EECS 16A Designing Information Devices and Systems I
Fall 2022 Homework 3

This homework is due September 23rd, 2022, at 23:59.
Self-grades are due September 26th, 2022, at 23:59.

Submission Format
Your homework submission should consist of a single PDF file that contains all of your answers (any handwritten answers should be scanned).

1. **Reading Assignment** For this homework, please read Notes 3,4, and 11A. Note 3 provides an overview of linear dependence and span, Note 4 gives an introduction to thinking about and writing proofs, and Note 11A gives an introduction to circuits.

   Please answer the following questions:

   (a) Why are there two definitions of linear dependence? What value does each definition provide?

   (b) Why is voltage "across" a circuit element?

2. **Linear Dependence**

   Learning Goal: *Evaluate the linear dependency of a set of vectors.*

   State if the following sets of vectors are linearly independent or dependent. If the set is linearly dependent, provide a linear combination of the vectors that sum to the zero vector.

   (a) $\left\{ \begin{bmatrix} -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \end{bmatrix} \right\}$

   (b) $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

   (c) $\left\{ \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$

   (d) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

3. **Linear Dependence in a Square Matrix**

   Learning Objective: *This is an opportunity to practice applying proof techniques. This question is specifically focused on linear dependence of rows and columns in a square matrix.*

Let A be a square $n \times n$ matrix, (i.e. both the columns and rows are vectors in $\mathbb{R}^n$). Suppose we are told that the columns of A are linearly dependent. Prove, then, that the rows of A must also be linearly dependent. You can use the following conclusion in your proof:

If Gaussian elimination is applied to a matrix A, and the resulting matrix (in reduced row echelon form) has at least one row of all zeros, this means that the rows of A are linearly dependent.

(**Hint**: Can you use the linear dependence of the columns to say something about the number of solutions to $A\vec{x} = \vec{0}$? How does the number of solutions relate to the result of Gaussian elimination?)

4. Image Stitching

Learning Objective: *This problem is similar to one that students might experience in an upper division computer vision course. Our goal is to give students a flavor of the power of tools from fundamental linear algebra and their wide range of applications.*

Often, when people take pictures of a large object, they are constrained by the field of vision of the camera. This means that they have two options to capture the entire object:

- Stand as far away as they need to include the entire object in the camera's field of view (clearly, we do not want to do this as it reduces the amount of detail in the image)
- (This is more exciting) Take several pictures of different parts of the object and stitch them together like a jigsaw puzzle.

We are going to explore the second option in this problem. Daniel, who is a professional photographer, wants to construct an image by using "image stitching". Unfortunately, Daniel took some of the pictures from different angles as well as from different positions and distances from the object. While processing these pictures, Daniel lost information about the positions and orientations from which the pictures were taken. Luckily, you and your friend Marcela, with your wealth of newly acquired knowledge about vectors and matrices, can help him!

You and Marcela are designing an iPhone app that stitches photographs together into one larger image. Marcela has already written an algorithm that finds common points in overlapping images. **It's your job to figure out how to stitch the images together using Marcela's common points to reconstruct the larger image.**

