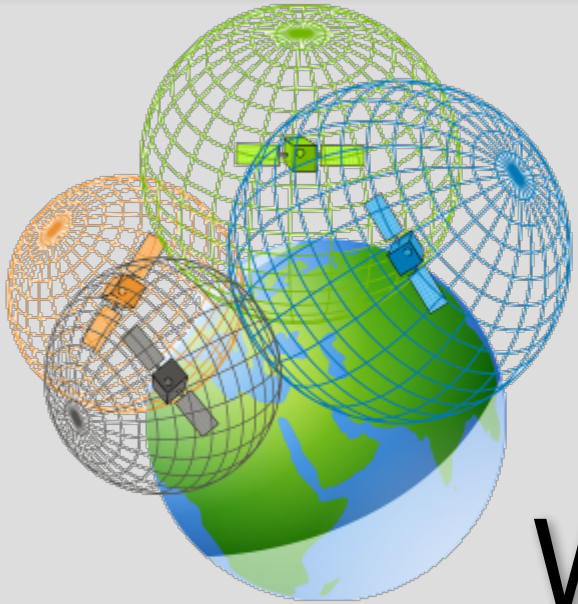
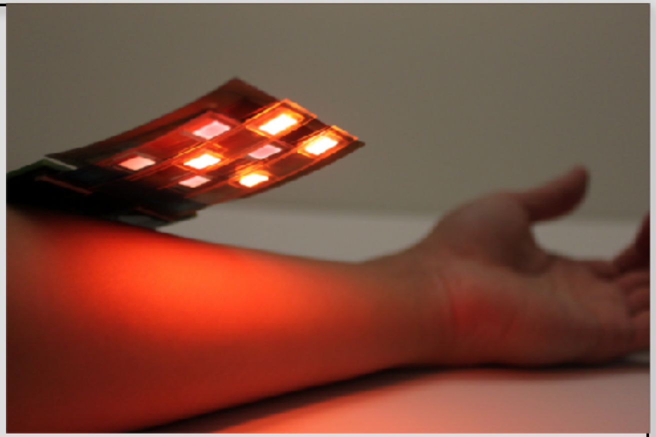




Welcome to EECS 16A!

Designing Information Devices and Systems I

Ana Arias and Miki Lustig

Fall 2022

Lecture 12A
Correlation and Tri-Lateration

Good morning!

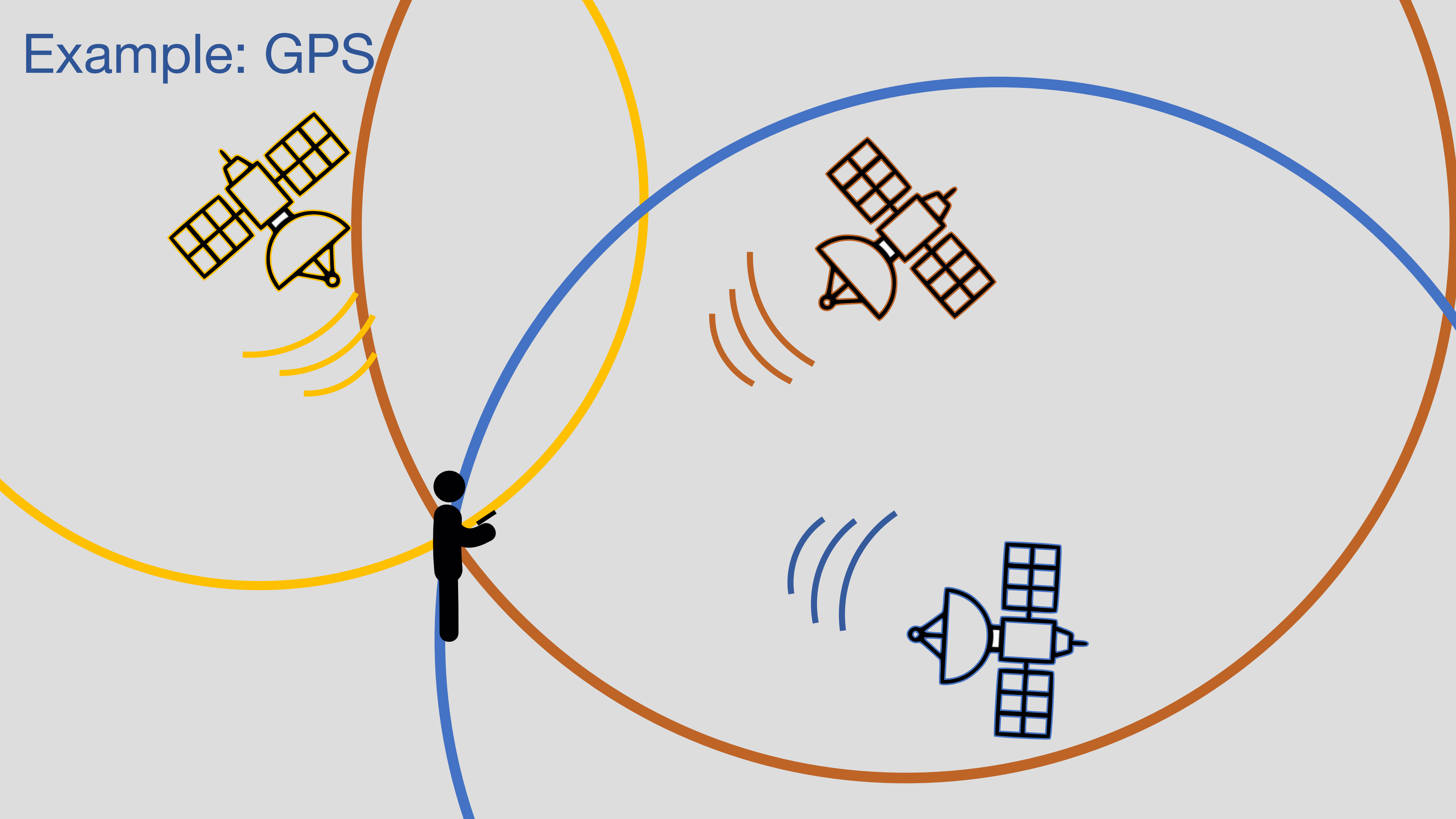
Last time:

