EECS 16A Designing Information Devices and Systems I
Summer 2020 Midterm 1

1. Pledge of Academic Integrity (2 points)

By my honor, I affirm that:

(1) this document, which I will produce for the evaluation of my performance, will reflect my original, bona fide work;

(2) as a member of the UC Berkeley community, I have acted and will act with honesty, integrity, and respect for others;

(3) I have not violated—nor aided or abetted anyone else to violate—nor will I—the instructions for this exam given by the course staff, including, but not limited to, those on the cover page of this document; and

(4) I have not committed, nor will I commit, any act that violates—nor aided or abetted anyone else to violate—the UC Berkeley Code of Student Conduct.

Write your name and the current date as an acknowledgement of the above. (See Gradescope)

2. Administrivia (2 points)

I know that I will lose 2^n points for every n minutes I submit after the exam submission grace period is over.

For example, if the exam becomes available at my personalized link at 7:10 p.m. PT; the grace period will expire at 9:15 p.m. PT. If my submission is timestamped at 9:16 p.m. PT, I will lose 2 points; if it is timestamped at 9:18 p.m. PT, I will lose 8 points.

○ Yes

3. What is one of your hobbies? (2 points)

4. Tell us about something you're proud of this summer. (2 points)

5. Eeveelution (20 points)

(a) With your newly acquired knowledge of linear algebra and system modelling, you've got the opportunity to work in the legendary Professor Oak's Lab. His latest research delves into the evolution of Pokémon.

Pokémon evolution is studied using hit points (HP) and combat power (CP). Thus, each Pokémon can be represented with a vector of the form $\begin{bmatrix} h \\ c \end{bmatrix}$ where h stands for HP and c stands for CP.

Our study considers Eevee's evolution into Vaporeon, Jolteon, and Flareon, which can be represented by the following transformations:

$$V\left(\begin{bmatrix} h \\ c \end{bmatrix}\right) = \begin{bmatrix} 15c+15h \\ 4c+6h+16 \end{bmatrix}$$

$$J\left(\begin{bmatrix} h \\ c \end{bmatrix}\right) = \begin{bmatrix} 15c \cdot h \\ 4c^2+10h \end{bmatrix}$$

$$F\left(\begin{bmatrix} h \\ c \end{bmatrix}\right) = \begin{bmatrix} 14c+8h \\ 7c+16h \end{bmatrix}$$

Which of the above transformations are linear in $\begin{bmatrix} h \\ c \end{bmatrix}$? Select all that apply.

(1) F

(2) V

(3) J

Solution: A linear transformation multiplies its input by scalars and adds them together. Thus, if $f(x)$ is a linear transformation, it must have the following properties.

- Preservation of addition $f(x+y) = f(x) + f(y)$ where $x, y \in$ Domain of f
- Preservation of scalar multiplication $f(\alpha x) = \alpha f(x)$ where $\alpha \in R$ and $x \in$ Domain of f

Let us test each transformation for these properties using the vectors $\begin{bmatrix} h_1 \\ c_1 \end{bmatrix}$ and $\begin{bmatrix} h_2 \\ c_2 \end{bmatrix}$.

$$V(\begin{bmatrix} h_1+h_2 \\ c_1+c_2 \end{bmatrix}) = \begin{bmatrix} 15(c_1+c_2)+15(h_1+h_2) \\ 4(c_1+c_2)+6(h_1+h_2)+16 \end{bmatrix}$$

$$V(\begin{bmatrix} h_1 \\ c_1 \end{bmatrix}) + V(\begin{bmatrix} h_2 \\ c_2 \end{bmatrix}) = \begin{bmatrix} 15c_1+15h_1 \\ 4c_1+6h_1+16 \end{bmatrix} + \begin{bmatrix} 15c_2+15h_2 \\ 4c_2+6h_2+16 \end{bmatrix} = \begin{bmatrix} 15(c_1+c_2)+15(h_1+h_2) \\ 4(c_1+c_2)+6(h_1+h_2)+32 \end{bmatrix}$$

$$V(\begin{bmatrix} h_1+h_2 \\ c_1+c_2 \end{bmatrix}) \neq V(\begin{bmatrix} h_1 \\ c_1 \end{bmatrix}) + V(\begin{bmatrix} h_2 \\ c_2 \end{bmatrix})$$

$\therefore$ V is not a linear transformation.

$$J(\begin{bmatrix} h_1+h_2 \\ c_1+c_2 \end{bmatrix}) = \begin{bmatrix} 15(c_1+c_2)\cdot(h_1+h_2) \\ 4(c_1+c_2)^2+10(h_1+h_2) \end{bmatrix}$$

$$J(\begin{bmatrix} h_1 \\ c_1 \end{bmatrix}) + J(\begin{bmatrix} h_2 \\ c_2 \end{bmatrix}) = \begin{bmatrix} 15c_1 \cdot h_1 \\ 4{c_1}^2+10h_1 \end{bmatrix} + \begin{bmatrix} 15c_2 \cdot h_2 \\ 4{c_2}^2+10h_2 \end{bmatrix} = \begin{bmatrix} 15(c_1 \cdot h_1 + c_2 \cdot h_2) \\ 4({c_1}^2+{c_2}^2)+10(h_1+h_2) \end{bmatrix}$$

$$J(\begin{bmatrix} h_1 + h_2 \\ c_1 + c_2 \end{bmatrix}) \neq J(\begin{bmatrix} h_1 \\ c_1 \end{bmatrix}) + J(\begin{bmatrix} h_2 \\ c_2 \end{bmatrix})$$

$\therefore$ J is not a linear transformation.

$$F(\begin{bmatrix} h_1 + h_2 \\ c_1 + c_2 \end{bmatrix}) = \begin{bmatrix} 14(c_1 + c_2) + 8(h_1 + h_2) \\ 7(c_1 + c_2) + 16(h_1 + h_2) \end{bmatrix}$$

$$F(\begin{bmatrix} h_1 \\ c_1 \end{bmatrix}) + F(\begin{bmatrix} h_2 \\ c_2 \end{bmatrix}) = \begin{bmatrix} 14c_1 + 8h_1 \\ 7c_1 + 16h_1 \end{bmatrix} + \begin{bmatrix} 14c_2 + 8h_2 \\ 7c_2 + 16h_2 \end{bmatrix} = \begin{bmatrix} 14(c_1 + c_2) + 8(h_1 + h_2) \\ 7(c_1 + c_2) + 16(h_1 + h_2) \end{bmatrix}$$

$$F(\begin{bmatrix} h_1 + h_2 \\ c_1 + c_2 \end{bmatrix}) = F(\begin{bmatrix} h_1 \\ c_1 \end{bmatrix}) + F(\begin{bmatrix} h_2 \\ c_2 \end{bmatrix})$$

$$F(\alpha \cdot \begin{bmatrix} h_1 \\ c_1 \end{bmatrix}) = \begin{bmatrix} 14\alpha c_1 + 8\alpha h_1 \\ 7\alpha c_1 + 16\alpha h_1 \end{bmatrix} = \alpha \cdot \begin{bmatrix} 14c_1 + 8h_1 \\ 7c_1 + 16h_1 \end{bmatrix} = \alpha \cdot F(\begin{bmatrix} h_1 \\ c_1 \end{bmatrix})$$

$\therefore$ F is a linear transformation.

(b) Professor Oak's prototype 'Eevolver' can only accept transformations in the form of matrices. Eevee can evolve into the rare psychic Pokémon Espeon through the following transformation:

$$E\left(\begin{bmatrix} h \\ c \end{bmatrix}\right) = \begin{bmatrix} 2h \\ -5c + 29h \end{bmatrix}$$

The Professor has tasked you to come up with a matrix P such that $E\left(\begin{bmatrix} h \\ c \end{bmatrix}\right) = P\begin{bmatrix} h \\ c \end{bmatrix}$

Find a correct representation of the P matrix.

