EECS 16A Designing Information Devices and Systems I
Spring 2020 Midterm 1

Exam Location: Soda 380

PRINT your student ID: ________________________

PRINT AND SIGN your name: ________________, ________________ ________________
(last name) (first name) (signature)

PRINT time of your Monday section and the GSI's name: ________________________

PRINT time of your Wednesday section and the GSI's name: ________________________

Name and SID of the person to your left: ________________________

Name and SID of the person to your right: ________________________

Name and SID of the person in front of you: ________________________

Name and SID of the person behind you: ________________________

1. What are you looking forward to over Spring Break? (3 points)

2. Approximately what % of lectures do you watch regularly, either online or in person? (0 points)
For statistical purposes only.

◯ 0% ◯ 25% ◯ 50% ◯ 75% ◯ 100%

3. Tell us about something that makes you happy. (3 points)

Do not turn this page until the proctor tells you to do so. You may work on the questions above.

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Extra page for scratchwork.
Work on this page will NOT be graded.

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4. Splotchy Writing (10 points)

Professor Courtade writes with a sharpie to accommodate the vision of as many people as possible. Unfortunately, some characters get smudged, which makes them difficult to read. The following is a (hypothetical) passage from lecture notes, and the smudges are labeled (1),(2),...,(10). Your task is to identify correct expressions for each of the smudges.

Let $A \in \mathbb{R}^{n \times (1)}$ be a matrix with rank r. It is always possible to write A in terms of its compact SVD

$$A = U\Sigma V^\top,$$

where Σ is a diagonal $r \times r$ matrix, and $U \in \mathbb{R}^{(2) \times (3)}$ and $V \in \mathbb{R}^{(4) \times (5)}$ have orthonormal columns. This means that $U^\top U = I_{(6)}$ and $V^\top V = I_{(7)}$, where we write I_m to denote the $m \times (8)$ identity matrix, for an integer m. The columns of U form a basis for the range of A, which is is defined as

$$\text{range}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^k\}.$$

Note that range(A) is a subspace of $\mathbb{R}^{(9)}$, which has dimension (10).

Select the values for each smudge from the multiple choice below. For each smudge, completely fill in the circle next to the correct answer. (Hint: Resist the temptation to get distracted by unfamiliar terminology... that isn't what this question is about.)

(1)	◯ k	◯ m	◯ n	◯ r
(2)	◯ k	◯ m	◯ n	◯ r
(3)	◯ k	◯ m	◯ n	◯ r
(4)	◯ k	◯ m	◯ n	◯ r
(5)	◯ k	◯ m	◯ n	◯ r
(6)	◯ k	◯ m	◯ n	◯ r
(7)	◯ k	◯ m	◯ n	◯ r
(8)	◯ k	◯ m	◯ n	◯ r
(9)	◯ k	◯ m	◯ n	◯ r
(10)	◯ k	◯ m	◯ n	◯ r

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5. Matrix Inversion (10 points)

You landed your first job at 16A*tech* (the Bay Area's newest and hottest tech company), and your first assignment is to invert a matrix $A \in \mathbb{R}^{n\times n}$. You say "no problem", and implement Gaussian elimination. You obtain the following reduction of the augmented matrix:

$$[A|I] \longrightarrow [I|P].$$

The dimension n is extremely large, so the computation takes several days to complete, and you give your boss the matrix $P \in \mathbb{R}^{n\times n}$ just minutes before the deadline.

(a) (2 points) Your boss panics, saying "Oh, no! Your procedure only guarantees that $AP = I$ and not necessarily that $PA = I$." In one sentence, concisely explain why your boss thinks this might be an issue.

(b) (8 points) You try to calm them down, saying "Don't worry, the matrix also satisfies $PA = I$, and therefore P is the inverse of A just like you wanted. I'll prove it to you..."

Your proof consists of the following two steps (fill in the details as your answer to this question):

Step 1: Argue that your matrix P is the unique $Q \in \mathbb{R}^{n\times n}$ satisfying $AQ = I$.

Step 2: Prove that $PA = AP = I$. (Hint: consider the matrix $A(P + PA - I)$)

As suggested by part (a), you should not assume that A^{-1} exists. Proving that it does is the point of this problem.

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6. Tomography (19 points)

Recall that in our simple tomography example of 4 pixels arranged into a 2×2 matrix, our initial set of measurements produced the following system of equations with unknowns $x_1, \ldots, x_4$ and measured intensities $b_1, \ldots, b_4$:

$$\begin{array}{ccccc} x_1 & +x_2 & & & = b_1 \\ & & x_3 & +x_4 & = b_2 \\ x_1 & & +x_3 & & = b_3 \\ & x_2 & & +x_4 & = b_4 \end{array}$$

(a) (3 points) Write the above system of equations in matrix-vector form $A\vec{x} = \vec{b}$.

(b) (8 points) Use Gaussian elimination to find a basis for the nullspace of your matrix in part (a). Show your work.

(c) (2 points) Suppose $\vec{x}_0$ denotes the correct pixel values, which of course satisfy $A\vec{x}_0 = \vec{b}$. Give another solution $\vec{x}_1$ to the system of equations $A\vec{x} = \vec{b}$, satisfying $\vec{x}_1 \neq \vec{x}_0$. Leave your answer in terms of $\vec{x}_0$.

(d) (2 points) Suppose we add the measurement

$$x_1 + x_4 = b_5.$$

Will the resulting new system of equations always have a solution for any values $b_1, b_2, \ldots, b_5$? Completely fill in the circle next to the correct answer.

◯ Yes ◯ No

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(e) (4 points) Assuming a solution exists for the new system of equations in part (d), will the solution be unique? Justify your answer by showing work to support your conclusion.