- Talked about GPS
  - Known position of satellites
  - Each satellite has its own signature
- Talked about inner product
  - Measure of similarity between vectors
  - When zero — orthogonal vectors
- Talked about using inner products for classification

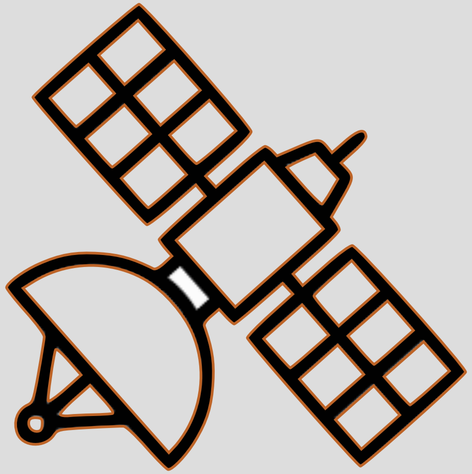
Today:

- Computing delay with cross-correlation
- Finding position with multi-lateration

Example: GPS



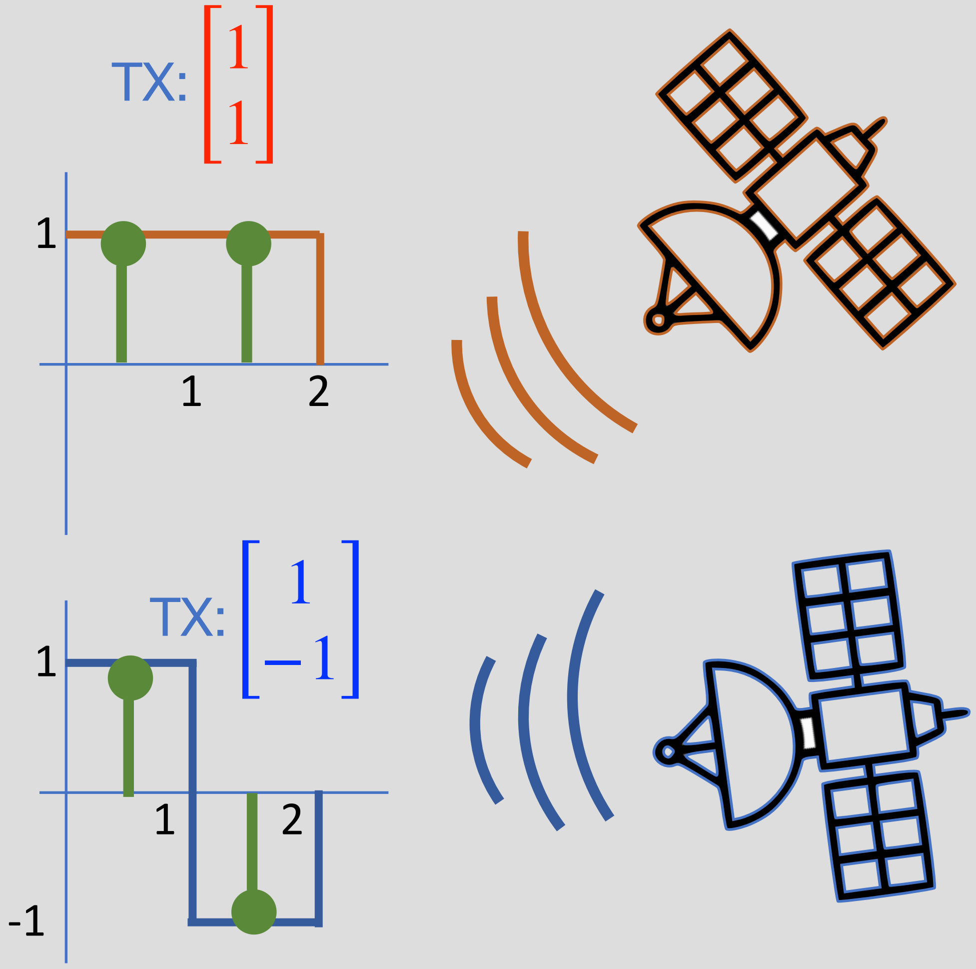
GPS



- 24 satellites
  - Known position
  - Time synchronized
  - 8 usually visible
- Problem:
  - Classify which satellite is transmitting
  - Estimate distance to GPS
  - Estimate position from noisy data
- Tools:
  - Inner product
  - Cross correlation
  - Least Squares

Localization

- Satellites transmit a unique code
  - Radio signal
- Signal is received and digitized by a receiver



Two problems:

1. Interference
2. Timing (Next!)

Interference

Possibility 1: Both sats are in TX

$$\vec{r} = \vec{s}_1 + \vec{s_2} + \vec{n}$$

Possibility 2: Only S1 is in Tx

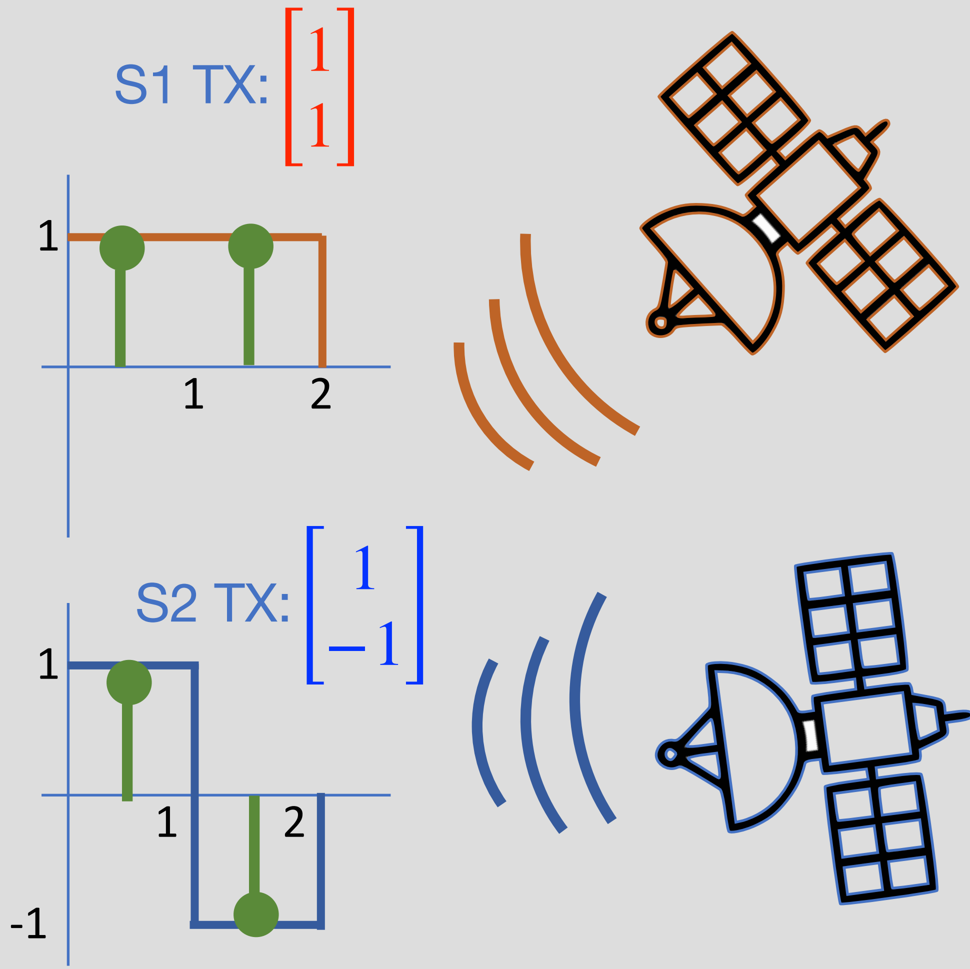
$$\vec{r} = \vec{s}_1 + \vec{n}$$

Possibility 3: Only S2 is in Tx

