EECS 16A Designing Information Devices and Systems I
Fall 2022 Homework 4

This homework is due September 30, 2022, at 23:59.
Self-grades are due October 3, 2022, at 23:59.

Submission Format

Your homework submission should consist of **one** file.

`hw4.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned).

Submit each file to its respective assignment on Gradescope.

1. Reading Assignment

For this homework, please read Notes 5, 6, 7, and 8. Note 5 provides an overview of multiplication of matrices with vectors, by considering the example of water reservoirs and water pumps. Note 6 introduces matrix inversion. Notes 7 and 8 give an overview of matrix vector spaces and subspaces, as well as column spaces and nullspaces. You are always welcome and encouraged to read beyond this as well.

Please answer the following questions:

(a) You have seen in Note 5 that the pump system can be represented by a state transition matrix. What constraint must this matrix satisfy in order for the pump system to obey water conservation?

(b) From Note 8, what are the three necessary properties for a vector space to be a *subspace*?

2. Feedback on your study groups

Please help us understand how your study groups are going! **Fill out the following survey** (even if you are not in a study group) to help us create better matchings in the future. In case you have not been able to connect with a study group, or would like to try a new study group, there will be an opportunity for you to request a new study group as well in this form.

https://forms.gle/TtPvVVZsgZ8W8iLC6

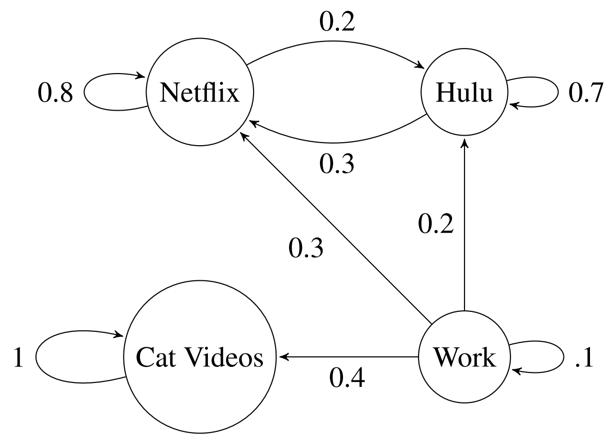
To get full credit for this question you must both

1. Fill out the survey (it will record your email)
2. Indicate in your homework submission that you filled out the survey.

3. Social Media

Learning Objective: *Practice setting up transition matrices from a diagram and understand how to compute subsequent states of the system.*

As a tech-savvy Berkeley student, the distractions of streaming services are always calling you away from productive stuff like homework for your classes. You're curious—are you the only one who spends hours switching between Netflix or Hulu? How do other students manage to get stuff done and balance staying up to date with the Bachelor? You conduct an experiment, collect some data, and notice Berkeley students tend to follow a pattern of behavior similar to the figure below. So, for example, if $x = 100$ students are on Netflix, in the next timestep, 20 (i.e., $0.2 \cdot x$) of them will click on a link and move to Hulu, and 80 (i.e, $0.8 \cdot x$) will remain on Netflix.



(a) Let us define $x_N[n]$ as the number of students on Netflix at time-step n; $x_H[n]$ as the number of students on Hulu at time-step n; $x_C[n]$ as the number of students watching any kind of cat video at time-step n; and $x_W[n]$ as the number of students working at time-step n.

Let the state vector be: $\vec{x}[n] = \begin{bmatrix} x_N[n] \\ x_H[n] \\ x_C[n] \\ x_W[n] \end{bmatrix}$. Derive the corresponding transition matrix $\mathbf{A}$.

Hint: A transition matrix, $\mathbf{A}$, is the matrix that transitions $\vec{x}[n]$, the vector at time-step n to $\vec{x}[n+1]$, the vector at time-step $n+1$. In other words: $\vec{x}[n+1] = \mathbf{A}\vec{x}[n]$.