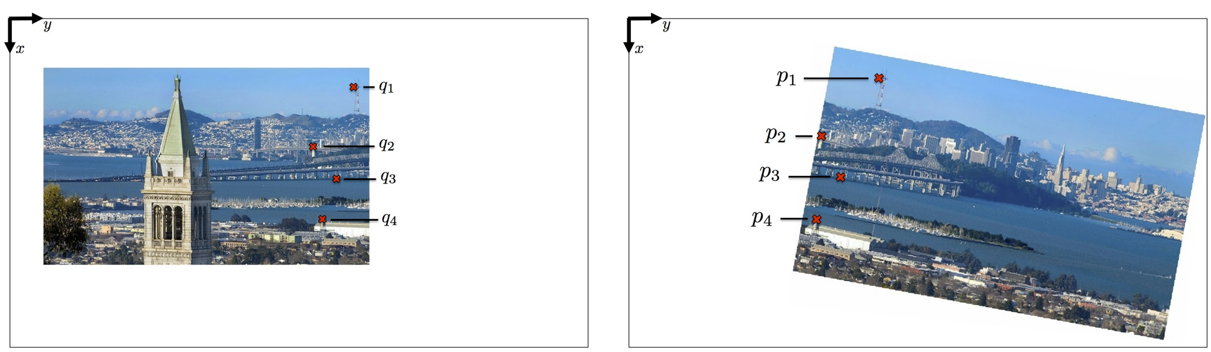


Figure 1: Two images to be stitched together with pairs of matching points labeled.

We will use vectors to represent the common points which are related by a affine transformation. Your idea is to find this affine transformation. For this you will use a single matrix, $\mathbf{R}$, and a vector, $\vec{t}$, that transforms every common point in one image to their corresponding point in the other image. Once you find $\mathbf{R}$ and $\vec{t}$ you will be able to transform one image so that it lines up with the other image.

Suppose $\vec{p} = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$ is a point in one image , which is transformed to $\vec{q} = \begin{bmatrix} q_x \\ q_y \end{bmatrix}$ is the corresponding point in the other image (i.e., they represent the same object in the scene). For example, Fig. 1 shows how the points $\vec{p}_1$, $\vec{p}_2$ ... in the right image are transformed to points $\vec{q}_1$, $\vec{q}_2$ ... on the left image. You write down the following relationship between $\vec{p}$ and $\vec{q}$.

$$\begin{bmatrix} q_x \\ q_y \end{bmatrix} = \underbrace{\begin{bmatrix} r_{xx} & r_{xy} \\ r_{yx} & r_{yy} \end{bmatrix}}_{\mathbf{R}} \begin{bmatrix} p_x \\ p_y \end{bmatrix} + \underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{\vec{t}} \tag{1}$$

This problem focuses on finding the unknowns (i.e. the components of $\mathbf{R}$ and $\vec{t}$), so that you will be able to stitch the image together. *Note that this is the opposite from our usual setting in which we would solve for $\vec{p}$ given all other variables.*

(a) To understand how the matrix $\mathbf{R}$ and vector $\vec{t}$ transforms any vector representing a point on a image, Consider this example equation similar to Equation (1),

$$\vec{v} = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \vec{u} + \vec{w} = \vec{v}_1 + \vec{w}. \tag{2}$$

Use $\vec{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for this part.

We want to find out what geometric transformation(s) can be applied on $\vec{u}$ to give $\vec{v}$.

Step 1: Find out how $\begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$ is transforming $\vec{u}$. Evaluate $\vec{v}_1 = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \vec{u}$.

What **geometric transformation(s)** might be applied to $\vec{u}$ to get $\vec{v}_1$? Choose the options that answers the question and explain your choice.

(i) Rotation
(ii) Scaling
(iii) Shifting/Translation

Drawing the vectors $\vec{u}$, and $\vec{v}_1$ in two dimensions on a single plot might help you to visualize the transformations.

Step 2: Find out $\vec{v} = \vec{v}_1 + \vec{w}$. Find out how **addition of $\vec{w}$ is geometrically transforming** $\vec{v}_1$. Choose the option that answers the question and explain your choice.

(i) Rotation
(ii) Scaling
(iii) Shifting/Translation

Drawing the vectors $\vec{v}$, $\vec{w}$, and $\vec{v}_1$ in two dimensions on a single plot might help you to visualize the transformations.

(b) Now back to the main problem. First, multiply Equation (1) out into **two equations**.

(i) What are the known values and what are the unknown values in each equation (recall what we are trying to solve for in Equation (1))?
(ii) How many unknown values are there?

(iii) How many independent equations do you need to solve for all the unknowns?