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7. Dynamical Systems (26 points)

Define matrices $Q, R \in \mathbb{R}^{2\times 2}$ according to

$$Q = \begin{bmatrix} 0 & 3/4 \\ 1 & 1/4 \end{bmatrix}, \qquad R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(a) (5 points) Find the eigenvalues for the matrix Q.

(b) (4 points) Consider a system with state vector $\vec{x}[n] \in \mathbb{R}^2$ at time $n \geq 1$ given by

$$\vec{x}[n] = Q\vec{x}[n-1].$$

Is there a non-zero vector $\vec{x}$ satisfying $\vec{x} = Q\vec{x}$? If yes, give one such vector.

(c) (3 points) Draw the state-transition diagram for the system in part (b). Label your nodes "A" and "B".

(d) (4 points) Now, consider a system with state vector $\vec{w}[n] \in \mathbb{R}^2$ at time $n \geq 1$ given by:

$$\vec{w}[n] = \begin{cases} Q\,\vec{w}[n-1] & \text{if } n \text{ is odd} \\ R\,\vec{w}[n-1] & \text{if } n \text{ is even.} \end{cases}$$

Write expressions for $\vec{w}[1]$, $\vec{w}[2]$, $\vec{w}[3]$ and $\vec{w}[4]$ in terms of $\vec{w}[0]$ and Q and R. Write each answer in the form of a matrix-vector product.

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(e) (10 points) Suppose we start the system of part (d) with state $\vec{w}[0] = \begin{bmatrix} 11/14 & 3/14 \end{bmatrix}^\top$. Find expressions for $\vec{w}_{\text{even}}$ and $\vec{w}_{\text{odd}}$, which are defined according to

$$\vec{w}_{\text{even}} = \lim_{k\to\infty} \vec{w}[2k], \qquad \vec{w}_{\text{odd}} = \lim_{k\to\infty} \vec{w}[2k+1].$$

In words, $\vec{w}_{\text{even}}$ and $\vec{w}_{\text{odd}}$ describe the long-term behavior of the system at even and odd time-instants, respectively. (Hint: you can avoid computation by thinking about the system at even time-instants in terms of a state-transition diagram.)

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8. Linearly Independent Solutions (5 points)

Let $A \in \mathbb{R}^{17\times 32}$ satisfy $\dim(C(A)) = 9$, where $C(A)$ denotes the column-space of A. How many linearly independent solutions can be found to the system of equations $A\vec{x} = \vec{0}$?

Note: Be careful. You are not being asked how many solutions exist for this system of equations, but rather how many *linearly independent solutions* can be found. You may just give a numerical answer; no work is required.

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9. Inverses and Transposes (8 points)

Given an invertible matrix $A \in \mathbb{R}^{n\times n}$, use the definition of matrix inverse to prove that

$$(A^T)^{-1} = (A^{-1})^T.$$

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10. Orthogonal Complements (16 points)

Consider the vector space $\mathbb{R}^n$, and let $\mathbb{U}$ be a subspace of $\mathbb{R}^n$. We define the set $\mathbb{U}^\perp \subset \mathbb{R}^n$, called the *orthogonal complement* of $\mathbb{U}$, according to

$$\mathbb{U}^\perp = \{\vec{x} \in \mathbb{R}^n \mid \vec{u}^\top \vec{x} = 0 \text{ for all } \vec{u} \in \mathbb{U}\}.$$

(a) (4 points) Show that $\mathbb{U}^\perp$ is a subspace of $\mathbb{R}^n$.

(b) (4 points) Find a concise expression for the intersection $\mathbb{U} \cap \mathbb{U}^\perp$. Justify your answer.

(c) (6 points) Working in dimension $n = 3$, consider the subspace

$$\mathbb{U} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Find a basis for $\mathbb{U}^\perp$.

(d) (2 points) For the subspaces $\mathbb{U}$ and $\mathbb{U}^\perp$ of part (c), show that $\mathbb{U} + \mathbb{U}^\perp = \mathbb{R}^3$.

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Extra page for scratchwork.
Work on this page will NOT be graded.

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Doodle page!
Draw us something if you want or give us suggestions, compliments, or complaints.
You can also use this page to report anything suspicious that you might have noticed.

EECS 16A Designing Information Devices and Systems I
Spring 2020 Midterm 1 Instructions

Read the following instructions before the exam.

There are 9 problems of varying numbers of points. You have 120 minutes for the exam. The problems are of varying difficulty, so pace yourself accordingly and avoid spending too much time on any one question until you have gotten all of the other points you can.

There are 18 pages on the exam, so there should be 9 sheets of paper in the exam. The exam is printed double-sided. Do not forget the problems on the back sides of the pages! Notify a proctor immediately if a page is missing. **Do not tear out or remove any of the pages. Do not remove the exam from the exam room.**

No collaboration is allowed, and do not attempt to cheat in any way. Cheating will not be tolerated.

Write your student ID on each page before time is called. If a page is found without a student ID, we are not responsible for identifying the student who wrote that page.

You may consult ONE handwritten 8.5” $\times$ 11” note sheet (front and back). No phones, calculators, tablets, computers, other electronic devices, or scratch paper are allowed.

Please write your answers legibly in the boxed spaces provided on the exam. The space provided should be adequate. **If you still run out of space, please use a boxed space for another part of the same problem and clearly tell us in the original problem space where to look.**

In general, show all of your work in order to receive full credit.

Partial credit will be given for substantial progress on each problem.

If you need to use the restrooms during the exam, bring your student ID card, your phone, and your exam to a proctor. You can collect them once you return from the restrooms.

Our advice to you: if you can’t solve the problem, state and solve a simpler one that captures at least some of its essence. You will perhaps find yourself on a path to the solution.

Good luck!

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