(b) There are 630 of you in the class. Suppose on a given Friday evening (the day when HW is due), there are 350 EECS16A students on Netflix, 200 on Hulu, 30 watching Cat Videos, and 50 actually doing work. In the next timestep, how many people will be doing each activity? In other words, after you apply the matrix once to reach the next timestep, what is the state vector?

(c) Compute the sum of each column in the state transition matrix. What is the interpretation of this?

4. Inverse Transforms

Learning Objectives: *Matrices represent linear transformations, and their inverses (if they exist) represent the opposite transformation. Here we practice inversion, but are also looking to develop an intuition. Visualizing the transformations might help develop this intuition.*

For each of the following values of matrix A:

i. Find the inverse, $\mathbf{A}^{-1}$, if it exists. If you find that the inverse does not exist, mention how you decided that. Solve this by hand.

ii. **For parts (a)-(d) only**, in addition to finding the inverse (if it exists), describe how the matrix $\mathbf{A}$ geometrically transforms an arbitrary vector $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \in \mathbb{R}^2$.

For example, if $\mathbf{A} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} 2x_0 \\ 2y_0 \end{bmatrix}$, then $\mathbf{A}$ could scale $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ by 2 to get $\begin{bmatrix} 2x_0 \\ 2y_0 \end{bmatrix}$. If $\mathbf{A} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_0 \\ -y_0 \end{bmatrix}$, then $\mathbf{A}$ could reflect $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ across the x-axis, etc. *Hint: It may help to plot a few examples to recognize the pattern.*

iii. **Again, for parts (a)-(d) only**, if we use $\mathbf{A}$ to geometrically transform $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ to get $\begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$, **is it possible to reverse the transformation geometrically,** i.e. is it possible to retrieve $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ from $\begin{bmatrix} u \\ v \end{bmatrix}$ geometrically?

(a) $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

(b) $\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(c) $\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

(d) $\mathbf{A}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ assuming $\cos\theta \neq 0$. *Hint:* Recall $\cos^2\theta + \sin^2\theta = 1$.

In addition to answering subparts i., ii., and iii. (using three-dimensional vector $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \in \mathbb{R}^3$), also prove $\mathbf{A}(\theta)^{-1} = \mathbf{A}(-\theta)$.

(e) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$

(f) $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 1 & 4 & 4 \end{bmatrix}$

(g) $\mathbf{A} = \begin{bmatrix} -1 & 1 & -\frac{1}{2} \\ 1 & 1 & -\frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix}$

(h) **(OPTIONAL)** $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

(i) **(OPTIONAL)** $\mathbf{A} = \begin{bmatrix} 3 & 0 & -2 & 1 \\ 0 & 2 & 1 & 3 \\ 3 & 1 & 0 & 4 \\ 1 & 0 & 0 & 1 \end{bmatrix}$

Hint 1: What do the linear (in)dependence of the rows and columns tell us about the invertibility of a matrix?
Hint 2: We're reasonable people!

5. Subspaces, Bases and Dimension

Learning Objective*: Explore how to recognize and show if a subset of a vector space is or is not a subspace. Further practice identifying a basis for (i.e., a minimal set of vectors which span) an arbitrary subspace.*

For each of the sets $\mathbb{U}$ (which are *subsets* of $\mathbb{R}^3$) defined below, state whether $\mathbb{U}$ is a *subspace* of $\mathbb{R}^3$ or not. If $\mathbb{U}$ is a subspace, find a basis for it and state the dimension.

Note:

- To show $\mathbb{U}$ is a subspace, you have to show that all three properties of a subspace hold.
- To show $\mathbb{U}$ is not a subspace, you only have to show at least one property of a subspace does not hold.