Solution: The first element of the transformed vector can be written as $2h$ + $0c$. Thus, the first row of the matrix P should be $\begin{bmatrix} 2 & 0 \end{bmatrix}$.

Similarly, the second element of the transformed vector can be written as $29h + (-5)c$. Thus, the second row of the matrix P should be $\begin{bmatrix} 29 & -5 \end{bmatrix}$.

$\therefore$ P = $\begin{bmatrix} 2 & 0 \\ 29 & -5 \end{bmatrix}$

(c) Professor Oak programs three new transformation matrices into his 'Eevolver':

- *Umbreon Transformation*, represented by the 2×2 matrix U
- *Leafeon Transformation*, represented by the 2×2 matrix L
- *Glaceon Transformation*, represented by the 2×2 matrix G

Starting with a Pokémon represented by $\vec{p} \in \mathbb{R}^2$, the Professor first applies the **Umbreon Transformation**. Then he applies the **Leafeon Transformation**, and last, he applies the **Glaceon Transformation**. What expression, in terms of L, G, U, and $\vec{p}$, represents the Pokémon after this series of transformations?

Solution: Let us represent the Pokémon obtained after the Umbreon, Leafeon and Glaceon Transformations by vectors $\vec{u}$, $\vec{l}$, $\vec{g}$ respectively.
First, we apply the Umbreon Transformation.
$\vec{u} = U\vec{p}$
Next, we apply the Leafeon Transformation to the result of the previous transformation, i.e., $\vec{u}$.
$\vec{l} = L\vec{u} = LU\vec{p}$
At last, we apply the Glaceon Transformation to the result of the previous transformation, i.e., $\vec{l}$.
$\vec{g} = G\vec{l} = GLU\vec{p}$
Thus, the solution is $GLU\vec{p}$.

Professor Oak wants to try repeatedly using his 'Eevolver' on the same Pokémon, but he isn't sure which transformation matrix to try. To help him decide, he asks to you find the eigenvalues and associated eigenspaces of the following matrix:

$$T = \begin{bmatrix} 16 & 6 \\ -30 & -11 \end{bmatrix}$$

(d) State the equation you would solve to find the eigenvalues.

Solution: Let λ be an eigenvalue of T and $\vec{x}$ be a corresponding eigenvector such that $\vec{x} \neq \vec{0}$.
$\therefore T\vec{x} = \lambda\vec{x}$
$T\vec{x} - \lambda\vec{x} = \vec{0}$
$(T - \lambda I)\vec{x} = \vec{0}$
Thus, concluding from the above equation, $\det(T - \lambda I) = 0$.
$\det(T - \lambda I) = (16 - \lambda)(-11 - \lambda) - (6)(-30) = \lambda^2 - 16\lambda + 11\lambda - 176 + 180$
$\lambda^2 - 5\lambda + 4 = 0$
This is the equation you would solve to find the eigenvalues.

(e) Professor Oak tells you that one of the eigenvalues is 1. Find the associated eigenspace.

Solution: The eigenspace associated with an eigenvalue is the span of the corresponding eigenvector. To find the eigenvector, we solve the equation $(T - \lambda I)\vec{x} = \vec{0}$ with $\lambda = 1$ for $\vec{x}$.

$$\begin{bmatrix} 15 & 6 \\ -30 & -12 \end{bmatrix} \vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 15 & 6 & 0 \\ -30 & -12 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & \frac{6}{15} & 0 \\ -2 & \frac{-12}{15} & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & \frac{6}{15} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\therefore \vec{x} = \begin{bmatrix} \frac{2}{5} \\ -1 \end{bmatrix}$$

The associated eigenspace is given by span $\left\{ \begin{bmatrix} \frac{2}{5} \\ -1 \end{bmatrix} \right\}$ = span $\left\{ \begin{bmatrix} 2 \\ -5 \end{bmatrix} \right\}$.

6. Snackable Linear Combinations (12 points)

(a) You want to make some snacks at home, with the ingredients you have. You limit your options to cookies, bread, muffins and donuts. The table below shows the ingredients needed for each snack.

Cookies		**Bread**		**Muffins**		**Donuts**	
INGREDIENTS		INGREDIENTS		INGREDIENTS		INGREDIENTS	
Eggs	2	Eggs	2	Eggs	2	Eggs	3
Flour (lb)	4	Flour (lb)	2	Flour (lb)	3	Flour (lb)	3
Butter (tbsp)	3	Butter (tbsp)	3	Butter (tbsp)	5	Butter (tbsp)	2
Sugar (tbsp)	2	Sugar (tbsp)	4	Sugar (tbsp)	5	Sugar (tbsp)	5

You have 21 eggs, 12 lb flour, 34 tbsps butter, 23 tbsps sugar available. You want to use all of these ingredients completely. Frame this problem in the form $A\vec{x} = \vec{b}$, where $\vec{x} = \begin{bmatrix} x_c \\ x_b \\ x_m \\ x_d \end{bmatrix}$, and x_c, x_b, x_m, x_d represent the quantity of cookies, bread, muffins, and donuts, respectively. Choose the A, $\vec{b}$ pair that can be used to solve for the correct quantities of each snack.

Solution:

First, we write a system of equations based on the table and given ingredient quantities.

$2x_c + 2x_b + 2x_m + 3x_d = 21$

$4x_c + 2x_b + 3x_m + 3x_d = 12$

$3x_c + 3x_b + 5x_m + 2x_d = 34$

$2x_c + 4x_b + 5x_m + 5x_d = 23$

Now, we can write this in the form $A\vec{x} = \vec{b}$.

$$\begin{bmatrix} 2 & 2 & 2 & 3 \\ 4 & 2 & 3 & 3 \\ 3 & 3 & 5 & 2 \\ 2 & 4 & 5 & 5 \end{bmatrix} \begin{bmatrix} x_c \\ x_b \\ x_m \\ x_d \end{bmatrix} = \begin{bmatrix} 21 \\ 12 \\ 34 \\ 23 \end{bmatrix}$$

(b) You are given a new A and $\vec{b}$. Shown below are 3 steps of Gaussian Elimination performed on the augmented matrix.

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 4 & 1 \\ 0 & 0 & 2 & 5 & 3 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 0 & 6 \end{array}\right] \overrightarrow{??} \left[\begin{array}{cccc|c} 1 & 0 & 3 & 4 & 1 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & \frac{5}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 5 & -3 \end{array}\right]$$

Find the correct row reduction steps.

Solution: The correct answer is (i) $R_2 \leftrightarrow R_3$ (ii) $R_3/2$ (iii) $R_4 = 2R_3 - R_4$.

We start with the following augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 4 & 1 \\ 0 & 0 & 2 & 5 & 3 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 0 & 6 \end{array}\right]$$

In order to have a leading 1 in the second column, we can swap R_2 with R_3.

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 4 & 1 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 5 & 3 \\ 0 & 0 & 2 & 0 & 6 \end{array}\right]$$

Then we look at R_3 and see that we can get a leading 1 in the third column by dividing the row by 2.

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 4 & 1 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & \frac{5}{2} & \frac{3}{2} \\ 0 & 0 & 2 & 0 & 6 \end{array}\right]$$

Finally, we want to have a leading coefficient in the fourth column so we can do this by subtracting R_4 from $2R_3$.

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 4 & 1 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & \frac{5}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 5 & -3 \end{array}\right]$$

This gives us the correct final matrix. By trying the row operations in the other answer choices, we see that they do not give us the correct final matrix.

(c) You later find a new recipe book that you like better. The new A matrix created from these recipes is shown below. Find the basis vector for the nullspace of this matrix. Note: You can assume that a and b are nonzero.

$$A = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 2 & 0 \\ 0 & 0 & b & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & a & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & b & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

First we will convert this matrix to a system of equations which gives us the following:

$x_1 + ax_4 = 0$

$x_2 + 2x_3 = 0$

$bx_3 + 6x_4 = 0$

We choose x_4 as the free variable, as there is no leading coefficient in the fourth column. Now we will express all the variables in terms of x_4.

