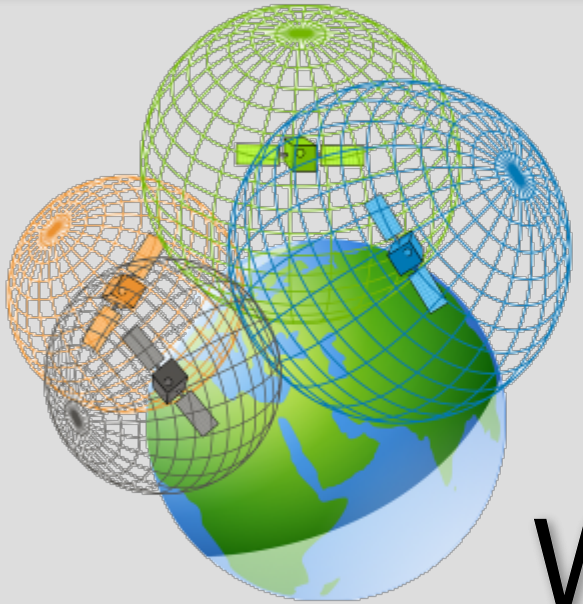
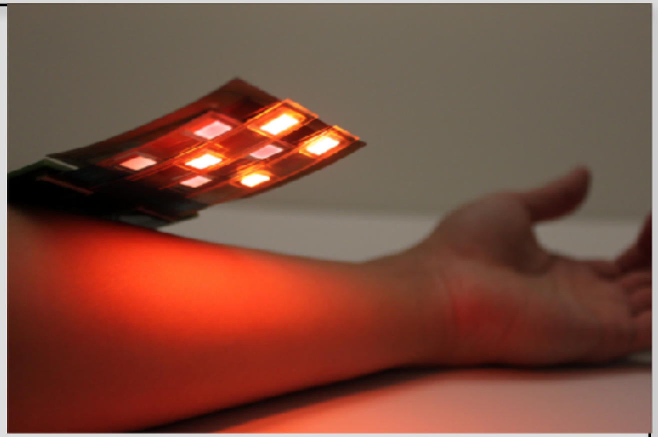




Welcome to EECS 16A!

Designing Information Devices and Systems I

Ana Arias and Miki Lustig

Fall 2022

Lecture 11B

GPS, APS, Inner Products and Norms

Announcements

Learning Goals

Not a survey class — rigorous and deep

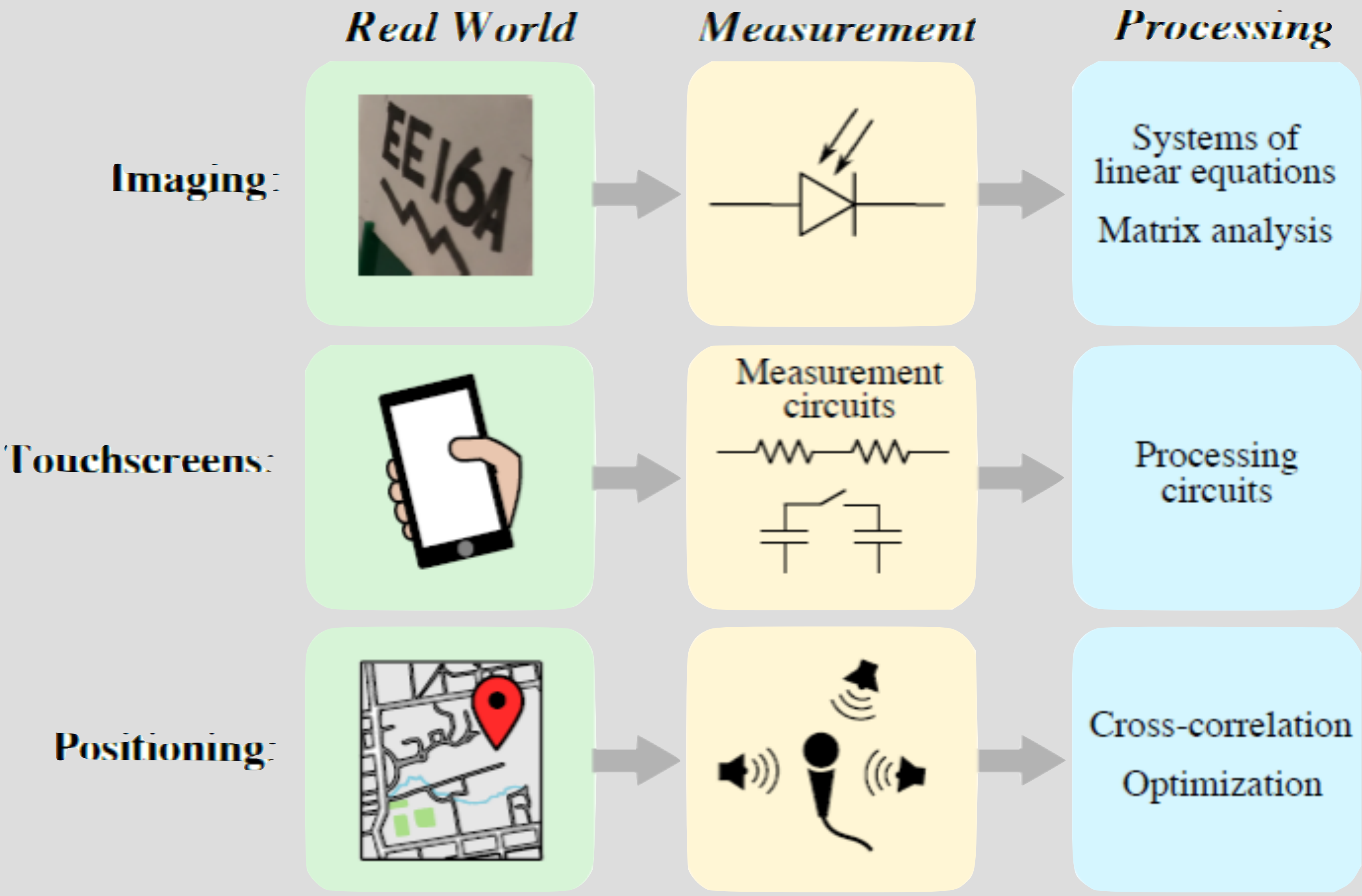
EECS 16A

- Module 1: Introduction to systems
  - How do we collect data? build a model?
- Module 2: Introduction to circuits and design
  - How do we use a model to solve a problem
- Module 3: Introduction Signal Processing and Machine Learning
  - How do we “learn” models from data, and make predictions?

EECS 16B

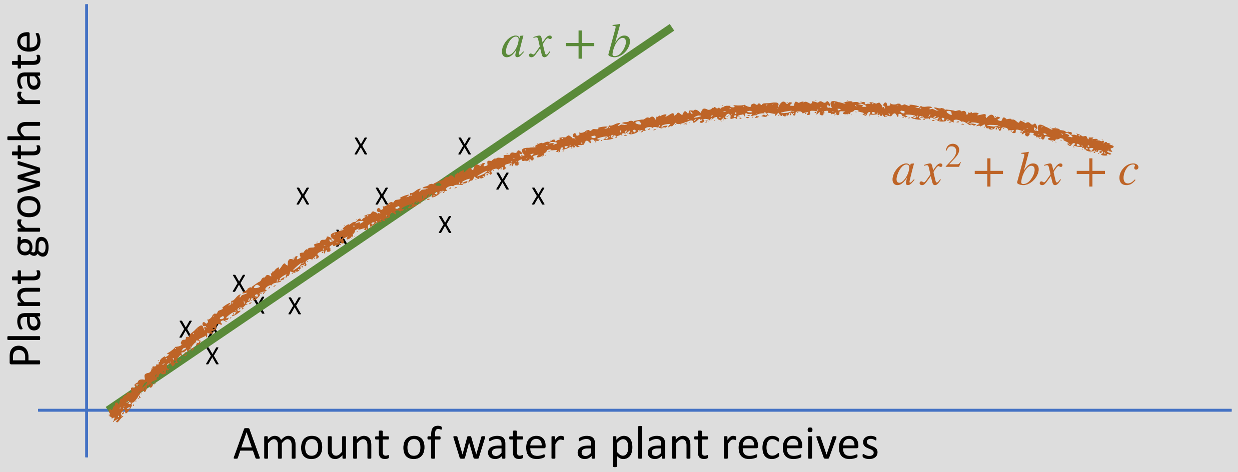
- Module 4: Advanced circuit design / analysis
- Module 5: Introduction to control and robotics
- Module 6: Introduction to data analysis and signal processing

16A Lab Examples

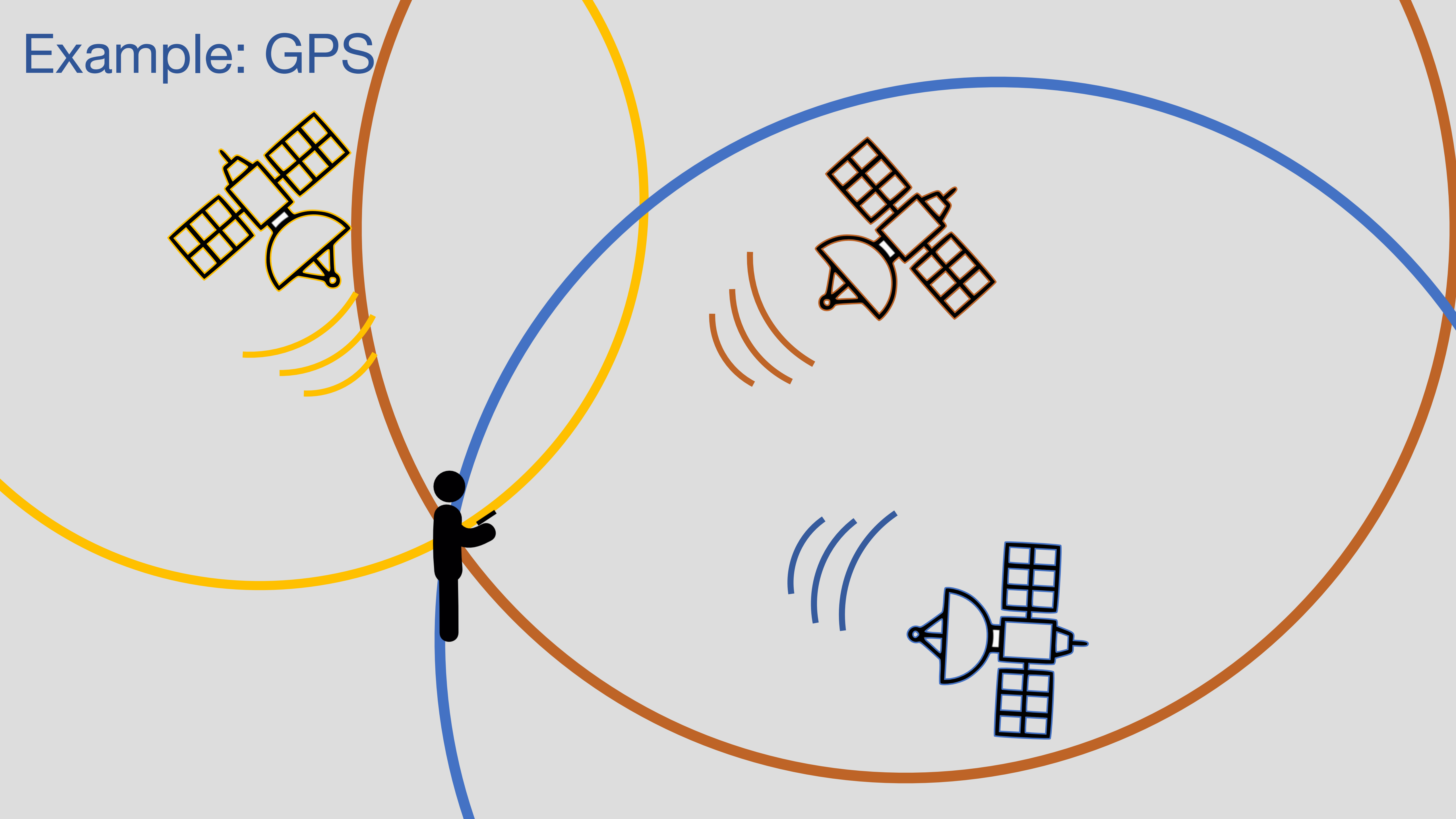


This module

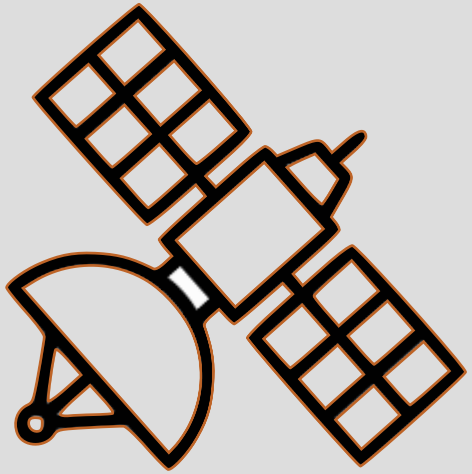
- Classification
  - Example: How can you tell if a picture is Miki or Ana
- Estimation
  - For example, how to estimate model parameters from data
- Prediction
  - How to predict stocks value tomorrow based on [past performance