(iv) How many pairs of common points $\vec{p}$ and $\vec{q}$ will you need in order to write down a system of equations that you can use to solve for the unknowns? ***Hint**: Remember that each pair of $\vec{p}$ and $\vec{q}$ is related by two equations, one for each coordinate.*

(c) Use what you learned in the above two subparts to explicitly write out **just enough** equations of these transformations as you need to solve the system. Assume that the four pairs of points from Fig. 1 are labeled as:

$$\vec{q}_1 = \begin{bmatrix} q_{1x} \\ q_{1y} \end{bmatrix}, \vec{p}_1 = \begin{bmatrix} p_{1x} \\ p_{1y} \end{bmatrix} \qquad \vec{q}_2 = \begin{bmatrix} q_{2x} \\ q_{2y} \end{bmatrix}, \vec{p}_2 = \begin{bmatrix} p_{2x} \\ p_{2y} \end{bmatrix} \qquad \vec{q}_3 = \begin{bmatrix} q_{3x} \\ q_{3y} \end{bmatrix}, \vec{p}_3 = \begin{bmatrix} p_{3x} \\ p_{3y} \end{bmatrix} \qquad \vec{q}_4 = \begin{bmatrix} q_{4x} \\ q_{4y} \end{bmatrix}, \vec{p}_4 = \begin{bmatrix} p_{4x} \\ p_{4y} \end{bmatrix}.$$

(d) Remember that we are ultimately solving for the components of the $\mathbf{R}$ matrix and the vector $\vec{t}$. This is different from our usual setting and so we need to reformulate the problem into something we are more used to (i.e., $A\vec{x} = \vec{b}$ where x is the unknown). In order to do this, let's view the components of $\mathbf{R}$ and $\vec{t}$ as the unknowns in the equations from from part c). We then have a system of linear equations which we should be able to write in the familiar matrix-vector form. Specifically, we can store the unknowns in a vector $\vec{\alpha} = \begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{yx} \\ r_{yy} \\ t_x \\ t_y \end{bmatrix}$ and specify 6×6 matrix $\mathbf{A}$ and vector $\vec{b}$ such that $A\vec{\alpha} = \vec{b}$. Please write out the entries of $\mathbf{A}$ and $\vec{b}$ to match your equations from part c). To get you started, we provide the first row of $\mathbf{A}$ and first entry of $\vec{b}$ which corresponds to one possible equation from part c):

$$\begin{bmatrix} p_{1x} & p_{1y} & 0 & 0 & 1 & 0 \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \\ ? & ? & ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} r_{xx} \\ r_{xy} \\ r_{yx} \\ r_{yy} \\ t_x \\ t_y \end{bmatrix} = \begin{bmatrix} q_{1x} \\ ? \\ ? \\ ? \\ ? \\ ? \end{bmatrix}.$$

Your job is the fill in the remaining entries according to the other equations.

(e) In the IPython notebook `prob4.ipynb`, you will have a chance to test out your solution. Plug in the values that you are given for p_x, p_y, q_x, and q_y for each pair of points into your system of equations to solve for the matrix, $\mathbf{R}$, and vector, $\vec{t}$. The notebook will solve the system of equations, apply your transformation to the second image, and show you if your stitching algorithm works. **You are NOT responsible for understanding the image stitching code or Marcela's algorithm.** What are the values for $\mathbf{R}$ and $\vec{t}$ which correctly stitch the images together?

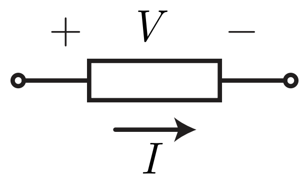
5. Basic Circuit Components

Learning Objectives: *Review basics of cucuit components and current-voltage relationships*

In the laboratory, you are tasked with identifying a single unknown component within a circuit. You can use a piece of electrical equipment called a multimeter to measure either voltage (voltmeter) across or the current (ammeter) through the component (you can measure both quantities simultaneously using two multimeters).