$x_4 = x_4$

$x_1 = -ax_4$

$x_3 = \frac{-6}{b}x_4$

$x_2 = -2x_3 = (-2)(\frac{-6}{b})x_4 = \frac{12}{b}x_4$

Therefore, the basis for the null-space of this matrix is

$$\left\{ \begin{bmatrix} -a \\ \frac{12}{b} \\ \frac{-6}{b} \\ 1 \end{bmatrix} \right\}.$$

7. Code Breaker (12 points)

You've been pranked! Your friends have stolen your things and have hidden them all over town. They are communicating messages back and forth that contain coordinates representing the locations of your missing stuff. Unfortunately, your friends are pretty smart, so they've used an encoding matrix to encode the vector of coordinates,

$$\vec{y} = A\vec{x},$$

where $\vec{y}$ is the encoded message, $\vec{x}$ is the original vector of coordinates, and A is the encoding matrix. But not to worry! Your 16A knowledge should help you break the code and find your stuff.

(a) You've successfully intercepted both the original and encoded versions of one of the locations. If the original vector, $\vec{x}$, belongs to $\mathbb{R}^2$ and the encoded vector, $\vec{y}$, belongs to $\mathbb{R}^6$, what are the dimensions of the encoding matrix, A?

Solution:

By the definition of matrix multiplication, A must be 6×2 to yield the proper dimensions for $\vec{y}$ and have a computable product with $\vec{x}$. It must have the same number of rows as $\vec{y}$, and the same number of columns as $\vec{x}$ has rows.

$$\underbrace{\vec{y}}_{6\times 1} = \underbrace{A}_{m\times n}\underbrace{\vec{x}}_{2\times 1} = \underbrace{A}_{6\times 2}\underbrace{\vec{x}}_{2\times 1} = \underbrace{A\vec{x}}_{6\times 1}$$

(b) What is the minimum number of pairs of original and encoded messages that you'd need to intercept in order to determine the encoding matrix?

Solution: The unknowns of this problem are the entries of A:

$$\underbrace{\vec{y}}_{\text{known}} = \underbrace{A}_{\text{unknown}}\underbrace{\vec{x}}_{\text{known}}$$

Writing out the equation in terms of the entries of $\vec{y}$, A, and $\vec{x}$, we can see that there are $6 \times 2 = 12$ unknowns.

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \\ a_{51} & a_{52} \\ a_{61} & a_{62} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Each pair of $\vec{y}$ and $\vec{x}$ give us 6 equations in the 12 unknowns:

$$y_i = x_1 \cdot a_{i1} + x_2 \cdot a_{i2} \text{ for } i = 1, \cdots, 6$$

To solve for 12 unknowns, we require 12 equations minimum. So we require at least two pairs.

(c) We have successfully intercepted n pairs of messages, where n is greater than or equal to the minimum number of pairs needed to determine the encoding matrix. Which of the following must be true about the n intercepted $\vec{x}$ vectors (original messages) if we want to recover the encoding matrix? Select all that apply.

(1) They must all be linearly independent

(2) They must span $\mathbb{R}^2$

(3) They must form a basis for $\mathbb{R}^2$

(4) None of these

(5) The $2 \times n$ matrix containing these vectors as columns must have rank 2

Solution: To check the effect of a simple linear dependence, consider the equations that one gets when one encodes $\vec{x}$ and $\alpha\vec{x}$ for $\alpha \neq 0$. If $\vec{y} = A\vec{x}$, then $A(\alpha\vec{x}) = \alpha A\vec{x} = \alpha\vec{y}$. The latter equation can be divided through by α, which means that $\vec{x}$ and $\alpha\vec{x}$ yield the same equations in the entries of A. Some notion of linear independence is required for the $\vec{x}$'s.

However, linear independence is not a strict requirement. Consider what happens for the following set of $\vec{x}$'s:

$$\{\vec{x}_1, \vec{x}_2, \vec{x}_1 + \vec{x}_2\} \text{ where } \vec{x}_1 \neq \vec{x}_2$$

While the set is linearly dependent, $\vec{x}_1$ and $\vec{x}_2$ yield different equations in the entries of A. Since there are 2 distinct pairs of $\vec{x}$ and $\vec{y}$, and thus 12 different equations, we can solve for the entries of A. The strict requirement is that a subset of the $\vec{x}$'s containing the minimum number of the $\vec{x}$'s is linearly independent. This eliminates (1) and (3) as linear independence is not a necessary condition, and bases are required to be linearly independent. (2) and (5) are valid options: the set $\{\vec{x}_1, \cdots, \vec{x}_n\}$ cannot span $\mathbb{R}^2$ without having a subset of two linearly independent vectors. If the rank of the matrix with $\vec{x}_i$ as its columns is 2, and $\vec{x}_i \in \mathbb{R}^2$, then the space spanned by the $\vec{x}_i$'s is simply $\mathbb{R}^2$. (4) is eliminated as there are valid conditions.

8. Fundamental Subspaces (12 points)

(a) You have a 3×3 matrix $A = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}$. After performing some row operations, the resulting matrix is:

$$\begin{bmatrix} * & * & 0 \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$$

Non-zero entries are indicated by $*$. Construct a basis for the columnspace of the matrix A using $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$.

Solution: From the given matrix after certain operations are performed, we can visually inspect and mathematically verify that the matrix's columns are linearly independent (all rows are nonzero) and the matrix is square. Hence, we know that when we find the reduced row echelon form of the matrix, we will end up with the 3x3 identity. This tells us that all three columns of the original matrix are needed to construct a valid basis for its columnspace. Thus, the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ forms a basis for the columnspace of A.

(b) What are the rank and nullity of the given matrix B where $*$ are non-zero values?

$$B = \begin{bmatrix} 0 & 0 & * & * & * & * \\ * & * & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution: The rank-nullity theorem states that the number of columns of a matrix must be equal to the sum of its nullity and rank. The rank of the matrix is 2, as all but 2 of the rows/columns are linearly dependent. Since B has 6 columns, the nullity of the matrix is 6 - 2 = 4.

(c) Let's turn our attention to a more general case. Let M be a matrix with m rows and n columns. You visualize the nullspace of this matrix using computer software and see a plane in a three-dimensional space. You then try to visualize the columnspace of the matrix, but cannot as the software reports that it is unable to plot 9-dimensional vectors. Determine m, n, and the rank of M.

Solution: The null space contains 3-dimensional vectors. Since when we find the null space we are solving $M\vec{x} = 0$, $n = 3$. Since the columnspace is a subspace in 9 dimensions, the matrix must have 9 rows and $m = 9$. As in the previous part, we use the rank-nullity theorem. The nullity is 2; this follows from the fact that you visualize a plane, which is two-dimensional even though it is visualized in three dimensions. Hence, the rank is 3 - 2 = 1.

9. Pumps Problem (16 points)

(a) For the transition diagram below, which values of a, b, c would lead to a conservative system? If this is not possible, please state why.