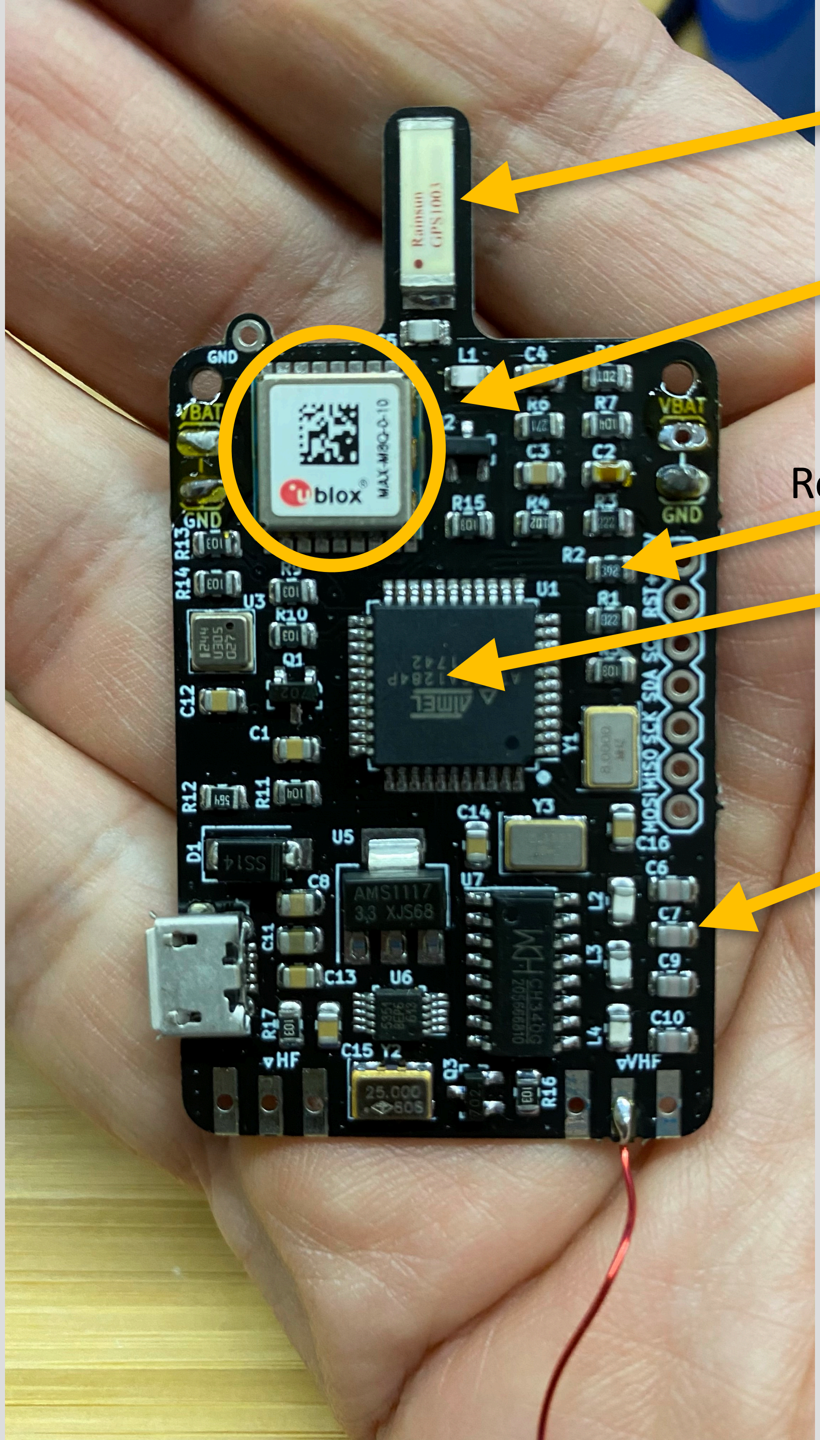
Example: GPS



GPS



- 24 satellites
  - Known position
  - Time synchronized
  - 8 usually visible
- Problem:
  - Classify which satellite is transmitting
  - Estimate distance to GPS
  - Estimate position from noisy data
- Tools:
  - Inner product
  - Cross correlation
  - Least Squares



