EECS 16A Designing Information Devices and Systems I
Fall 2022 Homework 2

This homework is due September 16th, 2022, at 23:59.
Self-grades are due September 19th, 2022, at 23:59.

Submission Format

Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned) as well as your IPython notebook saved as a PDF.

If you do not attach a PDF "printout" of your IPython notebook, you will not receive credit for problems that involve coding. Make sure that your results and your plots are visible. Assign the IPython printout to the correct problem(s) on Gradescope.

1. Reading Assignment

For this homework, please read Note 2A, Note 2B, Note 3, and Note 4. Notes 2A and 2B provide an overview of vectors, matrices, and operations among them. Note 3 provides an overview of linear dependence (not yet covered) and span. Note 4 introduces mathematical thinking and writing proofs.

Please answer the following questions:

(a) What is the span of a set of vectors?

(b) How can you check if a particular vector is in the span of a set of vectors?

(c) Given that $\vec{b} \in \text{span}\{\ \vec{a}_1, \vec{a}_2, \vec{a}_3\ \}$ and $\vec{a}_1, \vec{a}_2, \vec{a}_3$ are column vectors of $\mathbf{A}$, which *one* of the following statements does not make sense:
   i. $\vec{b}$ is in the span of matrix $\mathbf{A}$
   ii. $\vec{b}$ is in the range of $\mathbf{A}$
   iii. $\vec{b}$ is in the column space of $\mathbf{A}$

(d) Please write a few sentences about how you can use the strategies in the notes to tackle proof questions.

2. Filtering Out The Troll

Learning Goal: *The goal of this problem is to explore the problem of sound reconstruction by solving a system of linear equations.*

You were attending the 16A lecture the day before the first exam, and decided to record it using two directional microphones (one microphone receives sound from the x direction and the other from the y direction). However, someone (we have *no* idea who) in the audience was trolling around loudly, adding interference to the recording! The troll's interference dominates both of your microphones' recordings, so you cannot hear the recorded speech. Fortunately, since your recording device contained two microphones, you can combine the two individual microphone recordings to remove the troll's interference.

The diagram shown in Figure 1 shows the locations of the speaker, the troll, and you and your two microphones (at the origin).

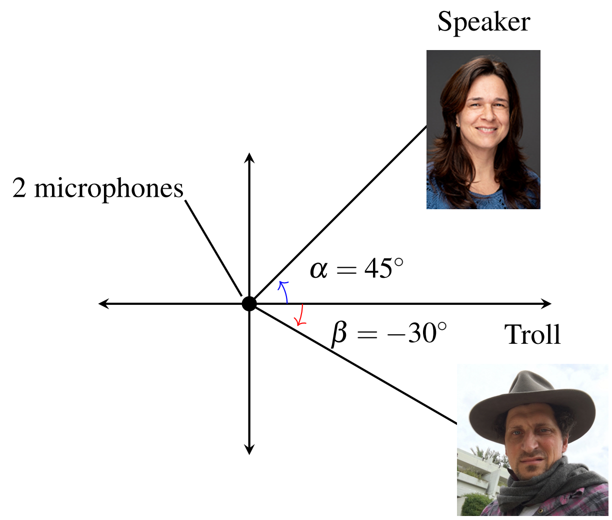

Figure 1: Locations of the speaker and the troll.

Since the microphones are directional, the strength of the recorded signal depends on the angle from which the sound arrives. Suppose that the sound arrives from an angle θ relative to the x-axis (in our case, these angles are 45°and $-30°$, labeled as α and β, respectively). The first microphone scales the signal by $\cos(\theta)$, while the second microphone scales the signal by $\sin(\theta)$. Each microphone records the weighted sum (or linear combination) of all received signals.