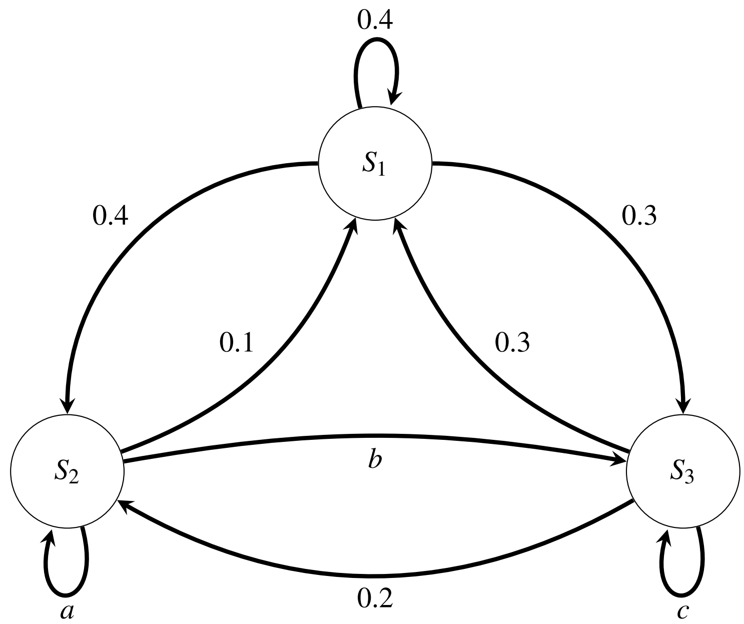

Figure 1: Transition Diagram

Solution: From this diagram, we can determine that our transition matrix is represented as:

$$\begin{bmatrix} 0.4 & 0.1 & 0.3 \\ 0.4 & a & 0.2 \\ 0.3 & b & c \end{bmatrix}$$

We can see that no matter what values are chosen for a, b, c, the elements in the first column does not add up to 1 so the system is not conservative.

(b) For a different system given by transition matrix:

$$T = \begin{bmatrix} 0.5 & 4.0 & 1.75 \\ 2.0 & 4.0 & -7.0 \\ 1.0 & 2.0 & -2.5 \end{bmatrix}$$

and state vector $\vec{x}[n] \in \mathbb{R}^3$ at time n ≥ 1 given by

$$\vec{x}[n] = T\vec{x}[n-1]$$

What is the vector $\vec{x}$ such that the current state equals the previous state? In other words what is the $\vec{x}$ such that $\vec{x} = \vec{x}[n] = \vec{x}[n-1]$?

Solution: To find the state $\vec{x}$ such that $\vec{x} = \vec{x}[n] = \vec{x}[n-1]$, we must find the eigenvector associated with the eigenvalue $\lambda = 1$. As a result we must determine the null space of the matrix $T - I$ which is equivalent to finding the solution to $(T-I)\vec{x} = \vec{0}$.

$$\left[\begin{array}{ccc|c} -0.5 & 4 & 1.75 & 0 \\ 2 & 3 & -7 & 0 \\ 1 & 2 & -3.5 & 0 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & -3.5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So solutions to $(T-I)\vec{x} = \vec{0}$ can be written as vectors $\vec{x}$ satisfying

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3.5 \\ 0 \\ 1 \end{bmatrix} x_3$$

where x_3 is a free variable. So any vector within $\text{span}\left\{\begin{bmatrix} 3.5 \\ 0 \\ 1 \end{bmatrix}\right\}$ satisfies our requirement that the current state equals the previous state.

(c) For a general system given by a transition matrix $B \in \mathbb{R}^{n \times n}$, which statements, if true, must imply that there exists a non-zero $\vec{x} \in \mathbb{R}^n$ such that $\vec{x} = B\vec{x}$? Choose all that apply.

(1) The columns of B are linearly dependent.

(2) $B - I$ has a non-trival nullspace.

(3) $B - I$ has a trivial nullspace.

(4) The columns of $B - I$ are linearly dependent.

(5) There is an eigenvalue $\lambda = 1$.

Solution: Our vector $\vec{x}$ must satisfy $\vec{x} = B\vec{x}$, therefore we can see that the matrix B has an eigenvector $\vec{x}$ with associated eigenvalue $\lambda = 1$. Rearranging this equation we find

$$B\vec{x} - \vec{x} = B\vec{x} - I\vec{x} = (B-I)\vec{x} = \vec{0}$$

Therefore, $\vec{x}$ must be in the nullspace of $B - I$. Because we know $\vec{x} \neq \vec{0}$, the matrix $B - I$ has a nontrivial nullspace, so it must also have linearly dependent columns. There is not enough information about the matrix B to determine if its columns are linearly dependent.

(d) Consider a state transition matrix $A \in \mathbb{R}^{3\times 3}$ with eigenvalue/eigenvector pairings:

$$\lambda_1 = 0.75, \vec{v}_1 \quad \lambda_2 = 1, \vec{v}_2 \quad \lambda_3 = 5, \vec{v}_3$$

where $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent. We can express any initial state vector $\vec{x}[0] \in \mathbb{R}^3$ as a linear combination of these vectors. In other words:

$$\vec{x}[0] = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix} \vec{\alpha}$$

For what values of $\vec{\alpha}$ will this system converge to a non-zero steady state? Choose all that apply.

(1) $\vec{\alpha} = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}$ (2) $\vec{\alpha} = \begin{bmatrix} 2 \\ 0.5 \\ 0.5 \end{bmatrix}$ (3) $\vec{\alpha} = \begin{bmatrix} -8 \\ 4 \\ 0 \end{bmatrix}$ (4) $\vec{\alpha} = \begin{bmatrix} 0 \\ 4 \\ -6 \end{bmatrix}$ (5) $\vec{\alpha} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$

Solution: We can represent our initial state as

$$\vec{x}[0] = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3$$

To find our steady state we must take the limit as time approaches infinity.

$$\begin{aligned}
\vec{x}_{ss} &= \lim_{n\to\infty} \vec{x}[n] \\
&= \lim_{n\to\infty} \mathbf{A}^n \vec{x}[0] \\
&= \lim_{n\to\infty} \mathbf{A}^n (\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3) \\
&= \lim_{n\to\infty} (\alpha_1\lambda_1^n\vec{v}_1 + \alpha_2\lambda_2^n\vec{v}_2 + \alpha_3\lambda_3^n\vec{v}_3) \\
&= \lim_{n\to\infty} (\alpha_1(0.75)^n\vec{v}_1 + \alpha_2(1)^n\vec{v}_2 + \alpha_3(5)^n\vec{v}_3)
\end{aligned}$$

We can see that in order for our limit to converge to a non-zero steady state $\alpha_2 > 0$ and $\alpha_3 = 0$. The only option that satisfies these criteria is (3).

10. Simple Functions of a Matrix (12 points)

Characterizing the eigenvalues and eigenvectors of a matrix A starts with solving the characteristic equation:

$$\det(A-\lambda I)=0$$

This expression tells us a great deal about the structure embedded in A, which we will explore below. Let

$$A=\begin{bmatrix}4 & 3\\ 3 & 1\end{bmatrix}$$

(a) Compute the characteristic equation, simplifying it to a polynomial function of λ.

Solution: This matrix follows the form $A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$. The characteristic polynomial of any matrix of this form is $(a-\lambda)(d-\lambda)-bc=\lambda^2-(a+d)\lambda+(ad-bc)$. The characteristic equation sets this polynomial equal to 0. Thus, $\lambda^2-5\lambda-5=0$.

In 1853, mathematicians Arthur Cayley and William Hamilton created a theorem centered around the characteristic equation:

Theorem 0.1: [Cayley-Hamilton] Every square matrix satisfies its own characteristic equation.

We state this theorem without proof. As you advance in your studies of linear algebra and linear systems, you will encounter this theorem again, as well as the tools necessary to prove its correctness, but for now just take this as a fact. For this section, we give you a new matrix B:

$$B=\begin{bmatrix}2 & 1\\ 1 & 3\end{bmatrix}$$

The characteristic equation for this matrix is $\lambda^2-5\lambda+5=0$. What does it mean that a matrix should satisfy its own characteristic equation? The equation above is a function of λ, but what if we made it a function of B?

(b) What does the Cayley-Hamilton Theorem imply about the expression $B^2-5B+5I$? Select all that apply.

