{z}_0^T \vec{y}}{7} \vec{z}_0 = \vec{y} - \frac{13}{(2)(7)} \vec{z}_0$

$$\vec{r}^{[1]} = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{3}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} - \begin{bmatrix} \frac{13}{14} \\ \frac{13}{14} \\ \frac{13}{14} \\ -\frac{13}{14} \\ -\frac{13}{14} \\ \frac{13}{14} \\ -\frac{13}{14} \end{bmatrix} = \begin{bmatrix} -\frac{3}{7} \\ \frac{4}{7} \\ \frac{4}{7} \\ \frac{3}{7} \\ -\frac{4}{7} \\ -\frac{3}{7} \\ \frac{3}{7} \end{bmatrix}$$

(d) Now let's complete the second step of OMP, once again with the help of the Grunge-Fighting Oracle providing the information below. Note that the vector $\vec{q}^T$ is the residue at the end of the first iteration. Which $\vec{z}_i$ is selected in the second step? Finally, what are the values of L, M, α_L, and α_M?

$$\underbrace{\begin{bmatrix} -\frac{3}{7} & \frac{4}{7} & \frac{4}{7} & \frac{3}{7} & -\frac{4}{7} & -\frac{3}{7} & \frac{3}{7} \end{bmatrix}}_{\vec{q}^{\mathsf{T}}} \mathbf{Z} = \begin{bmatrix} \langle \vec{q}, \vec{z}_0 \rangle & \langle \vec{q}, \vec{z}_1 \rangle & \cdots & \langle \vec{q}, \vec{z}_6 \rangle \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{24}{7} & -\frac{4}{7} & -\frac{4}{7} & -\frac{4}{7} & -\frac{4}{7} & -\frac{4}{7} \end{bmatrix}.$$

Solutions: In the next iteration, we look for the $\vec{z}_k$ that produces the largest $|\langle \vec{r}^{[1]}, \vec{z}_k \rangle|$. Clearly, this is produced by $\vec{z}_1$. So $\vec{r}^{[2]} = \vec{r}^{[1]} - \frac{\vec{z}_1 \vec{z}_1^T}{\vec{z}_1^T \vec{z}_1} \vec{r}^{[1]} = \vec{r}^{[1]} - \frac{24}{(7)(7)} \vec{z}_1$

Since we are told that $\vec{y}$ consists only of two scaled and shifted versions of $\vec{x}$, we need not compute $\vec{r}^{[2]}$. We have already identified $\vec{z}_0 = \vec{x}$ and $\vec{z}_1 = \mathbf{S}\vec{x}$ as those components. All we need to do now is to determine how much of each is needed. his is where we do least squares:

$$\underbrace{\begin{bmatrix} \vec{z}_0 & \vec{z}_1 \end{bmatrix}}_{A_2} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}}_{\vec{\alpha}} = \vec{y} \implies \vec{\alpha} = (\vec{A}_2^T \vec{A}_2)^{-1} \vec{A}_2^T \vec{y}$$

$$\vec{A}_2^T \vec{A}_2 = \begin{bmatrix} \langle \vec{z}_0, \vec{z}_0 \rangle & \langle \vec{z}_0, \vec{z}_1 \rangle \\ \langle \vec{z}_1, \vec{z}_0 \rangle & \langle \vec{z}_1, \vec{z}_1 \rangle \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ -1 & 7 \end{bmatrix} \implies (\vec{A}_2^T \vec{A}_2)^{-1} = \frac{1}{49-1} \begin{bmatrix} 7 & 1 \\ 1 & 7 \end{bmatrix} = \frac{1}{48} \begin{bmatrix} 7 & 1 \\ 1 & 7 \end{bmatrix}$$

$$\vec{A}_2^T \vec{y} = \begin{bmatrix} \langle \vec{z}_0, \vec{y} \rangle \\ \langle \vec{z}_1, \vec{y} \rangle \end{bmatrix} = \begin{bmatrix} \frac{13}{2} \\ \frac{5}{2} \end{bmatrix} \implies \vec{\alpha} = \frac{1}{48} \begin{bmatrix} 7 & 1 \\ 1 & 7 \end{bmatrix} \begin{bmatrix} \frac{13}{2} \\ \frac{5}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

$$\implies y = 1\vec{z}_0 + \frac{1}{2}\vec{z}_1 = 1 \cdot \vec{x} + \frac{1}{2}\mathbf{S}\vec{x}$$

PRINT your name and student ID: ______________________________

[If you are want the work on this page be graded, please state CLEARLY which problem(s) this space is for. You can also draw us something if you want or give us suggestions or complaints. You can also use this page to report anything suspicious that you might have noticed.]