1. Matrix Multiplication Proof

(a) Given that matrix A is square and has linearly independent columns, which of the following are true?
   
   i. A is full rank
   ii. A has a trivial nullspace
   iii. $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$
   iv. A is invertible
   v. The determinant of A is non-zero
(b) Let two square matrices $M_1, M_2 \in \mathbb{R}^{2 \times 2}$ each have linearly independent columns. Prove that $G = M_1M_2$ also has linearly independent columns.

2. The Romulan Ruse

While scanning parts of the galaxy for alien civilization, the starship USS Enterprise NC-1701D encounters a Romulan starship that is known for advanced cloaking devices.

(a) The Romulan illusion technology causes a point $(x_0, y_0)$ to transform or map to $(u_0, v_0)$. Similarly, $(x_1, y_1)$ is mapped to $(u_1, v_1)$. Figure 1 and Table 1 show these points.
Find a transformation matrix $A_0$ such that

$$
\begin{bmatrix}
u_0 \\ v_0
\end{bmatrix} = A_0 \begin{bmatrix}u_0 \\ v_0
\end{bmatrix}, \text{ and } \begin{bmatrix}u_1 \\ v_1
\end{bmatrix} = A_0 \begin{bmatrix}x_1 \\ y_1
\end{bmatrix}.
$$

Table 1: Original and Mapped Points

<table>
<thead>
<tr>
<th>Original Point</th>
<th>Mapped Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x_0, y_0) = (500, 500)$</td>
<td>$(u_0, v_0) = (500, 1500)$</td>
</tr>
<tr>
<td>$(x_1, y_1) = (1000, 500)$</td>
<td>$(u_1, v_1) = (1000, 1500)$</td>
</tr>
</tbody>
</table>
(b) In this scenario, every point on the Romulan ship \((x_m, y_m)\) is mapped to \((u_m, v_m)\), such that vector \[
\begin{bmatrix}
x_m \\
y_m
\end{bmatrix}
\]
is rotated counterclockwise by 30° and then scaled by 2 in the x- and y-directions. This transformation is shown in Figure 2.

![Figure 2: Figure for part (b)](image)

<table>
<thead>
<tr>
<th>(\theta)</th>
<th>(\sin \theta)</th>
<th>(\cos \theta)</th>
<th>(\tan \theta)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>30°</td>
<td>(\frac{1}{2})</td>
<td>(\frac{\sqrt{3}}{2})</td>
<td>(\frac{1}{\sqrt{3}})</td>
</tr>
<tr>
<td>45°</td>
<td>(\frac{1}{\sqrt{2}})</td>
<td>(\frac{1}{\sqrt{2}})</td>
<td>1</td>
</tr>
<tr>
<td>60°</td>
<td>(\frac{\sqrt{3}}{2})</td>
<td>(\frac{1}{2})</td>
<td>(\sqrt{3})</td>
</tr>
<tr>
<td>90°</td>
<td>1</td>
<td>0</td>
<td>(\infty)</td>
</tr>
</tbody>
</table>

Table 2: Trigonometric Table

**Find a transformation matrix \(R\) such that**
\[
\begin{bmatrix}
u_m \\
v_m
\end{bmatrix} = R \begin{bmatrix}x_m \\
y_m
\end{bmatrix}.
\]
The Romulan ship has launched a probe into space and the Enterprise is trying to destroy the probe by firing a photon torpedo along a straight line from point \((0, 0)\) towards the probe.

(c) The Romulan generals found a clever way to hide the probe by transforming (mapping) its position with a *cloaking* (transformation) matrix \(A_p\):

\[
A_p = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.
\]

They positioned the probe at \((x_p, y_p)\) so that it maps to \((u_p, v_p) = (0, 0)\), where

\[
\begin{bmatrix} u_p \\ v_p \end{bmatrix} = A_p \begin{bmatrix} x_p \\ y_p \end{bmatrix}.
\]

This scenario is shown in Figure 3. The initial position of the torpedo is \((0, 0)\) and the torpedo cannot be fired on its initial position! Impressive trick indeed!

**Find the possible positions of the probe** \((x_p, y_p)\) **so that** \((u_p, v_p) = (0, 0)\).
(d) It turns out the Romulan engineers were not as smart as the Enterprise engineers. Their calculations
did not work out and they positioned the probe at \((x_q, y_q)\) such that the *cloaking* (transformation) ma-
trix, \(A_p\), mapped it to \((u_q, v_q)\), where

\[
\begin{bmatrix}
u_q \\ v_q
\end{bmatrix} = A_p \begin{bmatrix} x_q \\ y_q
\end{bmatrix}, \text{ and } A_p = \begin{bmatrix} 1 & 3 \\ 2 & 6
\end{bmatrix}.
\]

As a result, the torpedo, while traveling straight from
\((0,0)\) to \((u_q, v_q)\), hit the probe at \((x_q, y_q)\) on the way!
The scenario is shown in Figure 4. For the torpedo to
hit the probe, we must have

\[
\begin{bmatrix} u_q \\ v_q
\end{bmatrix} = \lambda \begin{bmatrix} x_q \\ y_q
\end{bmatrix}, \text{ where } \lambda
\]
is a real number.

**Find the possible positions of the probe \((x_q, y_q)\) so that \((u_q, v_q) = (\lambda x_q, \lambda y_q)\). Remember that the
torpedo cannot be fired on \((0,0)\). This means that \((u_q, v_q) = (\lambda x_q, \lambda y_q)\) cannot be \((0,0)\).**

![Figure 4: Figure for part (d)](image-url)