(1) It equals λ

(2) It is composed of the eigenvectors of B

(3) It is undefined

(4) It equals B

(5) It equals $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

Solution: For matrices, what is the equivalent of the scalar 0? It is the matrix of all zeros: $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$, which we often write $\mathbf{0}$. If $\mathbf{B}$ *satisfies* the characteristic equation, then substituting $\mathbf{B}$ for λ must yield the matrix equivalent of 0, $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$. Option (5) is the only correct option.

(c) Use the Cayley-Hamilton Theorem to compute $2B^3-10B^2+10B-2I$.

Solution:

$$\begin{aligned}
2\mathbf{B}^3 - 10\mathbf{B}^2 + 10\mathbf{B} - 2\mathbf{I} &= 2\mathbf{B}(\mathbf{B}^2 - 5\mathbf{B} + 5\mathbf{I}) - 2\mathbf{I} \\
&= 2\mathbf{B}(\mathbf{0}) - 2\mathbf{I} \quad (Cayley - Hamilton) \\
&= \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}
\end{aligned}$$

11. A (Solar) System Far, Far Away (20 points)

As a wandering bounty hunter, you have recently purchased a spaceship, Swordfish I, initially positioned at (x_0, y_0, z_0), which can be moved to some final position (x_1, y_1, z_1), where the motion is characterized by matrix A.

(a) Assume that (x_0, y_0, z_0) and (x_1, y_1, z_1) are related by

$$A \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad \text{where} \quad A = \begin{bmatrix} 5 & 10 \\ -4 & -8 \\ 13 & 26 \end{bmatrix}.$$

$\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ does not depend on z_0. Choose **all** the options that are true.

(1) The rows of A are linearly independent.

(2) The columnspace of A is a subspace of $\mathbb{R}^3$.

(3) The columns of A are linearly independent.

(4) The rank of A is 2.

(5) The columns of A span the entire $\mathbb{R}^3$.

Solution: Option (2) is correct.

The relationship between (x_0, y_0, z_0) and (x_1, y_1, z_1) is not relevant for this subpart.

Looking closely at **A**, we notice that its dimensions are 3×2, indicating that it must necessarily have either linearly dependent rows or columns as it is not square (and therefore is non-invertible). The columns are the 3×1 vectors $(5, -4, 13)$ and $(10, -8, 26)$. Because there are only 2 such column vectors, they cannot be linearly dependent unless one is a multiple of the other. By inspection, the columns are linearly dependent, as $(10, -8, 26) = (5, -4, 13)$. We can eliminate (3) as a possible answer. The rows are also linearly dependent, as $(13, 26)$, $(5, 10)$, and $(-4, -8)$ are all multiples of $(1, 2)$, so we can also eliminate (1) as a possible answer.

Knowing this, we can say that the column space of **A** must be a subspace of $\mathbb{R}^3$ because each column vector has a maximum of 3 elements in it, so each vector will be a vector within $\mathbb{R}^3$ and the closure property implies that any linear combinations of these column vectors will also fall within $\mathbb{R}^3$. However, these vectors do not span all of $\mathbb{R}^3$ precisely because of the linear dependency of the row vectors mentioned above. A minimum of 3 unique linearly independent vectors would be necessary to span $\mathbb{R}^3$, but **A** in this subpart only has 1. This means (5) can also be eliminated, but that (2) still holds true.

The rank of a matrix is its maximum number of linearly independent vectors, and to find it, we can transform **A** to its row-echelon form.

$$\begin{bmatrix} 5 & 10 \\ -4 & -8 \\ 13 & 26 \end{bmatrix} = (R_1/5) \begin{bmatrix} 1 & 2 \\ -4 & -8 \\ 13 & 26 \end{bmatrix} = (-R_2/2) \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 13 & 26 \end{bmatrix} = (R_3/13) \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} = (R_2 =$$
$$R_2 - R_1) \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 1 & 2 \end{bmatrix} = (R_3 = R_3 - R_1) \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

After finding the row-echelon form, we see that there is only 1 non-zero row vector left over, indicating that the rank of **A** is 1, not 2. Hence, we can also eliminate (4) as an answer.

(b) You want to be able to navigate to any final position (x_1, y_1, z_1) by choosing your initial position (x_0, y_0, z_0). You start with the following equations representing your spaceship:

$$A \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad \text{where} \quad A = \begin{bmatrix} 6 & -6 \\ -5 & 0 \\ 15 & -15 \end{bmatrix}.$$

Consider the set of all possible final positions that can be reached from the set of all initial positions. What geometric object does this set form?

Solution: Expand out the matrix-vector multiplication on the left-hand side to see what x_1, y_1, and z_1 are equal to more clearly:

$$\mathbf{A} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$
$$\begin{bmatrix} 6 & -6 \\ -5 & 0 \\ 15 & -15 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$$
$$6x_0 - 6y_0 = x_1$$
$$-5x_0 = y_1$$
$$15x_0 - 15y_0 = z_1$$

Recall that you can only control the values of x_0, y_0 and start by assuming that you can reach all possible points in $\mathbb{R}^3$ using the matrix-vector operation. However, if you set x_0 and y_0 to certain values, you are automatically removing all degrees of freedom from x_1, y_1, and z_1 which now have to take on constant values and cannot range freely from $(-\infty, \infty)$ as it should be able to in order to reach any vector in $\mathbb{R}^3$. Because only 1 vector (x_1, y_1, z_1) is accessible per (x_0, y_0) pair, the possible values of the right-hand side will construct **a 2D plane**, specifically a plane with basis vectors $(6, -5, 15)$, $(-6, 0, -15)$ – the column vectors of the matrix. Intuitively, this is because the matrix **A**, by being multiplied by the input vector, geometrically transforms this input vector into a vector on the plane defined by the basis vectors of its column space.

(c) You notice that the final position (x_1, y_1, z_1) does not depend on the initial z-position, z_0. You upgrade your ship, modifying the matrix A by adding a third column. Your upgraded ship obeys the following equations:

$$A \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}, \quad \text{where} \quad A = \begin{bmatrix} 0 & 4 & 1 \\ 1 & 0 & 3 \\ 4 & 5 & 0 \end{bmatrix}$$

Choose **all** the options that are true.

(1) The matrix A is non-invertible.

(2) The ship can reach any final position in the $\mathbb{R}^3$ space, if the right initial position (x_0, y_0, z_0) is used.

(3) The columns of A form a basis for $\mathbb{R}^3$.

(4) No two unique initial positions (x_0, y_0, z_0) result in the same final position (x_1, y_1, z_1).

(5) The dimension of the geometric object formed by the set of all possible values of (x_1, y_1, z_1) exceeds rank of A.

Solution: The correct options are (2), (3), and (4).

The matrix **A** has changed between this subpart and the previous ones. We notice that for **A** to perform a transformation to the input vector that maps each input vector to a unique output vector and vice versa i.e. for **A** to be invertible, **A** should row-reduce to the identity matrix in $\mathbb{R}^3$.

$$\begin{bmatrix} 0 & 4 & 1 \\ 1 & 0 & 3 \\ 4 & 5 & 0 \end{bmatrix} \sim (\text{swap } R_1, R_2) \begin{bmatrix} 1 & 0 & 3 \\ 0 & 4 & 1 \\ 4 & 5 & 0 \end{bmatrix} \sim (R_3 = R_3 - 4R_1) \begin{bmatrix} 1 & 0 & 3 \\ 0 & 4 & 1 \\ 0 & 5 & -12 \end{bmatrix} \sim (R_2 = R_2/4) \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1/4 \\ 0 & 5 & -12 \end{bmatrix} \sim (R_3 = R_3 - 5R_2) \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1/4 \\ 0 & 0 & -53/4 \end{bmatrix} \sim (R_3 = -4/53R_3) \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{bmatrix} \sim (R_1 = R_1 - 3R_3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/4 \\ 0 & 0 & 1 \end{bmatrix} \sim (R_2 = R_2 - 1/4R_3) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is a lot of work. Another trick to find out whether **A** is invertible is to realize that if there's a 0 element in a different position in every column vector in the matrix, it's impossible for one column vector to be a linear combination of the others i.e. all of the column vectors will be linearly independent.
Because of invertibility, even though (1) will not hold, (2), (3), and (4) will. If each (x_0, y_0, z_0) initial position can be mapped to a unique (x_1, y_1, z_1) (if invertible, the transformation performed by **A** is one-to-one), all of the points within $\mathbb{R}^3$ are reachable. Similarly, from the RREF we found above, the columns of **A** reduce to the unit vectors in $\mathbb{R}^3$ and are thus guaranteed to form a basis.
Given what was found above, the geometric object formed by the set of all possible values (x_1, y_1, z_1) is all of 3-d space. The rank of **A**, found from the RREF, is also 3 because there are 3 linearly independent column vectors composing the matrix, and the matrix is square. Hence, (5) also does not hold.