GPS- antenna

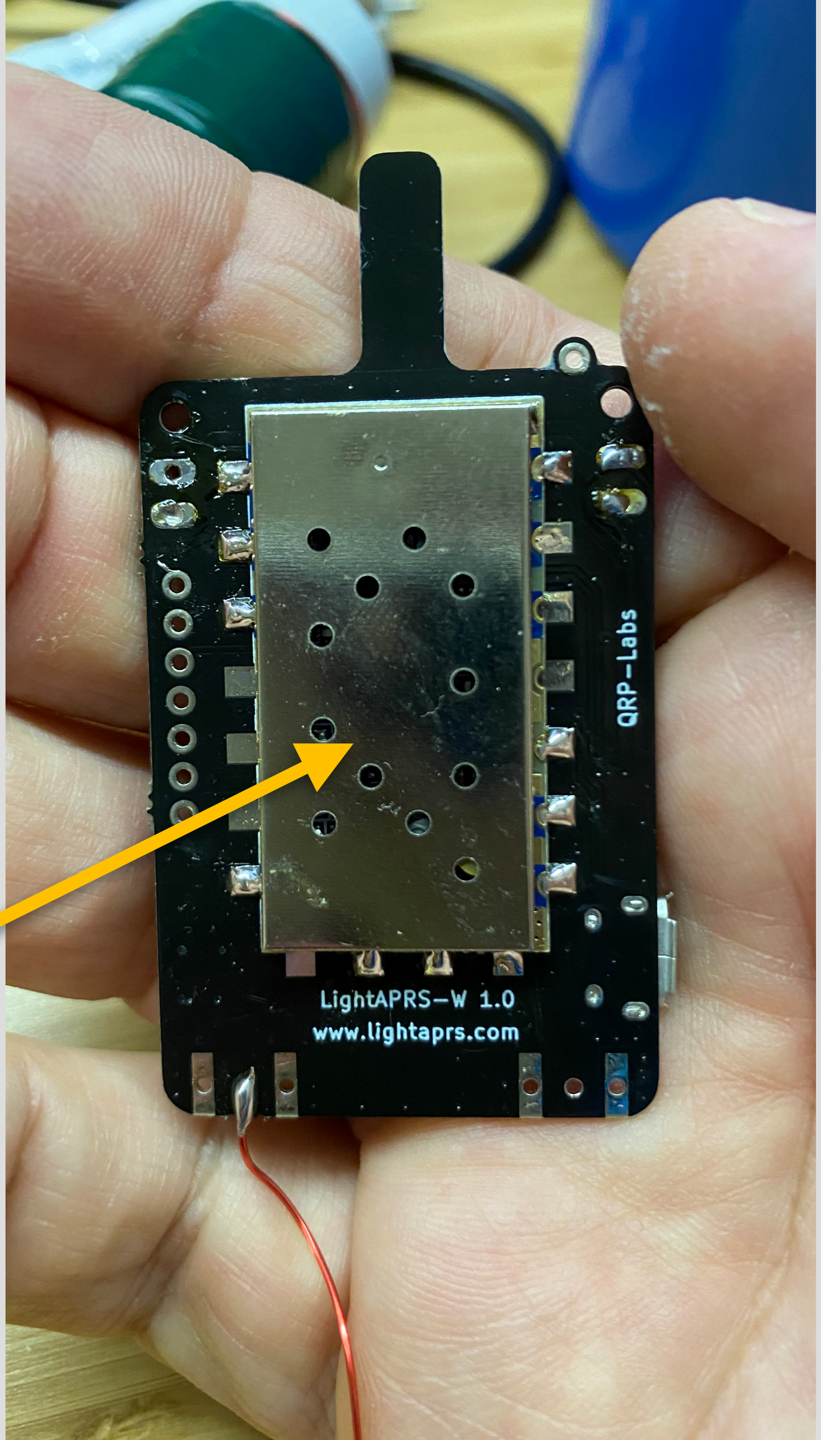
GPS- integrated circuit
~$10

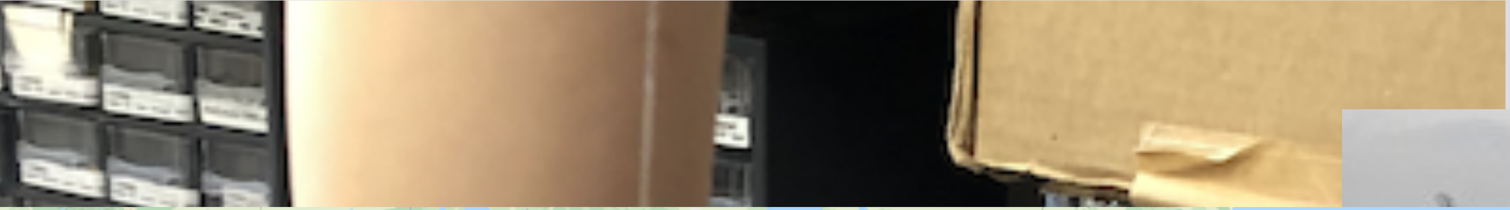
Resistors

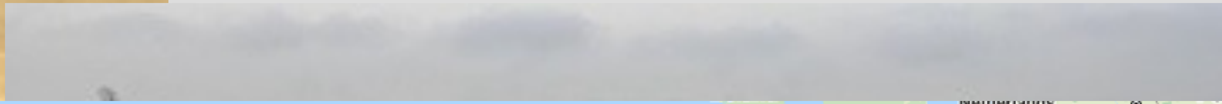
Arduino microprocessor

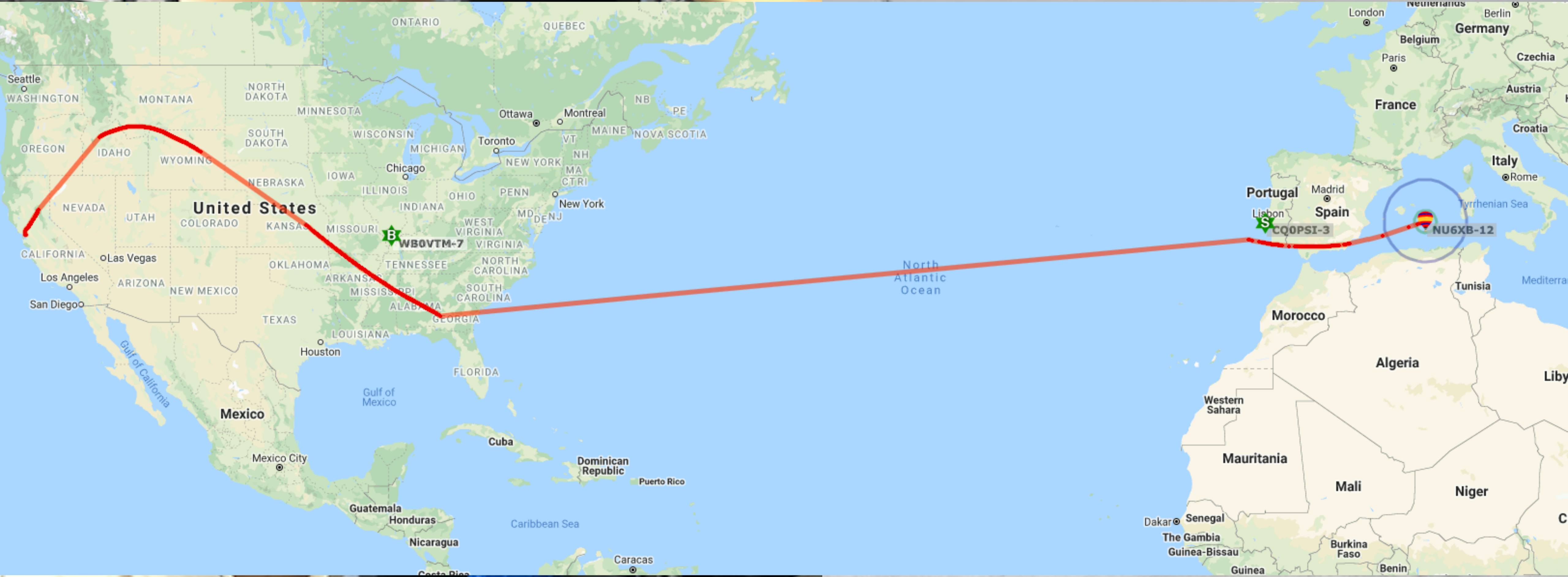
Capacitors

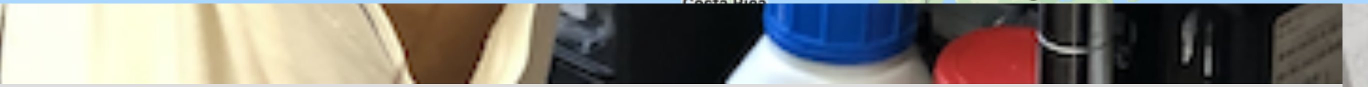
Radio integrated circuit

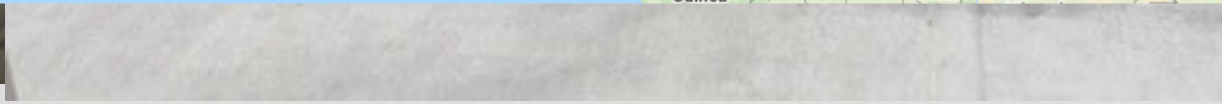




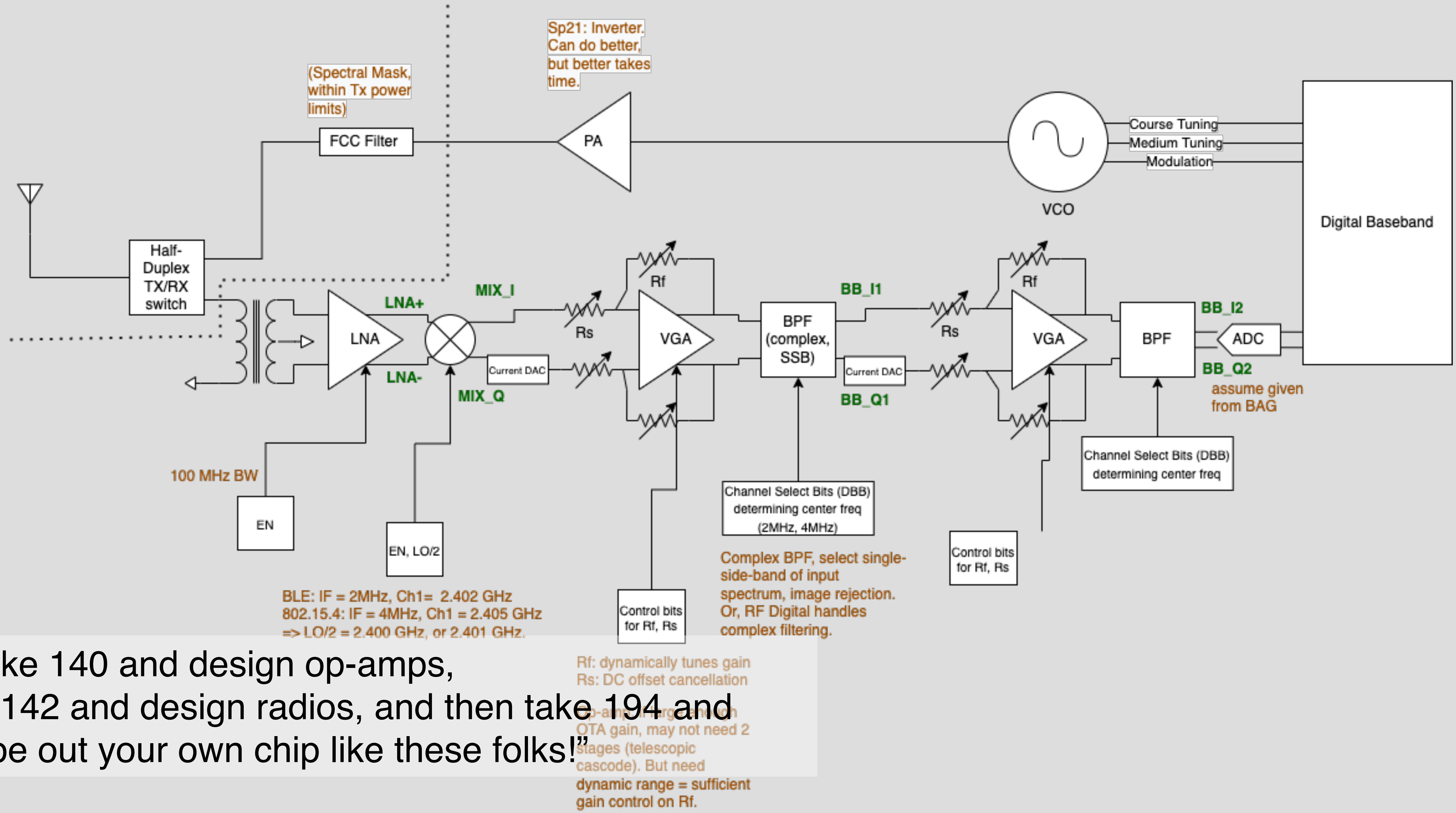



WB0VTM-7
CQ0PSI-3
NU6XB-12
United States
Mexico
North Atlantic Ocean
Gulf of Mexico
Caribbean Sea
Gulf of California
Tyrrhenian Sea
ONTARIO
QUEBEC
Seattle
WASHINGTON
OREGON
IDAHO
MONTANA
WYOMING
NORTH DAKOTA
SOUTH DAKOTA
NEBRASKA
MINNESOTA
WISCONSIN
MICHIGAN
IOWA
ILLINOIS
INDIANA
OHIO
PENN
NEW YORK
NB
PE
NOVA SCOTIA
MAINE
VT
NH
MA
CT
RI
NJ
DE
MD
WEST VIRGINIA
VIRGINIA
NORTH CAROLINA
SOUTH CAROLINA
GEORGIA
FLORIDA
ALABAMA
MISSISSIPPI
TENNESSEE
ARKANSAS
MISSOURI
KANSAS
COLORADO
UTAH
NEVADA
CALIFORNIA
ARIZONA
NEW MEXICO
OKLAHOMA
TEXAS
LOUISIANA
Ottawa
Montreal
Toronto
Chicago
New York
Las Vegas
Los Angeles
San Diego
Houston
Mexico City
Cuba
Dominican Republic
Puerto Rico
Guatemala
Honduras
Nicaragua
Caracas
London
Berlin
Germany
Belgium
Paris
France
Czechia
Austria
Croatia
Italy
Rome
Portugal
Lisbon
Madrid
Spain
Morocco
Algeria
Tunisia
Western Sahara
Mauritania
Mali
Niger
Dakar
Senegal
The Gambia
Guinea-Bissau
Guinea
Burkina Faso
Benin





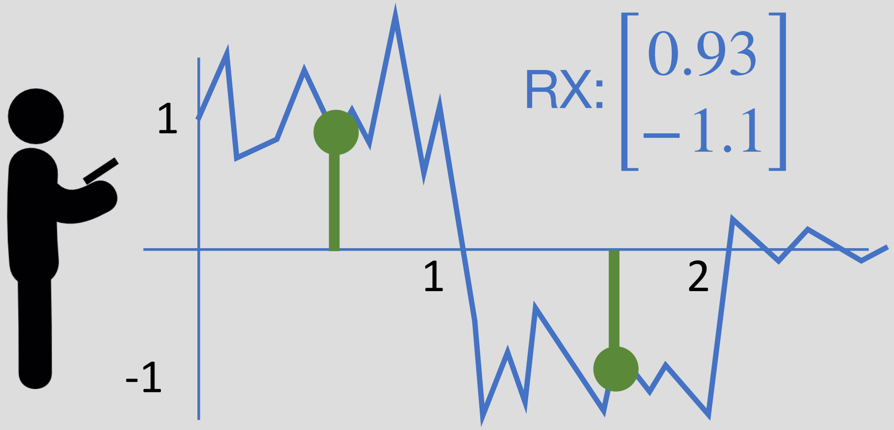
From Kris Pister:

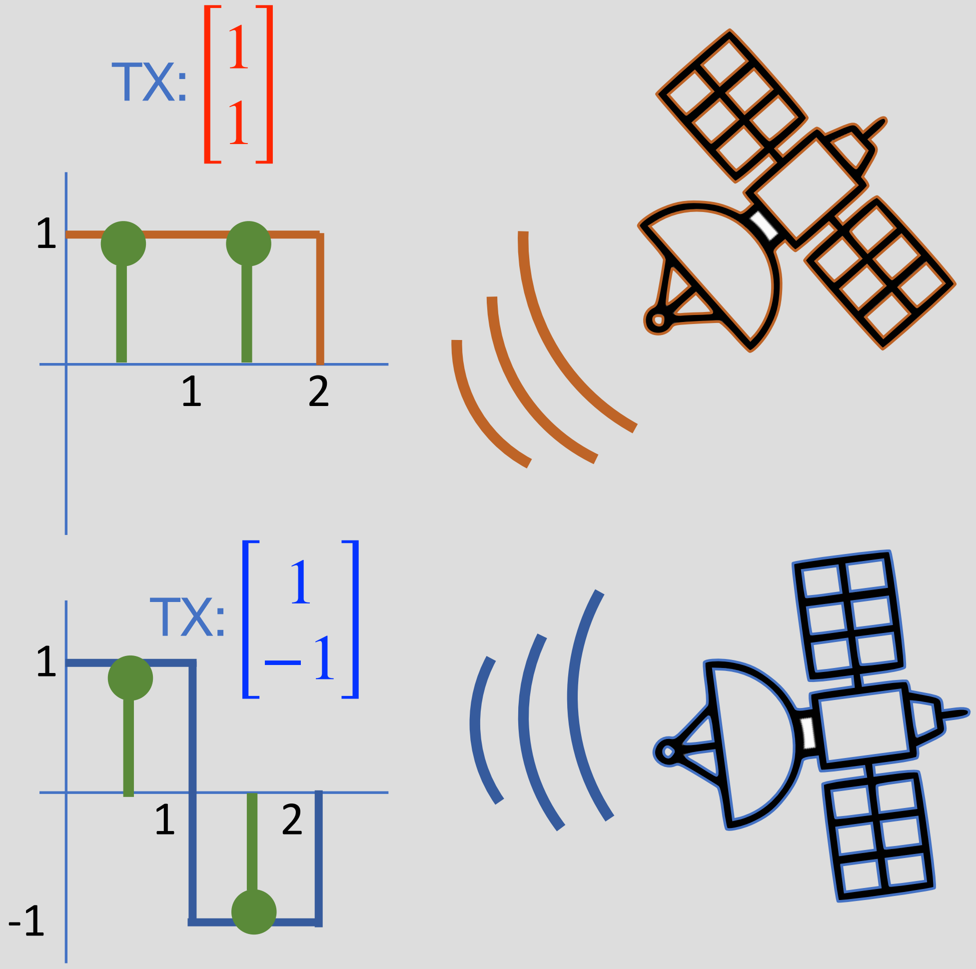


"take 140 and design op-amps,
or 142 and design radios, and then take 194 and
tape out your own chip like these folks!"

Problem 1: Classification

- Satellites transmit a unique code
  - Radio signal
- Signal is received and digitized by a receiver





Q: Which satellite was received?