(a) $\mathbb{U} = \left\{ \begin{bmatrix} 2(x+y) \\ x \\ y \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

(b) $\mathbb{U} = \left\{ \begin{bmatrix} x \\ y \\ z+1 \end{bmatrix} \mid x, y, z \in \mathbb{R} \right\}$

(c) $\mathbb{U} = \left\{ \begin{bmatrix} x \\ y \\ x+1 \end{bmatrix} \mid x, y \in \mathbb{R} \right\}$

6. Finding Null Spaces and Column Spaces

Learning Objectives*: Null spaces and column spaces are two fundamental vector spaces associated with matrices and they describe important attributes of the transformations that these matrices represent. This problem explores how to find and express these spaces.*

Definition (Null space): The null space of a matrix, $\mathbf{A} \in \mathbb{R}^{m \times n}$, is the set of all vectors $\vec{x} \in \mathbb{R}^n$ such that $\mathbf{A}\vec{x} = \vec{0}$. The null space is notated as Null($\mathbf{A}$) and the definition can be written in set notation as:

$$Null(\mathbf{A}) = \{\vec{x} \mid \mathbf{A}\vec{x} = \vec{0}, \vec{x} \in \mathbb{R}^n\}$$

Definition (Column space): The column space of a matrix, $\mathbf{A} \in \mathbb{R}^{m \times n}$, is the set of all vectors $\mathbf{A}\vec{x} \in \mathbb{R}^m$ for all choices of $\vec{x} \in \mathbb{R}^n$. Equivalently, it is also the span of the column vector of $\mathbf{A}$. The column space can be notated as $Col(\mathbf{A})$or range($\mathbf{A}$) and the definition can be written in set notation as:

$$Col(\mathbf{A}) = \{\mathbf{A}\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$$

Definition (Dimension): The dimension of a vector space is the number of basis vectors — i.e. the minimum number of vectors required to span the vector space.

(a) Consider a matrix $\mathbf{A} \in \mathbb{R}^{3 \times 5}$. What is the maximum possible number of linearly independent column vectors (i.e. the maximum possible dimension) of $Col(\mathbf{A})$?

(b) You are given the following matrix $\mathbf{A}$.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a *minimum* set of vectors that span $Col(\mathbf{A})$ (i.e. a basis for $Col(\mathbf{A})$). (This problem does not have a unique answer, since you can choose many different sets of vectors that fit the description here.) What is the dimension of $Col(\mathbf{A})$?

Hint: You can do this problem by observation. Alternatively, use Gaussian Elimination on the matrix to identify how many columns of the matrix are linearly independent. The columns with pivots (leading ones) in them correspond to the columns in the original matrix that are linearly independent.

(c) Find a *minimum* set of vectors that span $Null(\mathbf{A})$ (i.e. a basis for $Null(\mathbf{A})$), where $\mathbf{A}$ is the same matrix as in part (b). What is the dimension of $Null(\mathbf{A})$?

(d) For the following matrix $\mathbf{D}$, find $Col(\mathbf{D})$ and its dimension, and $Null(\mathbf{D})$ and its dimension. Using inspection or Gaussian elimination are both valid methods to solve the problem.

$$\mathbf{D} = \begin{bmatrix} 1 & -1 & -3 & 4 \\ 3 & -3 & -5 & 8 \\ 1 & -1 & -1 & 2 \end{bmatrix}$$

(e) Find the sum of the dimensions of $Null(\mathbf{A})$ and $Col(\mathbf{A})$. Also find the sum of the dimensions of $Null(\mathbf{D})$ and $Col(\mathbf{D})$. What do you notice about these sums in relation to the dimensions of $\mathbf{A}$ and $\mathbf{D}$, respectively?

7. Prelab Questions

These questions pertain to the Pre-Lab reading for the Imaging 3 lab. You can find the reading under the Imaging 3 Lab section on the 'Schedule' page of the website. We do not expect in-depth answers for the questions. Please limit your answers to a maximum of 2 sentences.

(a) What properties does the mask matrix H need to have for us to reconstruct the image?

(b) Briefly describe why averaging multiple signals/measurements is a good idea.

(c) How do we get the image back from the new equation that models our system? Note that your answer cannot contain the image vector, $\vec{i}$ (since it is an unknown). You may, however, give an answer that includes $\vec{i}_{est}$.

(d) What allows us to control the effect of noise in our system? *Hint*: Look at the terms in the equation that contains $\vec{i}_{est}$.

8. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.