(d) Your upgraded ship opens a lot of doors for you; you decide to go explore the ($\mathbb{R}^3$) universe. You don't want to miss a single sight, and hope that all points in the universe will be reachable by your ship. As before, you know the current position of your ship can be described by a 3-dimensional vector $\vec{s} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Your ship's navigation system can be represented by the following model which relates the position to a control input vector $\vec{u}$:

$$\vec{s} = B\vec{u},$$

You decide to modify the the matrix B from its default settings. You would like to choose B such that you can reach any possible final position $\vec{s}$ within $\mathbb{R}^3$ by controlling the input $\vec{u}$, which is not necessarily in $\mathbb{R}^3$. Which B, $\vec{u}$ pairs will allow you to explore the whole universe? Choose **all** the options that work.

(1) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 5 & 4 & 5 \\ 1 & 0 & 4 & 1 \\ 5 & 3 & 0 & 8 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$

(2) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 5 & 4 \\ 1 & 0 & 4 \\ 5 & 3 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

(3) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 5 & -5 \\ 1 & 0 & 1 \\ 5 & 3 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$

(4) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 5 & 5 & -5 \\ 1 & 0 & 1 & 1 \\ 5 & 3 & 8 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$

Solution: The correct options are (1) and (2).

For *any* final position (x, y, z) to be accessible from the input state, we need to check the row-reduction of each $\mathbf{B}$ to verify that the column vectors are linearly independent. For (2) and (3), this is relatively simple:

$$\begin{bmatrix} 0 & 5 & 4 \\ 1 & 0 & 4 \\ 5 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For (2), use the same trick from (c) – you don't need to row-reduce by hand all the way! Because $\mathbf{B}$ can be row-reduced to find the identity matrix composed of unique unit vectors for $\mathbb{R}^3$, we know that this pair will allow you explore the whole universe. This is because the transformation applied to the input will map each input vector to a unique output, thereby spanning all of $\mathbb{R}^3$. Similarly, for (3), we find the following:

$$\begin{bmatrix} 0 & 5 & -5 \\ 1 & 0 & 1 \\ 5 & 3 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Because $\mathbf{B}$ will only reduce to form 2 unique basis vectors, it will only span a plane, not the entirety of $\mathbb{R}^3$. For (1) and (4), we can also perform row-reduction and check on the resulting equations to determine whether each input position triplet maps to a unique output position triplet.

$$\begin{bmatrix} 0 & 5 & 4 & 5 \\ 1 & 0 & 4 & 1 \\ 5 & 3 & 0 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Similarly, for (4):

$$\begin{bmatrix} 0 & 5 & 5 & -5 \\ 1 & 0 & 1 & 1 \\ 5 & 3 & 8 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Because there are 4 controllable variables and 3 valid equations in this row-reduced case, there are actually more degrees of freedom than needed to span all of 3D space, so (1) will allow the ship to go anywhere in $\mathbb{R}^3$; its RREF has all the basis vectors needed – $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ along the columns. On the other hand, (4) row-reduces to have 1 row zero out completely. This means there is no valid set of 3 equations to map each input vector to an output vector, and not all of the set of 3D vectors will be accessible from the ship.

(e) You decide to upgrade to a new ship, the Swordfish II, a ship with a system that can be expressed by the following model:

$$\vec{s}[t+1] = \vec{s}[t] + \vec{b}u[t],$$

where $\vec{s}[t]$ is the current position of the ship at time t, $\vec{s}[t+1]$ is the next position of the ship (at time $t+1$), vector $\vec{b}$ helps define the motion of the ship, and $u[t]$ represents a user-controlled scalar input at time t. As you fly, you notice another ship, the Hammerhead, with the same flight system getting closer on your navigation display – at time $t = 0$ it is at $(x, y, z) = (-4, 2, 4)$, that is, $\vec{s}_{\text{other ship}}[0] = \begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix}$.

You communicate with the captain of the other ship, and realize that:

$$\vec{b}_{\text{your ship}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, u_{\text{your ship}}[t] = -2$$
$$\vec{b}_{\text{other ship}} = \begin{bmatrix} -3 \\ -4 \\ 1 \end{bmatrix}, u_{\text{other ship}}[t] = 2$$

Both ships are on autopilot, so $u_{\text{your ship}}$ and $u_{\text{other ship}}$ will not change over time. For which of the following starting coordinates for your ship, $\vec{s}_{\text{your ship}}[0]$, if any, is it possible that your ship and the other will collide at some time in the future ($t > 0$)? Select all that apply.

(1) $(-8, -6, 5)$

(2) $(0, 10, 4)$

(3) $(-12, -14, 4)$

(4) $(-8, -6, 4)$

(5) $(4, 18, 4)$

Solution: For your ship and the other ship to collide, at some $t \geq 0$, the positions $\vec{s}_{\text{you}}$ and $\vec{s}_{\text{other}}$ should be exactly the same. First, write out the equations that define the positions of these ships at each timestep, abbreviating by the parameters given in the subpart:

$$\vec{s}_{\text{you}}[t+1] = \vec{s}_{\text{you}}[t] + \vec{b}_{\text{you}}\vec{u}_{\text{you}}[t]$$
$$\vec{s}_{\text{other}}[t+1] = \vec{s}_{\text{other}}[t] + \vec{b}_{\text{other}}\vec{u}_{\text{other}}[t]$$

Substituting with the given constants, we notice that the recursive equation in which $\vec{s}[t+1]$ is defined as a function of $\vec{s}[t]$ is simply a vector with constant components being added in each timestep to the previous position.

$$\vec{s}_{\text{you}}[t+1] = \vec{s}_{\text{you}}[t] + \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$
$$\vec{s}_{\text{other}}[t+1] = \vec{s}_{\text{other}}[t] + \begin{bmatrix} -6 \\ -8 \\ 2 \end{bmatrix}$$

We can then write $\vec{s}[t]$ in terms of t, rather than recursively:

$$\vec{s}_{\text{you}}[t] = \vec{s}_{\text{you}}[0] + \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} t = \begin{bmatrix} \vec{s}[0]_x - 2t \\ \vec{s}[0]_y \\ \vec{s}[0]_z + 2t \end{bmatrix}$$

$$\vec{s}_{\text{other}}[t] = \vec{s}_{\text{other}}[0] + \begin{bmatrix} -6 \\ -8 \\ 2 \end{bmatrix} t = \begin{bmatrix} -4-6t \\ 2-8t \\ 4+2t \end{bmatrix}$$

Because a collision requires $\vec{s}_{\text{you}} = \vec{s}_{\text{other}}$ at the same t, the corresponding components of the two position vectors can be equated:

$$\vec{s}[0]_x - 2t = -4-6t$$
$$\vec{s}[0]_y = 2-8t$$
$$\vec{s}[0]_z + 2t = 4+2t$$

Substituting the choices provided to check which starting coordinates satisfy the above equations, we find that both (3) at $t = 2$ and (4) at $t = 1$ satisfy the system, and are potential collision points.