The speech signal can be represented as a vector, $\vec{s}$, and the troll's interference as vector $\vec{r}$, with each entry representing an audio sample at a given time. The recordings of the two microphones are given by $\vec{m}_1$ and $\vec{m}_2$:

$$\vec{m}_1 = \cos(\alpha) \cdot \vec{s} + \cos(\beta) \cdot \vec{r} \tag{1}$$

$$\vec{m}_2 = \sin(\alpha) \cdot \vec{s} + \sin(\beta) \cdot \vec{r} \tag{2}$$

where α and β are the angles at which the professor and the troll respectively are located with respect to the x-axis, and variables $\vec{s}$ and $\vec{r}$ are the audio signals produced by the professor and the troll respectively.

(a) Plug in $\alpha = 45° = \frac{\pi}{4}$ and $\beta = -30° = -\frac{\pi}{6}$ to Equations 1 and 2 to write the recordings of the two microphones $\vec{m}_1$ and $\vec{m}_2$ as a linear combination (i.e. a weighted sum) of $\vec{s}$ and $\vec{r}$.

(b) Solve the system from the earlier part using any convenient method you prefer to recover the important speech $\vec{s}$ as a weighted combination of $\vec{m}_1$ and $\vec{m}_2$. In other words, write $\vec{s} = c \cdot \vec{m}_1 + k \cdot \vec{m}_2$ (where c and k are scalars). What are the values of c and k?

(c) Partial IPython code can be found in `prob2.ipynb`, which you can access through the Datahub link associated with this assignment on the course website. Complete the code to get the signal of the important speech. Write out what the speaker says. (Optional: Where is the speech taken from?)

Note: You may have noticed that the recordings of the two microphones sound remarkably similar. This means that you could recover the real speech from two "trolled" recordings that sound almost identical! Leave out the fact that the recordings are actually different, and have some fun with your friends who aren't lucky enough to be taking EECS16A.

3. Gaussian Elimination

Learning Goal: *Understand the relationship between Gaussian elimination and the graphical representation of linear equations, and explore different types of solutions to a system of equations. You will also practice determining the parametric solutions when there are infinitely many solutions.*

(a) In this problem we will investigate the relationship between Gaussian elimination and the geometric interpretation of linear equations. You are welcome to draw plots by hand or using software. Please be sure to label your equations with a legend on the plot.

i. Plot the following set of linear equations in the x-y plane. If the lines intersect, write down the point or points of intersection.

$$x + 2y = 4 \tag{1}$$
$$2x - 4y = 4 \tag{2}$$
$$3x - 2y = 8 \tag{3}$$

ii. Write the above set of linear equations in augmented matrix form and do the first step of Gaussian elimination to eliminate the x variable from equation 2. Now, the second row of the augmented matrix has changed. Plot the corresponding new equation created in this step on the same graph as above. What do you notice about the new line you draw?

iii. Complete all of the steps of Gaussian elimination including back substitution. Now plot the new equations represented by the rows of the augmented matrix in the last step (after completing back substitution) on the same graph as above. What do you notice about the new line you draw?

(b) Write the following set of linear equations in augmented matrix form and use Gaussian elimination to determine if there are no solutions, infinite solutions, or a unique solution. If any solutions exist, determine what they are. You may do this problem by hand or use a computer. We encourage you to try it by hand to ensure you understand Gaussian elimination. Remember that it is possible to end up with fractions during Gaussian elimination.

$$\begin{aligned} x + 2y + 5z &= 3 \\ x + 12y + 6z &= 1 \\ 2y + z &= 4 \\ 3x + 16y + 16z &= 7 \end{aligned}$$

(c) Consider the following system:

$$\begin{aligned} 4x + 4y + 4z + w + v &= 1 \\ x + y + 2z + 4w + v &= 2 \\ 5x + 5y + 5z + w + v &= 0 \end{aligned}$$

If you were to write the above equations in augmented matrix form and use Gaussian elimination to solve the system, you would get the following (for extra practice, you can try and do this yourself):

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 3 & 16 \\ 0 & 0 & 1 & 0 & -3 & -17 \\ 0 & 0 & 0 & 1 & 1 & 5 \end{array}\right]$$

How many variables are free variables? Which ones? Find the general form of the solutions in terms of arbitrary real numbers.