Inner Product

- Provide a measure of "similarity" between vectors

- Definition: For a **real-valued** vector space, $\mathbb{V}$, the mapping

$$\vec{u}, \vec{v} \in \mathbb{V} \quad \to \quad < \vec{u}, \vec{v} > \in \mathbb{R}$$

is called an inner product if it satisfies:

1. Symmetry: $< \vec{u}, \vec{v} > = < \vec{v}, \vec{u} >$ (not true for $\mathbb{V} \in \mathbb{C}^N$)
2. Linearity: $< \alpha \vec{u}, \vec{v} > = \alpha < \vec{u}, \vec{v} > \qquad \alpha \in \mathbb{R}$

$$< \vec{u} + \vec{w}, \vec{v} > = < \vec{u}, \vec{v} > + < \vec{w}, \vec{v} >$$

3. Positive-definitness:

$$< \vec{v}, \vec{v} > \geq 0,$$

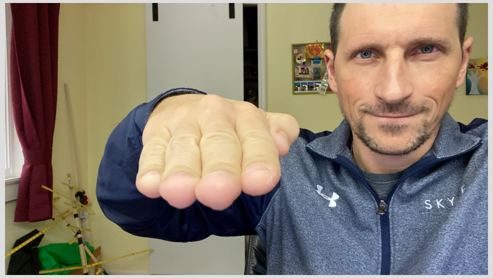
iff $< \vec{v}, \vec{v} > = 0 \quad \Leftrightarrow \quad \vec{v} = 0$

Inner Products

Example 1: Euclidean inner product (or dot product)

$$\vec{x}, \vec{y} \in \mathbb{R}^N, \qquad < \vec{x}, \vec{y} > = \vec{x}^T \vec{y}$$

Like this.... and like that!



$$\vec{y}^T \vec{x} = \underbrace{\begin{bmatrix} y_1 & y_2 & \cdots & y_N \end{bmatrix}}_{1 \times N} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}}_{N \times 1} = \underbrace{y_1 x_1 + y_2 x_2 + y_3 x_3 + \cdots + y_N x_N}_{\text{scalar } 1 \times 1} = \underbrace{\boxed{}}_{1 \times 1}$$

Example 1: Euclidean inner product

$$\vec{x}, \vec{y} \in \mathbb{R}^N, \qquad < \vec{x}, \vec{y} > = \vec{x}^T \vec{y}$$

Test:

Symmetry:

$$< \vec{x}, \vec{y} > = \vec{x}^T \vec{y}$$
$$< \vec{y}, \vec{x} > = \vec{y}^T \vec{x}$$

Linearity

$$< a\vec{x}, \vec{y} > = (a\vec{x})^T \vec{y} = a\vec{x}^T \vec{y}$$
$$< \vec{x} + \vec{z}, \vec{y} > = (\vec{x} + \vec{z})^T \vec{y} = \vec{x}^T \vec{y} + \vec{z}^T \vec{y}$$

Positive Definitness

$$< \vec{x}, \vec{x} > = \vec{x}^T \vec{x} = x_1^2 + x_2^2 + \cdots + x_N^2 \geq 0$$

Example 2: Weighted Inner Product

$\vec{x}, \vec{y} \in \mathbb{R}^N, \;\; Q \in \mathbb{R}^{N\times N}$ symmetric with positive eigenvalues

Define:

$$< \vec{x}, \vec{y} > = \vec{x}^T Q \vec{y}$$

Specific example:

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \qquad \vec{x}, \vec{y} \in \mathbb{R}^2$$

Symmetry:

$$\vec{x}^T Q \vec{y} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 & 3x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1 y_1 + 3 x_2 y_2$$

$$\vec{y}^T Q \vec{x} = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 & 3y_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 y_1 + 3 x_2 y_2$$

😍

Example 2: Weighted Inner Product

Specific example: $Q = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ $\quad \vec{x}, \vec{y} \in \mathbb{R}^2$

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Linearity: obvious!

Positive Definitness:

$$\vec{x}^T Q \vec{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1^2 + 3x_2^2 \geq 0$$

Norms

- For each inner product there's an associated norm
  - A measure of **<u>a length</u>** of elements in the vector space

$$||\vec{v}|| = \sqrt{< \vec{v}, \vec{v} >}$$

- Properties of norms:
  1. Homogeneity $\|\alpha \vec{v}\| = |\alpha| \|\vec{v}\|$ $\alpha \in \mathbb{R}$
  2. Non-negativity $\|\vec{v}\| \geq 0$
  3. Triangle Inequality $\|\vec{v} + \vec{u}\| \leq \|v\| + \|u\|$

Euclidian Norm

- Euclidean inner-product induces the euclidean norm

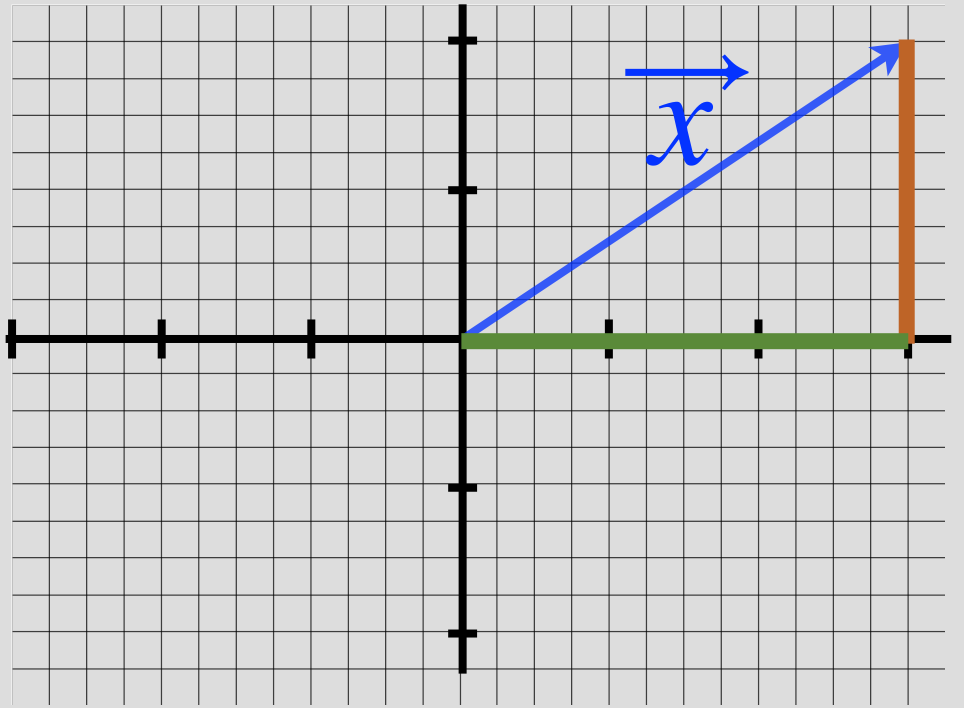
$$\vec{x} \in \mathbb{R}^N, \quad < \vec{x}, \vec{x} > = \vec{x}^T\vec{x}$$

$$\|\vec{x}\| = \sqrt{\vec{x}^T\vec{x}}$$

Specific example:

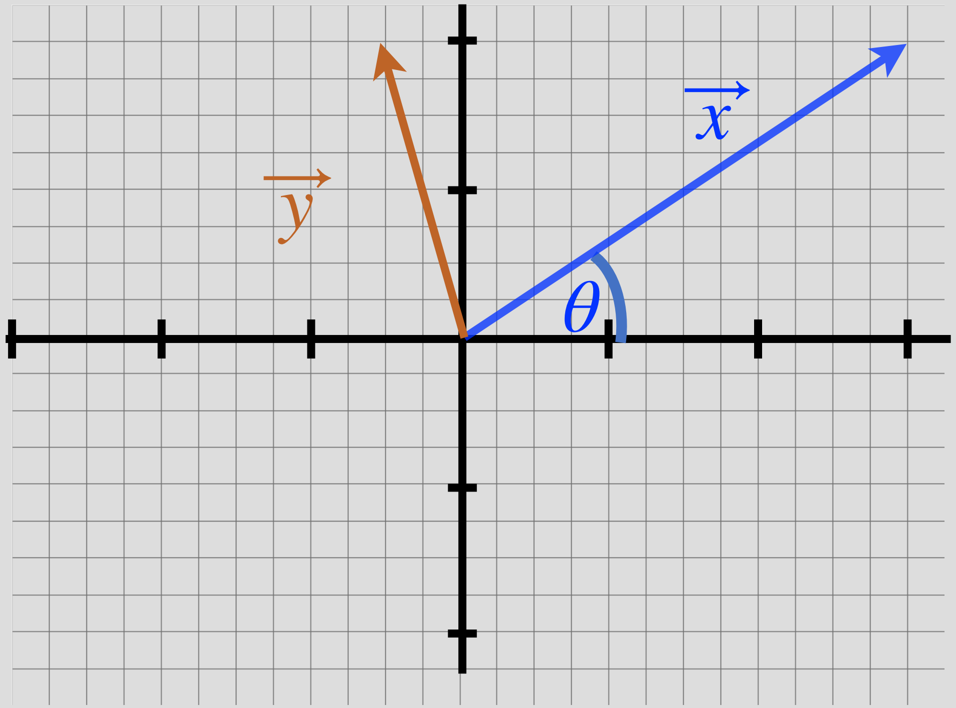
$$\vec{x} \in \mathbb{R}^2$$

$$\|\vec{x}\| = \sqrt{\vec{x}^T\vec{x}} = \sqrt{x_1^2 + x_2^2}$$



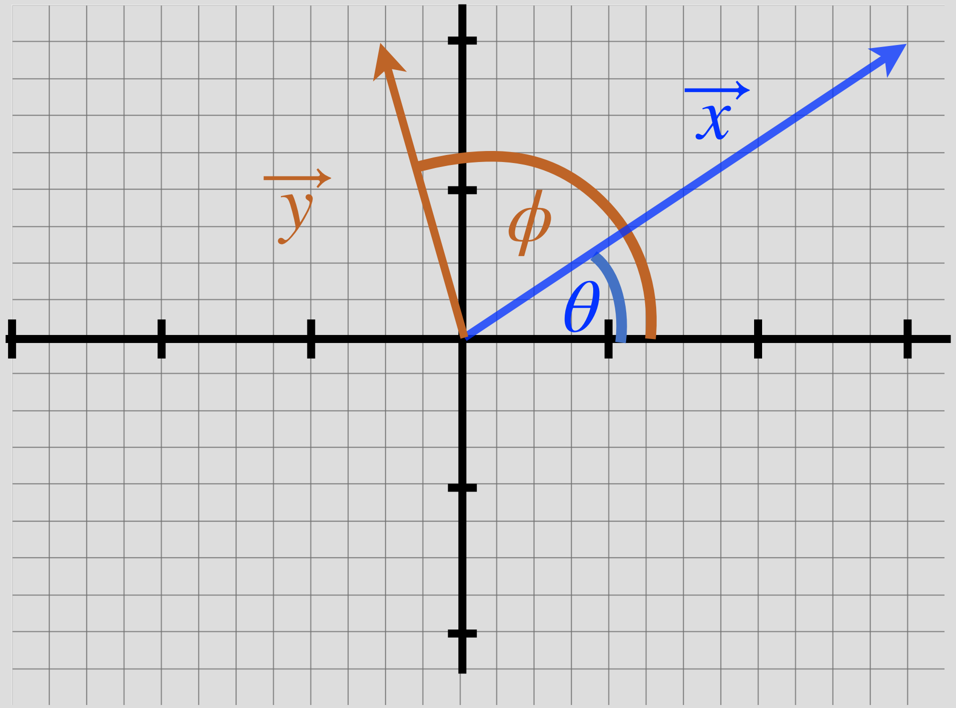
Geometrical Interpretation of Inner Product

$$\vec{x} = \|\vec{x}\| \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$



Geometrical Interpretation of Inner Product

$$\vec{x} = \|\vec{x}\| \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \qquad \vec{y} = \|\vec{y}\| \begin{bmatrix} \cos(\phi) \\ \sin(\phi) \end{bmatrix}$$