12. Image Transformation (12 points)

Laura, an aspiring photographer, took a picture of the Golden Gate Bridge last week. However, it didn't turn out as perfect as she wanted it. So, she started using Adobe Photoshop to edit the image. Unfortunately, she left her computer alone for a few minutes and her dog accidentally sat on the keyboard! The image now looks like this.

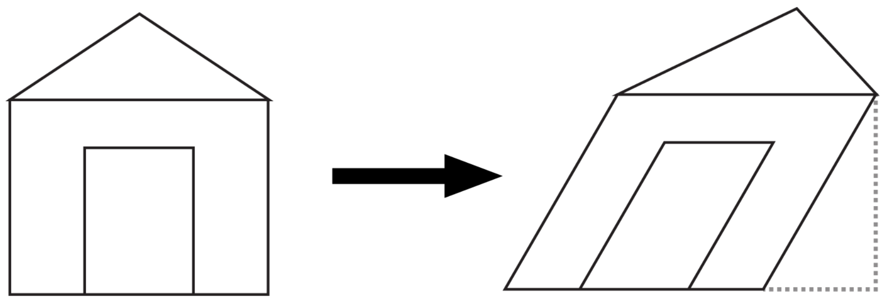

Figure 2: Image Transformation

No matter what she does, Laura can't seem to get the original image back. However, she figures out that her dog applied a transformation, T, to the original image of this form:

$$\begin{bmatrix} x_{new} \\ y_{new} \\ 1 \end{bmatrix} = T \begin{bmatrix} x_{orig} \\ y_{orig} \\ 1 \end{bmatrix} = \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{orig} \\ y_{orig} \\ 1 \end{bmatrix}$$

(a) Laura asks her friend for help and together they determine 3 out of the 6 unknowns of the the transformation matrix that was applied:

$$T = \begin{bmatrix} 3 & t_1 & t_2 \\ t_3 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}$$

Laura measures the following pairs of pixel locations:

(x_{orig}, y_{orig})	(x_{new}, y_{new})
$(1,1)$	$(14,9)$
$(2,1)$	$(17,11)$
$(5,2)$	$(32,18)$
$(3,4)$	$(38,16)$

Use this information to solve for the remaining unknowns in the matrix T.

Solution: We can solve for the missing values of the transformation matrix by setting up a system of equations where t_1, t_2, and t_3 are the unknowns. The transformation is applied on an individual pixel basis like so:

$$\begin{bmatrix} 3 & t_1 & t_2 \\ t_3 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{orig} \\ y_{orig} \\ 1 \end{bmatrix} = \begin{bmatrix} x_{new} \\ y_{new} \\ 1 \end{bmatrix}$$

We can expand this matrix-vector equation to get individual equations:

$$3x_{orig} + t_1 y_{orig} + t_2 = x_{new}$$

$$t_3 x_{orig} + y_{orig} + 6 = y_{new}$$

$$0x_{orig} + 0y_{orig} + 1 = 1$$

Now, we want to rearrange these individual equations to isolate the unknowns t_1, t_2, and t_3. The first equation becomes:

$$(1) : y_{orig}t_1 + t_2 = x_{new} - 3x_{orig}$$

The second equation becomes:

$$(2) : x_{orig}t_3 = y_{new} - y_{orig} - 6$$

The third equation does not help us because it doesn't contain any of the unknowns. We know values for x_{orig}, y_{orig}, x_{new}, and y_{new} that we can plug into the equations above. Each point can give us 2 new equations. Given that we have 3 unknowns, we need at least 2 points for 4 equations. Using the first two points above, we get the following 4 equations:

$$\text{Equation (1) with Point (1, 1)} : t_1 + t_2 = 11$$

$$\text{Equation (2) with Point(1, 1)} : t_3 = 2$$

$$\text{Equation (1) with Point (2, 1)} : t_1 + t_2 = 11$$

$$\text{Equation(2) with Point (2,1)} : 2t_3 = 4$$

The first two points give us 4 equations, but the second set of equations is a linear combination of the first. By trying an additional point, we can get another unique piece of information to use instead.

$$\text{Equation (1) with Point (5, 2)} : 2t_1 + t_2 = 17$$

$$\text{Equation(2) with Point (5,2)} : 5t_3 = 10$$

Now, we can set up these equations as a matrix-vector equation and solve for t_1, t_2, and t_3 using Gaussian Elimination or algebraic substitution.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix} = \begin{bmatrix} 11 \\ 2 \\ 17 \\ 10 \end{bmatrix}$$

Solving the system yields $t_1 = 6$, $t_2 = 5$, $t_3 = 2$.

(b) Now that Laura knows the transformation her dog applied, she decides to try to undo the transformation using its inverse. However, she's not sure if the inverse is unique.

Prove that if R and X are both inverses of T, then $R = X$. Assume that R, X, and T are all $n \times n$ matrices. Finish the proof by filling in the blanks from the options below. Remember, each equation should follow directly from the previous equation in the proof. Options can be used more than once.

$$XT = I$$
$$\boxed{?}\boxed{?}\boxed{?} = \boxed{?}\boxed{?}$$
$$XI = R$$
$$X = R$$

(1) T (2) R (3) X (4) I

Solution: We are trying to prove that if R and X are both inverses of T, then $R = X$. If R and X are both inverses of T, then $T^{-1} = X = R$. Consider the first line of the proof:

$$XT = I$$

We know this is true because $XT = T^{-1}T = I$. Now consider the third line of the proof:

$$XI = R$$

How did we get from line (1) to line (3)? We know that $I = T^{-1}T = TT^{-1}$ from the first line of the proof, so we can plug that in for I in the third line:

$$XTT^{-1} = R$$

We also know that $T^{-1} = R$, so we can plug that in as well and rewrite the equation:

$$(X)(T)(R) = (I)(R)$$

Therefore, we've shown the elements that make up line (2) are $XTR = IR$.

(c) Laura sets up an augmented matrix of the form

$$\left[\begin{array}{c|c} T & X \end{array}\right]$$

where T, R, and X are all 4×4 matrices. After applying Gaussian Elimination, Laura gets the following augmented matrix:

$$\left[\begin{array}{c|c} I & R \end{array}\right]$$

What equation did she show to be true?

Solution: Consider a matrix vector equation that we're the most familiar with:

$$\left[\begin{array}{c|c} A & \vec{b} \end{array}\right]$$

We set up this augmented matrix when we are trying to solve for $\vec{x}$ in the equation $A\vec{x} = \vec{b}$. To solve via Gaussian Elimination, we row-reduce the augmented matrix until we get:

$$\left[\begin{array}{c|c} \quad I \quad & \vec{x} \end{array}\right]$$

Therefore, following that same logic for the situation above, this augmented matrix is set up to solve for Y in the equation $TY = X$:

$$\left[\begin{array}{c|c} \quad T \quad & \quad X \quad \end{array}\right]$$

$$\left[\begin{array}{c|c} \quad I \quad & \quad R \quad \end{array}\right]$$

Therefore, the row-reduced matrix indicates that $Y = R$. If we plug in R for Y in the above equation we get $TR = X$.

13. Blurry Images (16 points)

Let $\vec{x} \in \mathbb{R}^n$ be a sharp image. We use a camera to take a picture of $\vec{x}$, but our camera is out of focus so our captured image, $\vec{y} \in \mathbb{R}^n$, is blurry. We can represent the blurring function as a linear transformation, B:

$$\vec{y} = B\vec{x}$$

Can we get back the sharp image from our blurry measurement? Examining the nullspace of B, denoted $N(B)$, will help us find out!

