EECS 16A Designing Information Devices and Systems I
Fall 2022 Homework 13

This homework is due Saturday, December 3rd, 2022, at 23:59.
Self-grades are due December 5th, 2022, at 23:59.

Submission Format
Your homework submission should consist of **one** file.

- `hw13.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned).

Submit the file to the appropriate assignment on Gradescope.

1. **Course Evaluation**

   Please fill out the course evaluation for EECS 16A. If at least 70% of the class fills out the course evaluation, everyone will receive one point of extra credit on the final, and if at least 80% fills it out, then everyone will receive two extra points.

   Also please fill out this study group feedback survey found here: https://docs.google.com/forms/d/e/1FAIpQLSf_rgvJmmUZCwprNsvcT-ggqh1xTWIyk6flbh9FiBzQShVHig/viewform?usp=sf_link

2. **Reading Assignment**

   For this homework, please read Note 22 (Trilateration and Correlation) and Note 23 (Least Squares).

3. **Audio File Matching**

   Learning Goal: *This problem motivates the application of correlation for pattern matching applications such as Shazam. Note: Shazam is an application that identifies songs playing around you.*

   Many audio processing applications rely on representing audio files as vectors, referred to as audio *signals*. Every component of the vector determines the sound we hear at a given time. We can use inner products to determine if a particular audio clip is part of a longer song, similar to an application like *Shazam*.

   Let us consider a very simplified model for an audio signal, $\vec{x}$. At each timestep k, the audio signal can be either $x[k] = -1$ or $x[k] = 1$.

   (a) Say we want to compare two audio files of the same length N to decide how similar they are. First, consider two vectors that are exactly identical, namely $\vec{x}_1 = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$ and $\vec{x}_2 = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$. What is the inner product of these two vectors? What if $\vec{x}_1 = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}^T$ but $\vec{x}_2$ oscillates between 1 and -1? Assume that N, the length of the two vectors, is an even number.

   Use this to suggest a method for comparing the similarity between a generic pair of length-N vectors.

   (b) Next, suppose we want to find a short audio clip in a longer one. We might want to do this for an application like *Shazam*, which is able to identify a song from a short clip. Consider the vector of length 8, $\vec{x} = \begin{bmatrix} -1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \end{bmatrix}^T$.

We want to find the short segment $\vec{y} := \begin{bmatrix} y[0] & y[1] & y[2] \end{bmatrix}^T = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}^T$ in the longer vector. To do this, perform the linear cross correlation between these two finite length sequences and identify at what shift(s) the linear cross correlation is maximized. Apply the same technique to identify what shift(s) gives the best match for $\vec{y} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$.

(If you wish, you may use iPython to do this part of the question, but you do not have to.)

(c) Now suppose our audio vector is represented using integers beyond simply just 1 and -1. Find the short audio clip $\vec{y} = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^T$ in the song given by $\vec{x} = \begin{bmatrix} 1 & 2 & 3 & 1 & 2 & 2 & 3 & 10 \end{bmatrix}^T$. Where do you expect to see the peak in the correlation of the two signals? Is the peak where you want it to be, i.e. does it pull out the clip of the song that you intended? Why?

(If you wish, you may use iPython to do this part of the question, but you do not have to.)

(d) Let us think about how to get around the issue in the previous part. We applied cross-correlation to compare segments of $\vec{x}$ of length 3 (which is the length of $\vec{y}$) with $\vec{y}$. Instead of directly taking the cross correlation, we want to normalize each inner product computed at each shift by the magnitudes of both segments, i.e. we want to consider the inner product $< \frac{\vec{x}_k}{\|\vec{x}_k\|}, \frac{\vec{y}}{\|\vec{y}\|} >$, where $\vec{x}_k$ is the length 3 segment starting from the k-th index of $\vec{x}$. This is referred to as normalized cross correlation. Using this procedure, now which segment matches the short audio clip best?

(e) We can use this on a more ‘realistic’ audio signal – refer to the IPython notebook, where we use normalized cross-correlation on a real song. Run the cells to listen to the song we are searching through, and add a simple comparison function `vector_compare` to find where in the song the clip comes from (i.e. write down the matching timestamp of the long audio clip). Running this may take a couple minutes on your machine, but note that this computation can be highly optimized and run super fast in the real world! Also note that this is not exactly how Shazam works, but it draws heavily on some of these basic ideas.

4. Mechanical Projections

Learning Goal: *The objective of this problem is to practice calculating projection of a vector and the corresponding squared error.*

(a) Find the projection of $\vec{b} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$ onto $\vec{a} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. What is the squared error between the projection and $\vec{b}$, i.e. $\|e\|^2 = \|\text{proj}_{\vec{a}}(\vec{b}) - \vec{b}\|^2$?

(b) Find the projection of $\vec{b} = \begin{bmatrix} 1 \\ 4 \\ -5 \end{bmatrix}$ onto the column space of $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$. What is the squared error between the projection and $\vec{b}$, i.e. $\|e\|^2 = \|\text{proj}_{\text{Col}(\mathbf{A})}(\vec{b}) - \vec{b}\|^2$?