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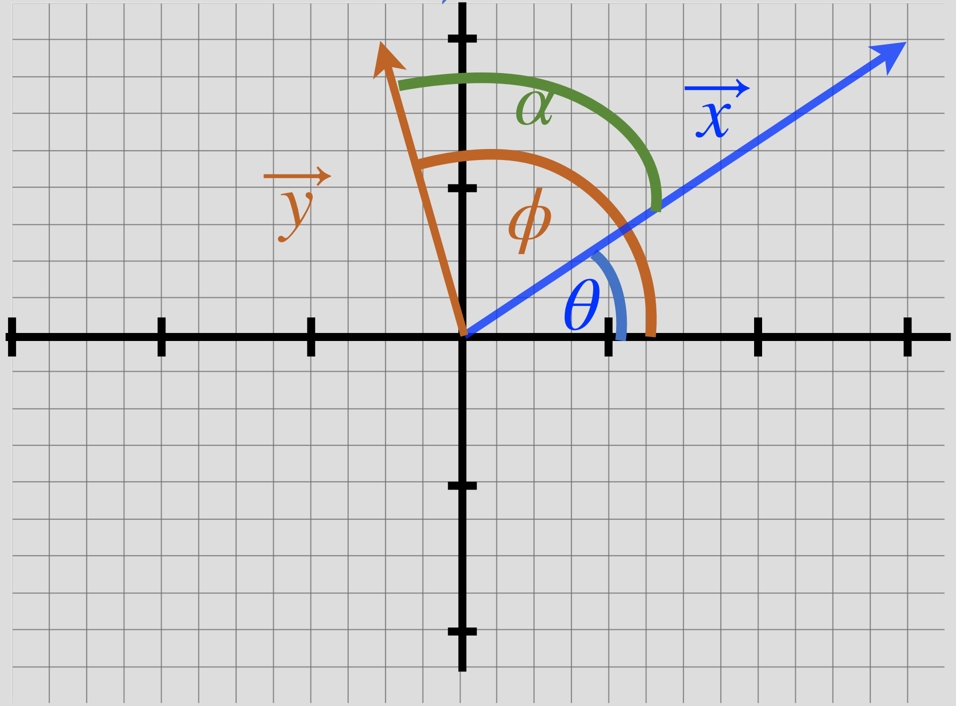
- For Euclidian inner product:

$$\begin{aligned} \vec{x}^T\vec{y} &= \|\vec{x}\|\|\vec{y}\| \left(\cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi)\right) \\ &= \|\vec{x}\|\|\vec{y}\| \cos(\phi - \theta) \\ &= \|\vec{x}\|\|\vec{y}\| \cos(\alpha) \end{aligned}$$



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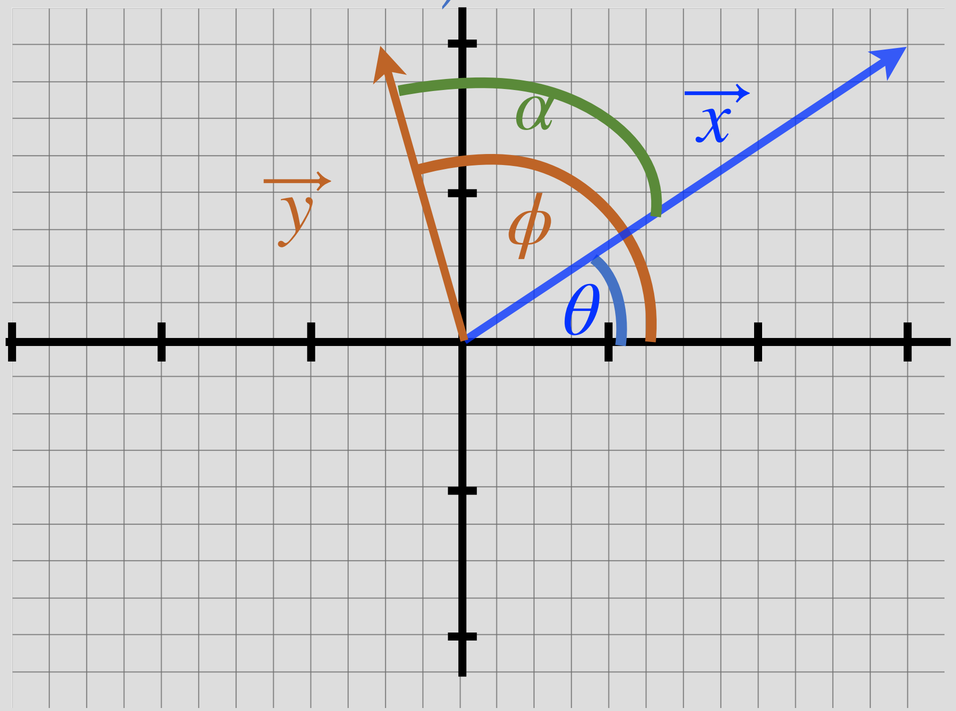


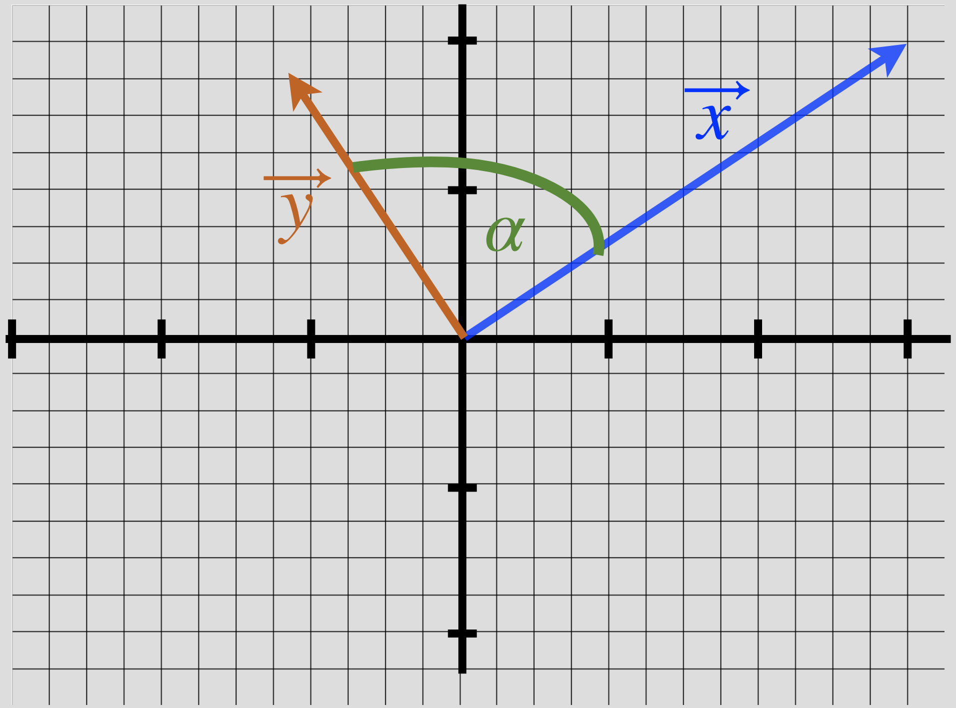
Orthogonality

- For an inner product $< \cdot , \cdot >$, two vectors $\overrightarrow{x}, \overrightarrow{y}$ are said to be orthogonal if $< \overrightarrow{x}, \overrightarrow{y} > = 0$

$$< \overrightarrow{x}, \overrightarrow{y} > = \| \overrightarrow{x} \| \| \overrightarrow{y} \| \cos(\alpha)$$

$$\Rightarrow \cos(\alpha) = 0$$

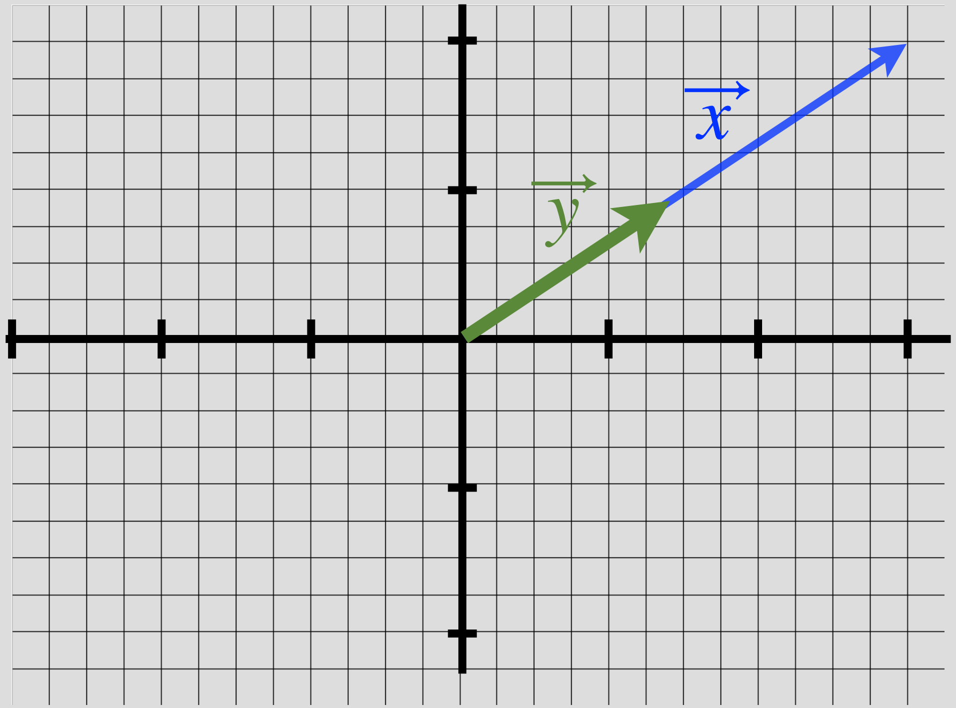
$$\Rightarrow \alpha = \frac{\pi}{2}$$



Cauchy-Schwarz Inequality

- Consider: $|<\overrightarrow{x},\overrightarrow{y}>|$

$$|<\overrightarrow{x},\overrightarrow{y}>| = \|\overrightarrow{x}\|\|\overrightarrow{y}\||\cos(\alpha)|$$



Cauchy-Schwarz Inequality

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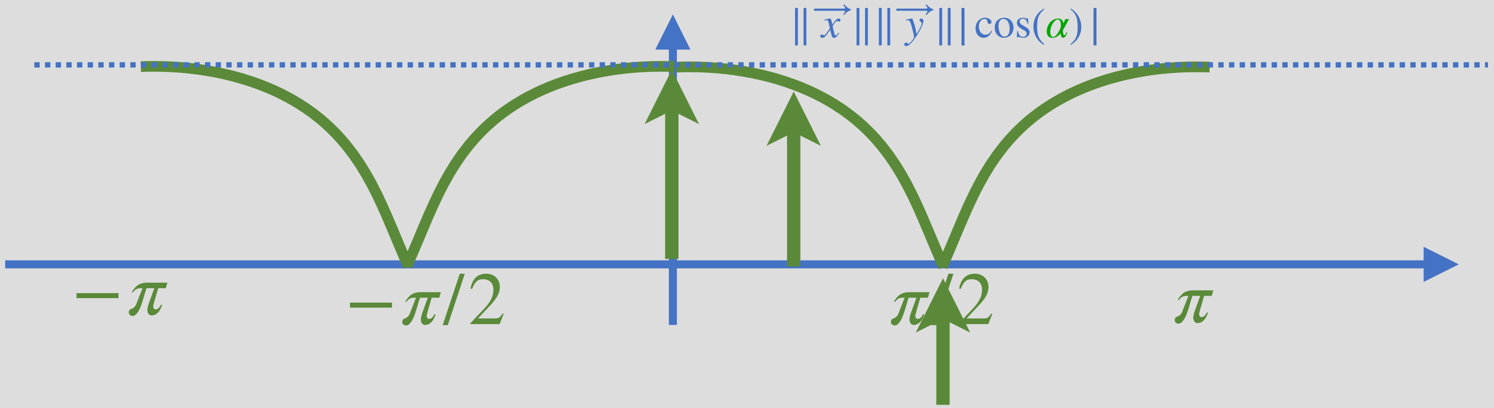