For each part of the problem, deduce the most likely type of circuit component based on the provided voltage and current measurements. Also draw the component circuit symbol and sketch the IV curve.

*Hint: You are told the possible choices are **short circuit (wire)**, **open circuit**, **resistor**, **voltage source**, and **current source**. Moreover, each part in this problem has a unique component, there are no repeats.*



(a) First, to familiarize yourself with common quantities of *voltage*, *current*, and *resistance*, fill in the unit name and unit symbol for each:

Quantity	Symbol	Unit Name	Unit Symbol
Voltage	V		
Current	I		
Resistance	R		

(b) You take one measurement and find the voltage is $V = 10$ V and current is $I = 1$ A. After changing a part of the circuit (not the part you are measuring), you take another measurement and find $V = 10$ V and $I = 2$ A. What is the most likely component type? Draw the component symbol and sketch the IV curve.

(c) You take one measurement and find the voltage is $V = 0$ V and current is $I = 1$ A. What is the most likely component type? Draw the component symbol and sketch the IV curve.

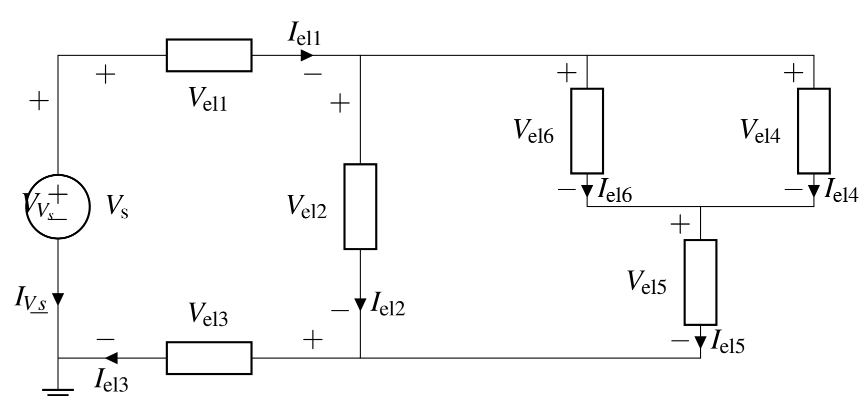
(d) You take one measurement and find the voltage is $V = 5$ V and current is $I = 0$ A. What is the most likely component type? Draw the component symbol and sketch the IV curve.

(e) You take one measurement and find the voltage is $V = 10$ V and current is $I = 2$ A. After changing a part of the circuit (not the part you are measuring), you take another measurement and find $V = -5$ V and $I = -1$ A. What is the most likely component type? Draw the component symbol and sketch the IV curve.

(f) You take one measurement and find the voltage is $V = 10$ V and current is $I = 2$ A. After changing a part of the circuit (not the part you are measuring), you take another measurement and find $V = -3$ V and $I = 2$ A. What is the most likely component type? Draw the component symbol and sketch the IV curve.

6. Intro to Circuits

Learning Goal: *This problem will help you practice labeling circuit elements and setting up KVL equations.*



(a) How many nodes does the above circuit have? Label them.
Note: The reference/0V (ground) node has been selected for you, so you don't need to label that, but you need to include it in your node count.

(b) Express all element voltages (including the element voltage across the source, V_s) as a function of node voltages. This will depend on the node labeling you chose in part (a).

(c) Write a KVL equation for all the loops that contain the voltage source V_s. These equations should be a function of element voltages and the voltage source V_s.

7. Prelab Questions These questions pertain to the Pre-Lab reading for the Imaging 2 lab. You can find the reading under the Imaging 2 Lab section on the 'Schedule' page of the website. We do not expect in-depth answers for the questions. Please limit your answers to a maximum of 2 sentences.

(a) Briefly explain (in 1-2 sentences) what the H matrix signifies.

(b) How will we get the image vector $\vec{i}$ from $\vec{s} = H\vec{i}$, the equation representing our imaging system?

8. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.