5. Mechanical Trilateration

Learning Goal: *The objective of this problem is to practice using trilateration to find the position based on the distance measurements and known beacon locations.*

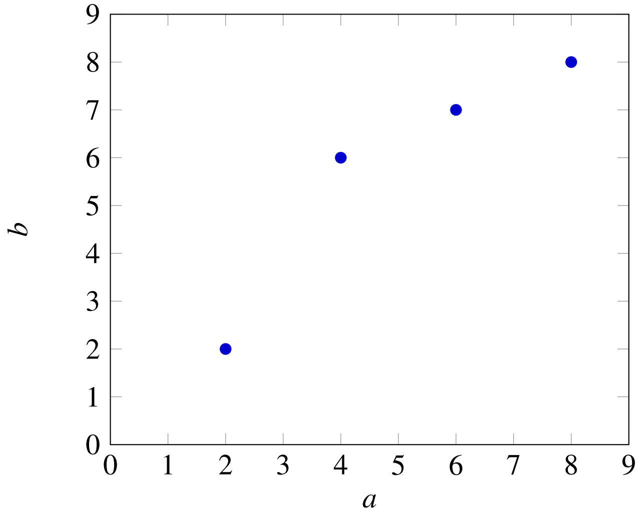
Trilateration is the problem of finding one’s coordinates given distances from known beacon locations. For each of the following trilateration problems, you are given 3 beacon locations ($\vec{s}_1$, $\vec{s}_2$, $\vec{s}_3$) and the corresponding distance (d_1, d_2, d_3) from each beacon to your location.

(a) $\vec{s}_1 = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$, $d_1 = 5$, $\vec{s}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $d_2 = 2$, $\vec{s}_3 = \begin{bmatrix} -11 \\ 6 \end{bmatrix}$, $d_3 = 13$. First, use any graphing calculator or ipython to graph the set of constraints given by $(\vec{s}_1, \vec{s}_2, \vec{s}_3)$ and (d_1, d_2, d_3), and take note of the number of solutions, or possible locations that you could be. Then **use trilateration** to find your location or possible locations. If a solution does not exist, state that it does not.

(b) $\vec{s}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $d_1 = 5\sqrt{2}$, $\vec{s}_2 = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$, $d_2 = 5\sqrt{2}$, $\vec{s}_3 = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$, $d_3 = 5$. First, use any graphing calculator or ipython to graph the set of constraints given by $(\vec{s}_1, \vec{s}_2, \vec{s}_3)$ and (d_1, d_2, d_3), and take note of the number of solutions, or possible locations that you could be. Then **use trilateration** to find your location or possible locations. Why can't we precisely determine our location, even though we have the same number of measurements as part (a)? Can we use our original constraints to narrow down our set of possible solutions we got from trilateration?

(c) $\vec{s}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $d_1 = 5$, $\vec{s}_2 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$, $d_2 = 2$, $\vec{s}_3 = \begin{bmatrix} -12 \\ 5 \end{bmatrix}$, $d_3 = 12$. First, use any graphing calculator or ipython to graph the set of constraints given by $(\vec{s}_1, \vec{s}_2, \vec{s}_3)$ and (d_1, d_2, d_3), and take note of the number of solutions, or possible locations that you could be. Then **use trilateration** to find your location or possible locations. If a solution does not exist, state that it does not.

6. Mechanical: Linear Least Squares



a	2	4	6	8
b	2	6	7	8

(a) Consider the above data points. Find the linear model of the form

$$\vec{b} = \vec{a}x$$

that best fits the data, i.e. find the scalar value of x that minimizes the squared error

$$\|\vec{e}\|^2 = \left\| \begin{bmatrix} b_1 \\ \vdots \\ b_4 \end{bmatrix} - \begin{bmatrix} a_1 \\ \vdots \\ a_4 \end{bmatrix} x \right\|^2 = \|\vec{b} - \vec{a}x\|^2. \tag{1}$$

Note: by using this linear model, we are implicitly forcing the fit equation to go through the origin, i.e. $0 = x0$ for all x.

Do not use IPython for this calculation and show your work. Once you've computed x, compute the squared error between your model's prediction and the actual $\vec{b}$ values as shown in Equation 1. Plot the best fit line along with the data points to examine the quality of the fit. (It is okay if your plot of $\vec{b} = \vec{a}x$ is approximate.)

Reminder: $\hat{x} = (\vec{a}^T\vec{a})^{-1}\vec{a}^T\vec{b}$

(b) You will notice from your graph that you can get a better fit by adding a b-intercept. That is we can get a better fit for the data by assuming a linear model of the form

$$\vec{b} = x_1\vec{a} + x_2.$$

In order to do this, we need to augment our **A** matrix for the least squares calculation with a column of 1's (do you see why?), so that it has the form

$$\mathbf{A} = \begin{bmatrix} a_1 & 1 \\ \vdots & \vdots \\ a_4 & 1 \end{bmatrix}.$$

Find x_1 and x_2 that minimize the squared error

$$\|\vec{e}\|^2 = \left\| \begin{bmatrix} b_1 \\ \vdots \\ b_4 \end{bmatrix} - \begin{bmatrix} a_1 & 1 \\ \vdots & \vdots \\ a_4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\|^2. \tag{2}$$

Do not use IPython for this calculation and show your work.

Reminder: $\hat{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b}$

Reminder: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Compute the squared error between your model's prediction and the actual $\vec{b}$ values as shown in Equation 2. Plot your new linear model. Is it a better fit for the data?

7. Proof: Least Squares

Let $\vec{\hat{x}}$ be the solution to a linear least squares problem.

$$\vec{\hat{x}} = \underset{\vec{x}}{\text{argmin}} \left\|\vec{b} - \mathbf{A}\vec{x}\right\|^2$$

Show that the error vector $\vec{b} - \mathbf{A}\vec{\hat{x}}$ is orthogonal to the columns of **A** by direct manipulation (i.e. plug the formula for the linear least squares estimate into the error vector and then check if $\mathbf{A}^T$ times the vector is the zero vector.)

8. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.