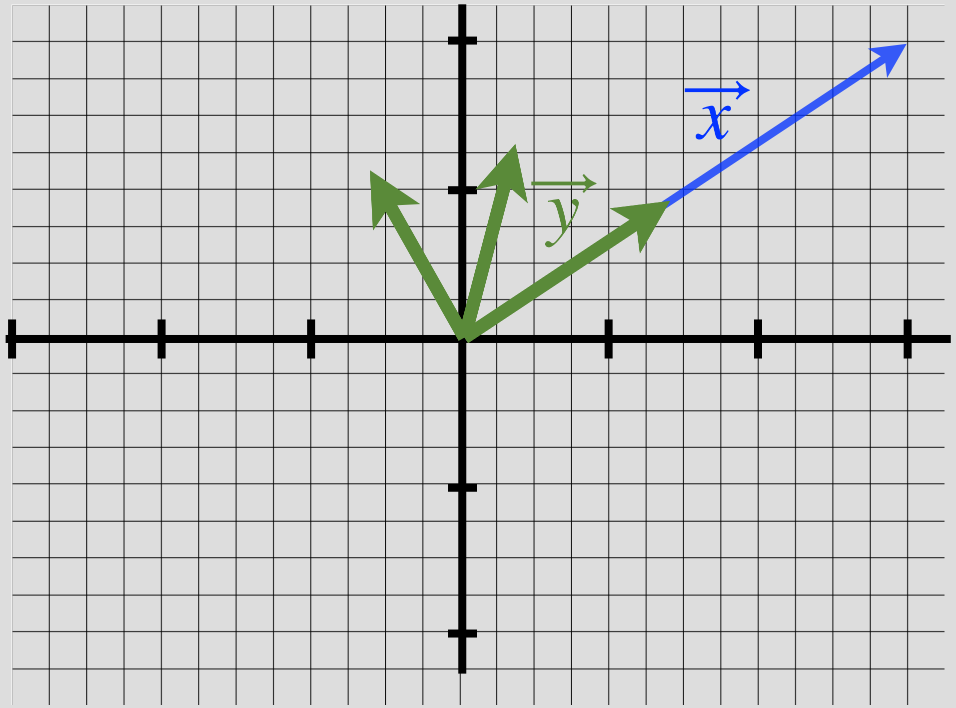

Cauchy-Schwarz Inequality

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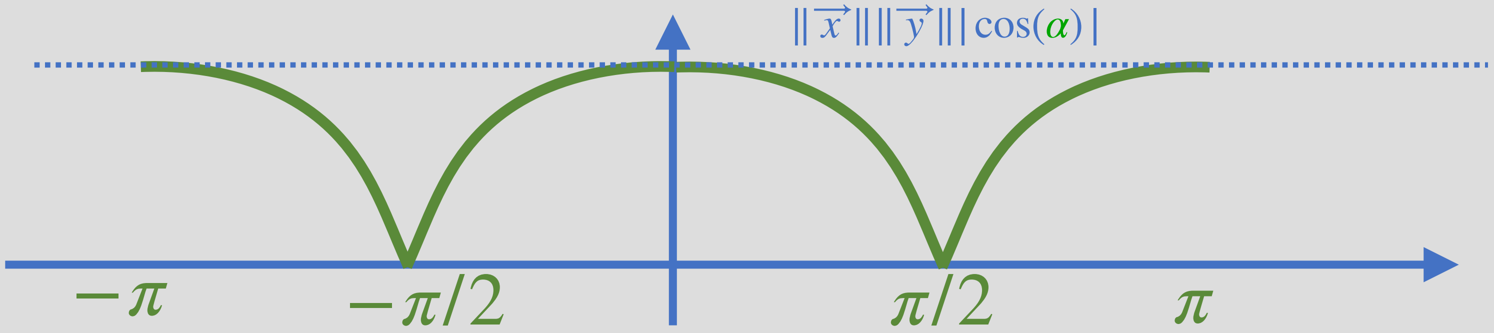
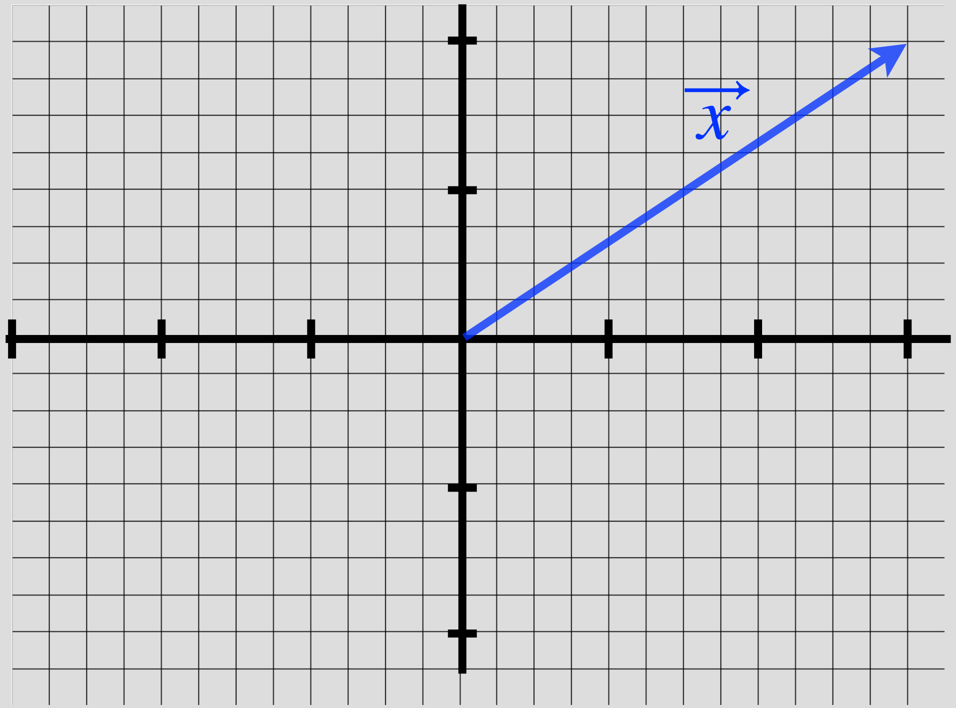
$$| < \vec{x}, \vec{y} > | = \|\vec{x}\|\|\vec{y}\|| \cos(\alpha) |$$

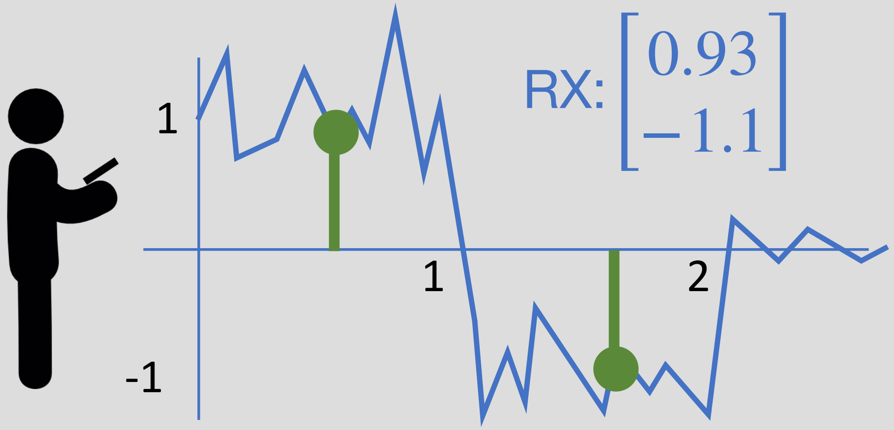


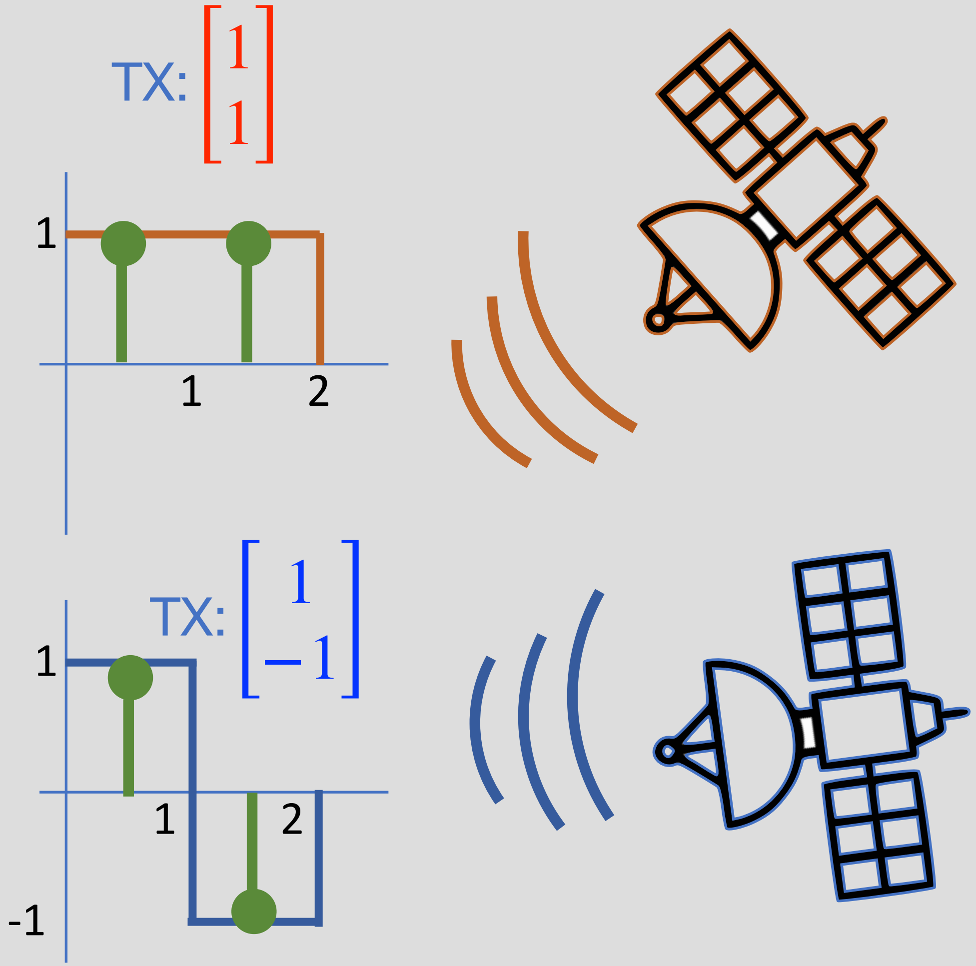


$$| < \vec{x}, \vec{y} > | \leq \|\vec{x}\|\|\vec{y}\|$$

Problem 1: Classification

- Satellites transmit a unique code
  - Radio signal
- Signal is received and digitized by a receiver



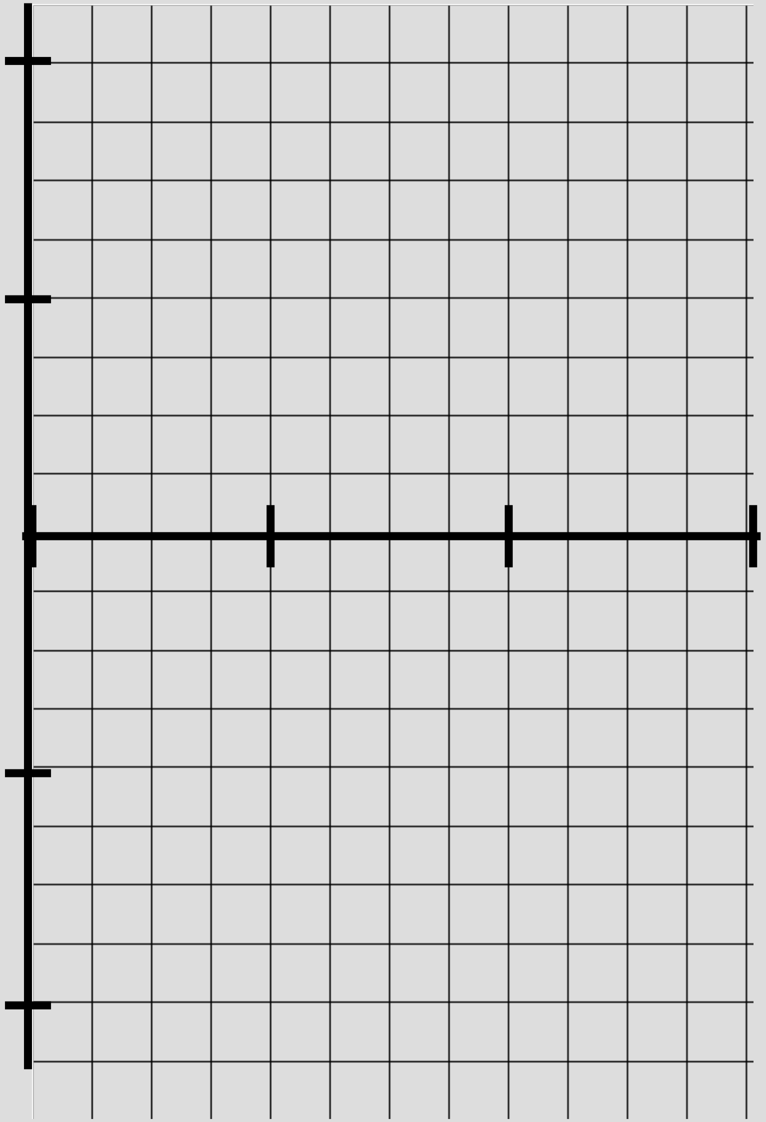


Q: Which satellite was received?