(a) Suppose we know that there are two sharp images, $\vec{x}_1$ and $\vec{x}_2$, that both result in the same blurry image $\vec{y}$.

$$\vec{y} = B\vec{x}_1 \qquad \vec{y} = B\vec{x}_2$$

Select all of the statements that are always true for any $\vec{x}_1$ and $\vec{x}_2$. Assume $\vec{y}$ is nonzero.

(1) $(4\vec{x}_1 + 10\vec{x}_2) \in N(B)$

(2) $(-8\vec{x}_1 + 8\vec{x}_2) \in N(B)$

(3) $(2\vec{x}_2 - 2\vec{x}_1) \in N(B)$

(4) $(2\vec{x}_2 - 18\vec{x}_1) \in N(B)$

(5) $\vec{x}_2 \in N(B)$

Solution: We're interested in the nullspace of B. Notice that we can subtract the two equations as follows:

$$\begin{aligned} \vec{y} - \vec{y} &= B\vec{x}_1 - B\vec{x}_2 \\ \vec{0} &= B(\vec{x}_1 - \vec{x}_2) \end{aligned}$$

Therefore, $(\vec{x}_1 - \vec{x}_2)$ is in the nullspace of B. But we also know that the any multiple of a vector in the nullspace is also in the nullspace. Consider a scalar, α:

$$\begin{aligned} \vec{0} &= B(\vec{x}_1 - \vec{x}_2) \\ \alpha\vec{0} &= \alpha B(\vec{x}_1 - \vec{x}_2) \\ \vec{0} &= B\alpha(\vec{x}_1 - \vec{x}_2) \end{aligned}$$

So any vector of the form $(\alpha\vec{x}_1 - \alpha\vec{x}_2)$ is in the null space of B.
Note that other generic linear combinations of $\vec{x}_1$ and $\vec{x}_2$ are not in the null space:

$$\begin{aligned} B(\alpha\vec{x}_1 - \beta\vec{x}_2) &= \alpha B\vec{x}_1 + \beta B\vec{x}_2 \\ &= \alpha\vec{y} + \beta\vec{y} \\ &= (\alpha + \beta)\vec{y} \end{aligned}$$

Since we assumed $\vec{y}$ to be nonzero, $B(\alpha\vec{x}_1 - \beta\vec{x}_2)$ is only equal to zero if $\alpha = -\beta$, which is the situation described above.

(b) Suppose we know that

$$\begin{bmatrix} -3 \\ 5 \\ 3 \end{bmatrix} \in N(B) \qquad \text{and} \qquad \vec{x} = \begin{bmatrix} 13 \\ -11 \\ 10 \end{bmatrix} \text{ is a solution to } \vec{y} = B\vec{x}.$$

Find a different image $\vec{w}$, where $\vec{w} \neq \vec{x}$, that is guaranteed to satisfy the measurement (e.g. $\vec{y} = B\vec{w}$).

Solution: Let $\vec{v} \in \text{Null}(B)$. This means that $B\vec{v} = \vec{0}$. We can add this to a known solution of $\vec{y} = B\vec{x}$ as follows:

$$\begin{aligned} \vec{y} &= B\vec{x} \\ \vec{y} + \alpha\vec{0} &= B\vec{x} + \alpha B\vec{v} \quad \text{(here we use the fact that } B\vec{v} = \vec{0}\text{)} \\ \vec{y} &= B(\vec{x} + \alpha\vec{v}) \end{aligned}$$

Therefore any vector of the form $\vec{x} + \alpha\vec{v}$ is another solution to the system of equations. For example, if $\alpha = 1$, we get

$$\vec{w} = \begin{bmatrix} 13 \\ -11 \\ 10 \end{bmatrix} + \begin{bmatrix} -3 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ -6 \\ 13 \end{bmatrix}.$$

Your friend suggests you try changing the nullspace of the blurring operator by applying an invertible linear transformation after capturing the image.

In the next two parts, you'll prove that

$$N(AB) = N(B)$$

for an invertible matrix A where $A, B \in \mathbb{R}^{n \times n}$.

(c) First we'll show that any vector $\vec{v}$ in the nullspace of B is also in the nullspace of AB. Complete the proof by filling in the question marks from the bank of options below. Each step should follow logically from the previous step, and each step can only be used once at most.

$\vec{v} \in N(B)$
?
?
?
$\vec{v} \in N(AB)$

Options:

A:	$A\vec{v} = \vec{0}$	**B:**	$A\vec{v}B = \vec{0}$	**C:**	$B\vec{v} = \vec{0}$
D:	$\vec{v} = B^{-1}\vec{0}$	**E:**	$AB\vec{v} = A\vec{0}$	**F:**	$BA\vec{v} = \vec{0}$
G:	$A\vec{0} = \vec{v}$	**H:**	$B^{-1}B\vec{v} = B^{-1}\vec{0}$	**I:**	$B\vec{v} = A^{-1}\vec{0}$
J:	$A^{-1}AB\vec{v} = A^{-1}\vec{0}$	**K:**	$B^2\vec{v} = B\vec{0}$	**L:**	$B\vec{v} = \vec{0}A^{-1}$
M:	$AB\vec{v} = \vec{0}$	**N:**	$A\vec{v} = B^{-1}\vec{0}$	**O:**	$B\vec{0} = \vec{v}$

To enter your answer in Gradescope, type the 3 letters corresponding to your choices in order. For example, if your answer is the first row of choices (in order), type "ABC" in Gradescope. To leave a spot blank, type "?" in that spot. For example, "A?B" indicates that you are not making a choice for the second unknown. Remember, there will be no additional penalty for incorrect answers.

Solution:

The completed proof is filled in below with explanations for each step.

$\vec{v} \in N(B)$	Given
$B\vec{v} = \vec{0}$	Definition of nullspace
$AB\vec{v} = A\vec{0}$	Left multiply both sides by A
$AB\vec{v} = \vec{0}$	A matrix times the zero vector is zero
$\vec{v} \in N(AB)$	Definition of nullspace

(d) Next, we'll show that any vector $\vec{u}$ in the nullspace of AB is also in the nullspace of B. Complete the proof by filling in the question marks from the bank of options below. Each step should follow logically from the previous step, and each step can only be used once at most.

$\vec{u} \in N(AB)$
?
?
?
?
$\vec{u} \in N(B)$

Options:

A:	$B\vec{u} = \vec{0}A^{-1}$	**B:**	$A^{-1}AB\vec{u} = A^{-1}\vec{0}$	**C:**	$A\vec{u} = \vec{0}$
D:	$AB\vec{u} = \vec{0}$	**E:**	$B\vec{0} = \vec{u}$	**F:**	$B^2\vec{u} = B\vec{0}$
G:	$A\vec{0} = \vec{u}$	**H:**	$B^{-1}B\vec{u} = B^{-1}\vec{0}$	**I:**	$AB\vec{u} = A\vec{0}$
J:	$\vec{u} = B^{-1}\vec{0}$	**K:**	$B\vec{u} = \vec{0}$	**L:**	$B\vec{u} = A^{-1}\vec{0}$
M:	$A\vec{u} = B^{-1}\vec{0}$	**N:**	$A\vec{u}B = \vec{0}$	**O:**	$BA\vec{u} = \vec{0}$

To enter your answer in Gradescope, type the 4 letters corresponding to your choices in order. For example, "ABCD." To leave a spot blank, type "?" in that spot. For example, "A?BC" indicates that you are not making a choice for the second unknown. Remember, there will be no additional penalty for incorrect answers.

Solution: The completed proof is filled in below with explanation for each step.

$\vec{u} \in N(AB)$	Given
$AB\vec{u} = \vec{0}$	Definition of nullspace
$A^{-1}AB\vec{u} = A^{-1}\vec{0}$	Left multiply both sides by A^{-1}. This is ok because A is invertible.
$B\vec{u} = A^{-1}\vec{0}$	Definition of inverse
$B\vec{u} = \vec{0}$	A matrix times the zero vector is zero
$\vec{u} \in N(AB)$	Definition of null space