4. Linearity

Learning Goal: *Review the definition of linearity and further develop the ability to recognize and identify linear functions.*

In this question, we will explore in further detail what exactly it means for a function to be linear. For each of the following, please specify the value(s) of $a \in \mathbb{R}$, if any, for which the function is linear. Here, x and y are variables.

(a)

$$f(x,y) = (3-a)x + 2ay$$

(b)

$$f(x,y) = a^2x + 8y$$

(c)

$$f(x,y) = y + axy - 3x$$

(d)

$$f(x,y) = (x+ay)^2$$

5. Vector-Vector, Matrix-Vector, and Matrix-Matrix Multiplication

Learning Objective: *Practice evaluating vector-vector, matrix-vector, and matrix-matrix multiplication.*

(a) For the following multiplications, state the dimensions of the result. If the product is not defined and thus has no solution, state this and justify your reasoning. For this problem $\vec{x} \in \mathbb{R}^N, \vec{y} \in \mathbb{R}^N, \vec{z} \in \mathbb{R}^M$, with $N \neq M$.

i. $\vec{x}^T \cdot \vec{z}$

ii. $\vec{x} \cdot \vec{x}^T$

iii. $\vec{x} \cdot \vec{y}^T$

iv. $\vec{x} \cdot \vec{z}^T$

For questions (b) through (d), complete the matrix-vector multiplication. If the product is not defined and thus has no solution, state this and justify your reasoning:

(b)

$$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

(d)

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

(e) Compute **AB** by hand, where **A** and **B** are given by

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}, \text{ and } \qquad \mathbf{B} = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -3 & 0 & 2 & -1 \end{bmatrix}$$

What are the dimensions of **AB**? Compute **BA** too if the operation is valid. If it is invalid, explain why. Make sure you show the work for your calculations.

(f) Compute **AB** by hand, where **A** and **B** are given by

$$\mathbf{A} = \begin{bmatrix} 3 & 21 & 9 \\ -1 & 14 & 4 \\ 7 & -8 & 2 \end{bmatrix}, \text{ and } \qquad \mathbf{B} = \begin{bmatrix} -2 & 4 \\ -1 & 2 \\ 3 & -6 \end{bmatrix}$$

Compute **BA** too if the operation is valid. If it is invalid, explain why. Make sure you show the work for your calculations.

6. Vectors in the Span

Learning Goal: *Practice determining whether a vector is in the span of a set of vectors.*

Determine whether a vector $\vec{v}$ is in the span of the given set of vectors. If it is in the span of given set, write $\vec{v}$ as a linear combination of given set of vectors (you will need to find the scalar coefficients in the linear combination).

(a) $\vec{v} = \begin{bmatrix} -10 \\ 4 \end{bmatrix}$ and $\left\{ \begin{bmatrix} -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$

(b) $\vec{v} = \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ and $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \\ -1 \end{bmatrix} \right\}$

(c) $\vec{v} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ and $\left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix} \right\}$

(d) $\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \\ 0 \end{bmatrix} \right\}$

7. Span Proofs

Learning Objectives: *This is an opportunity to practice your proof development skills.*

(a) Given some set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, show the following:

$$\operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} = \operatorname{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$$

In other words, we can replace one vector with the sum of itself and another vector and not change their span.

In order to show this, you have to prove the two following statements:

- If a vector $\vec{q}$ belongs in $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, then it must also belong in $\operatorname{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$.
- If a vector $\vec{r}$ belongs in $\operatorname{span}\{\vec{v}_1 + \vec{v}_2, \vec{v}_2, \ldots, \vec{v}_n\}$, then it must also belong in $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$.

In summary, you have to prove the problem statement from both directions.

(b) Consider the span of the set $(\vec{v}_1, ..., \vec{v}_n, \vec{u})$. Suppose $\vec{u}$ is in the span of $\{\vec{v}_1, ..., \vec{v}_n\}$. Then, show that any vector $\vec{r}$ in $span\{\vec{v}_1, ..., \vec{v}_n, \vec{u}\}$ is in $span\{\vec{v}_1, ..., \vec{v}_n\}$.

8. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.