Classification

- Q: How to mathematically formulate the classification problem?

$$\vec{r} = \begin{bmatrix} 0.93 \\ -1.1 \end{bmatrix} \qquad \vec{s}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{Or?} \quad \vec{s}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



Classification

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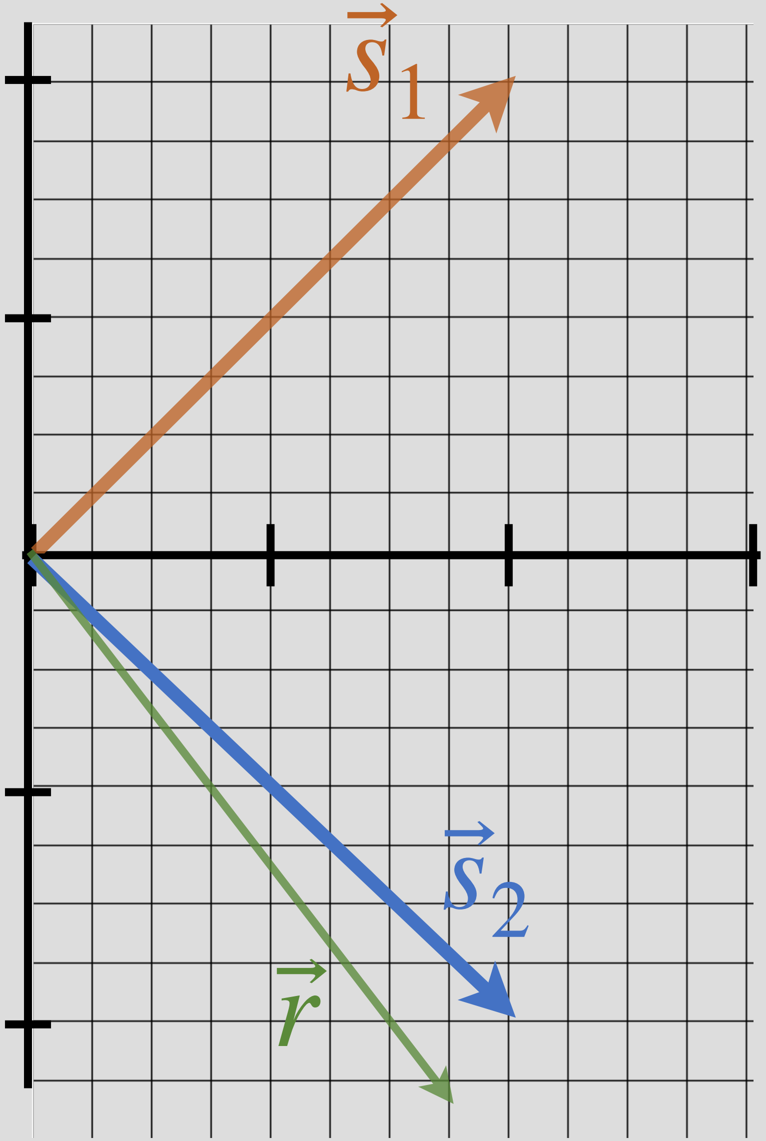
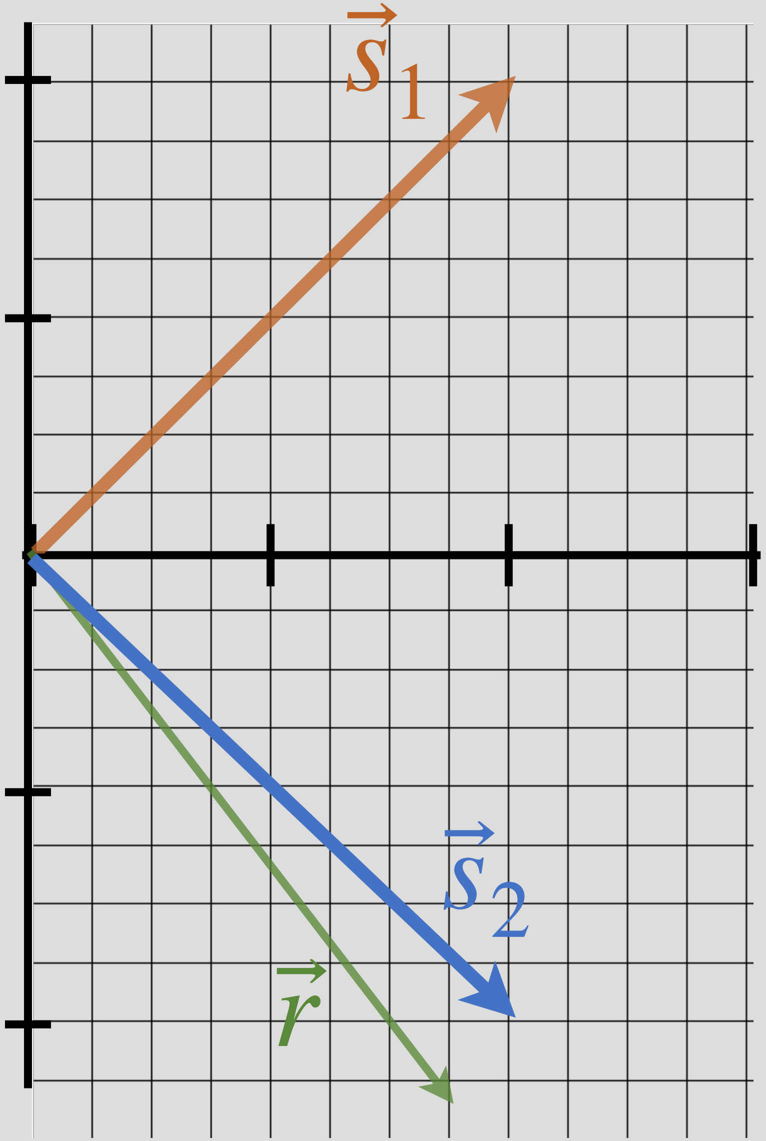


Classification

- Q: How to mathematically formulate the classification problem?

$$\vec{r} = \begin{bmatrix} 0.93 \\ -1.1 \end{bmatrix} \qquad \vec{s}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{Or?} \quad \vec{s}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



- A: Look at the length of the error vector

$$i^* = \underset{i \in \{1,2\}}{\operatorname{argmin}} \|\vec{r} - \vec{s}_i\|$$

Classification

$$i^* = \underset{i \in \{1,2\}}{\operatorname{argmin}} \|\vec{r} - \vec{s}_i\|^2$$

$$\vec{s}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{s}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{aligned} \|\vec{r} - \vec{s}_i\|^2 &= \langle \vec{r} - \vec{s}_i, \vec{r} - \vec{s}_i \rangle \\ &= \langle \vec{r}, \vec{r} - \vec{s}_i \rangle - \langle \vec{s}_i, \vec{r} - \vec{s}_i \rangle \end{aligned}$$

Classification

$$i^* = \underset{i \in \{1,2\}}{\operatorname{argmin}} \|\vec{r} - \vec{s}_i\|^2$$

$$\vec{s}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{s}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{aligned}
\|\vec{r} - \vec{s}_i\|^2 &= \langle \vec{r} - \vec{s}_i, \vec{r} - \vec{s}_i \rangle \\
&= \langle \vec{r}, \vec{r} - \vec{s}_i \rangle - \langle \vec{s}_i, \vec{r} - \vec{s}_i \rangle \\
&= \langle \vec{r}, \vec{r} \rangle - \langle \vec{r}, \vec{s}_i \rangle - \langle \vec{s}_i, \vec{r} \rangle + \langle \vec{s}_i, \vec{s}_i \rangle \\
&= \underbrace{\|\vec{r}\|^2}_{\text{Fixed!}} + \underbrace{\|\vec{s}_i\|^2}_{=2} - 2 \langle \vec{r}, \vec{s}_i \rangle
\end{aligned}$$

Classification

$$\|\vec{r} - \vec{s}_i\|^2 = \|\vec{r}\|^2 + \|\vec{s}_i\|^2 - 2\langle \vec{r}, \vec{s}_i \rangle$$

Fixed! =2

If $\langle \vec{r}, \vec{s}_i \rangle$ is maximized, then $\|\vec{r} - \vec{s}_i\|^2$ is minimized

Classification procedure:

for $i \in \{1,2\}$
compute $\langle \vec{r}, \vec{s}_i \rangle$

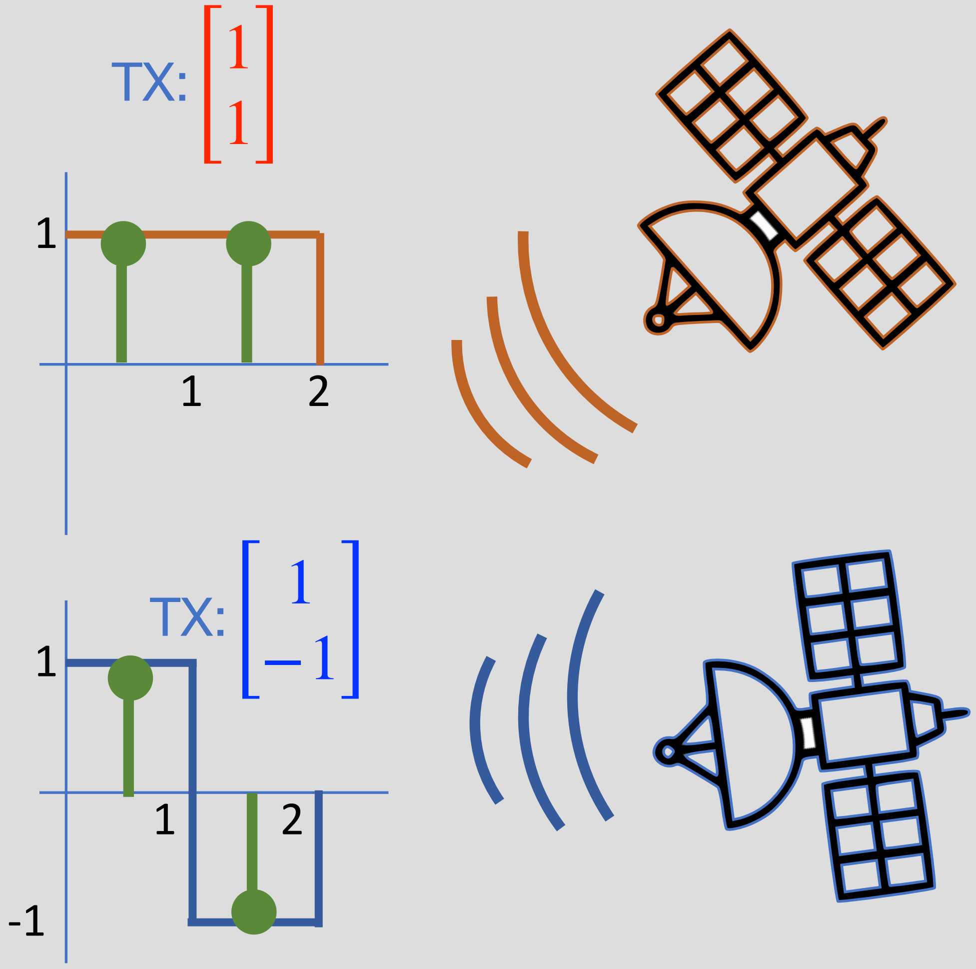
$\langle \vec{r}, \vec{s}_1 \rangle = -0.17$

$\langle \vec{r}, \vec{s}_2 \rangle = 2.03$

Return index i the maximizes the above $i^* = 2$

Localization

- Satellites transmit a unique code
  - Radio signal
- Signal is received and digitized by a receiver



Two problems:

1. Interference
2. Timing (next week)

Interference

Possibility 1: Both sats are in TX

$$\vec{r} = \vec{s}_1 + \vec{s_2} + \vec{n}$$

Possibility 2: Only S1 is in Tx

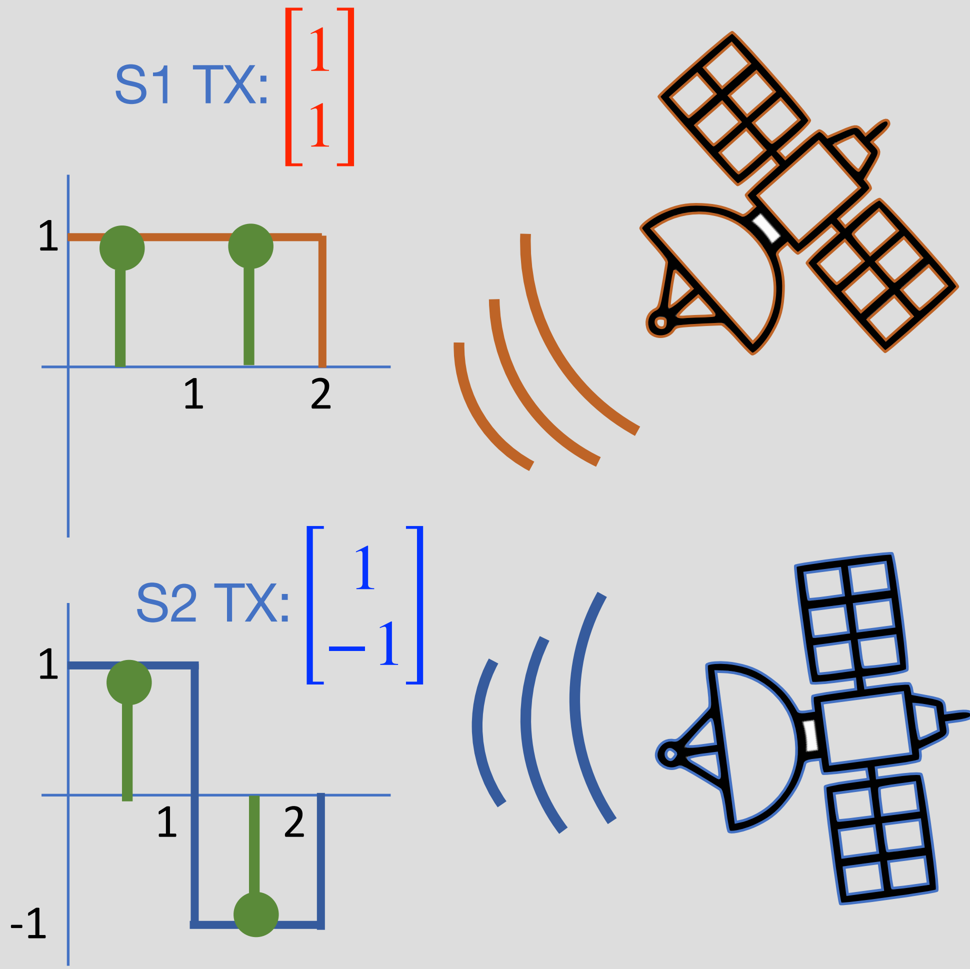
$$\vec{r} = \vec{s}_1 + \vec{n}$$

Possibility 3: Only S2 is in Tx

$$\vec{r} = \vec{s}_2 + \vec{n}$$

Possibility 4: None is in Tx

$$\vec{r} = \vec{n}$$



Interference

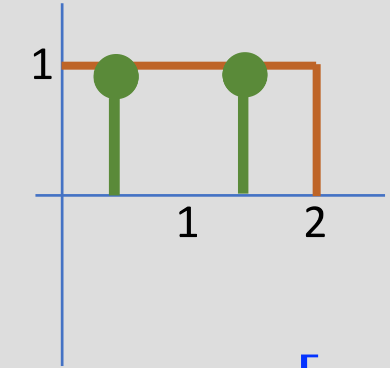
Possibility 1: Both sats are in TX

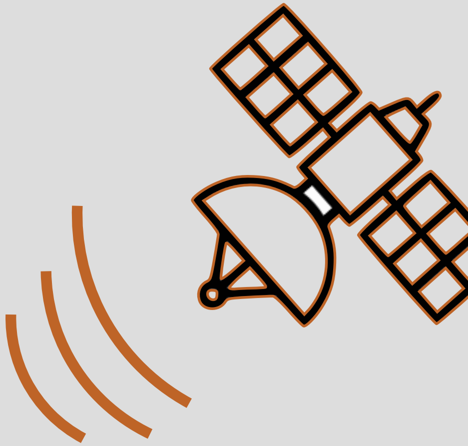
$$\vec{r} = \vec{s}_1 + \vec{s_2} + \vec{n}$$

$$\langle \vec{r}, \vec{s}_1 \rangle = \langle \vec{s}_1 + \vec{s}_2 + \vec{n}, \vec{s}_1 \rangle$$

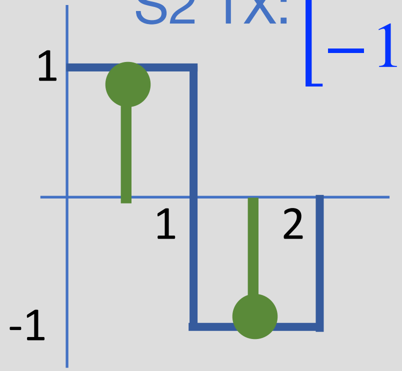
$$= \underbrace{\langle \vec{s}_1, \vec{s}_1 \rangle}_{\text{Desired}} + \underbrace{\langle \vec{s}_2, \vec{s}_1 \rangle}_{\text{Interference}} + \underbrace{\langle \vec{n}, \vec{s}_1 \rangle}_{\text{Small}}$$

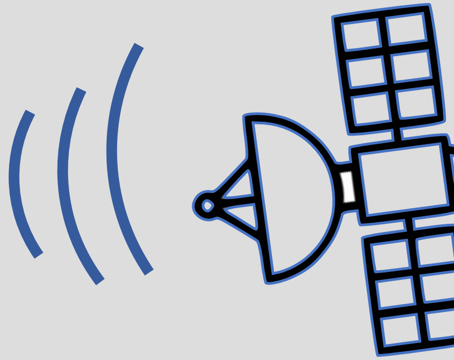
S1 TX: $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$





S2 TX: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

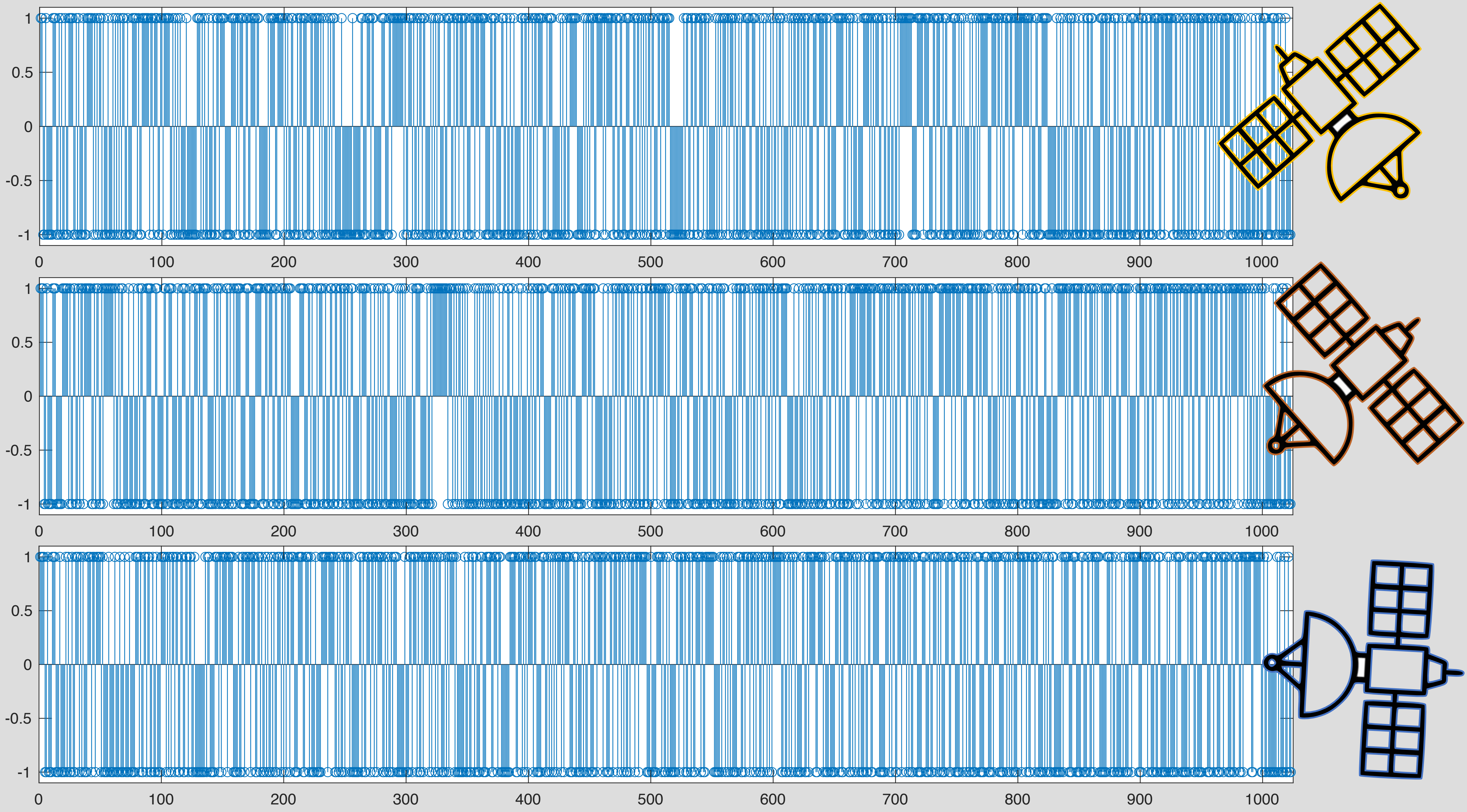




Q: How to design codes that don't interfere?

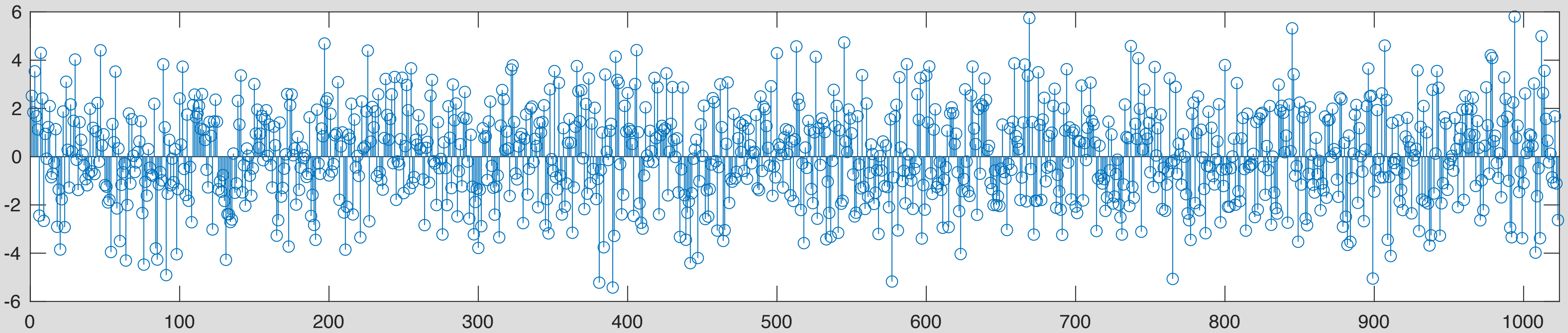
A: Make them orthogonal! $\langle \vec{s}_2, \vec{s}_1 \rangle = 0$

GPS Gold Codes

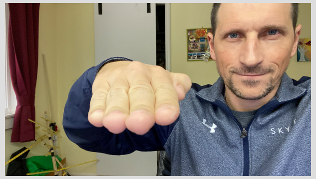


Example:

$\vec{r} =$



$$\langle \vec{r}, \vec{s}_i \rangle = \vec{r}^T \vec{s}_i$$



$$\begin{bmatrix} \quad \vec{r}^T \quad \end{bmatrix} \begin{bmatrix} \vec{s}_1 & \vec{s}_2 & \vec{s}_3 & \cdots & \vec{s}_{24} \end{bmatrix} = \begin{bmatrix} \vec{r}^T\vec{s}_1 & \vec{r}^T\vec{s}_2 & \cdots & \vec{r}^T\vec{s}_{24} \end{bmatrix}$$