$$\vec{r} = \vec{s}_2 + \vec{n}$$

Possibility 4: None is in Tx

$$\vec{r} = \vec{n}$$



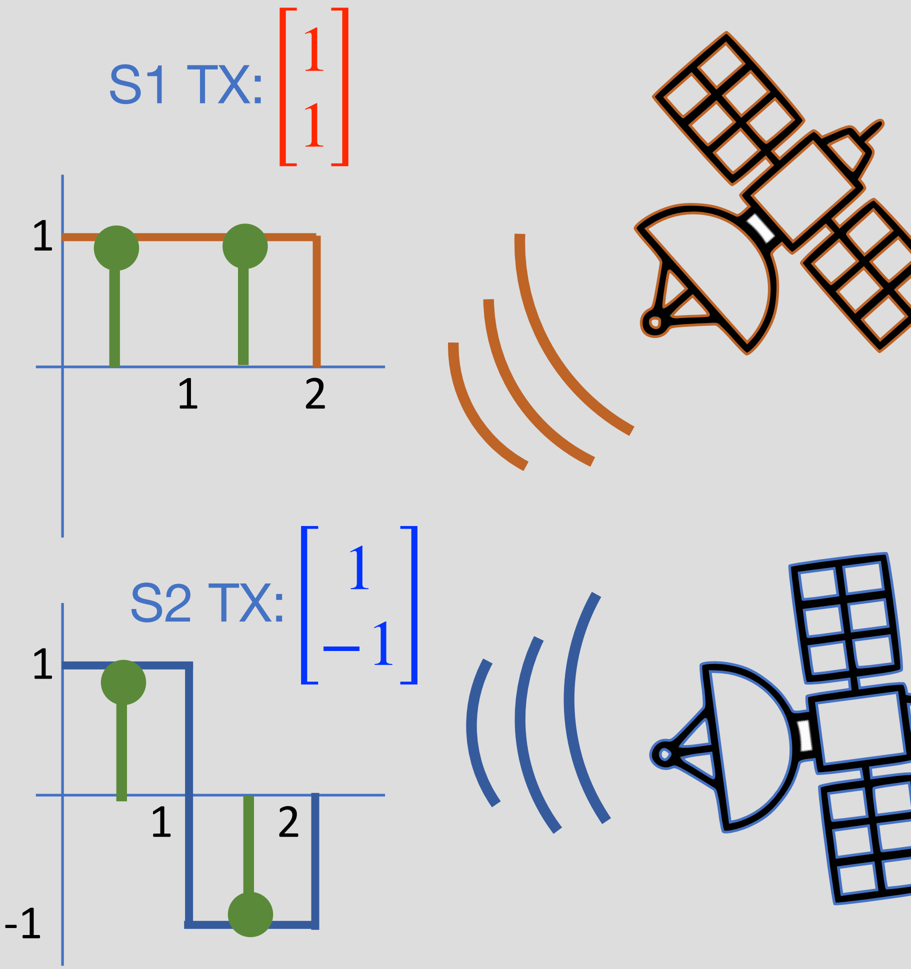
Interference

Possibility 1: Both sats are in TX

$$\vec{r} = \vec{s}_1 + \vec{s}_2 + \vec{n}$$

$$\langle \vec{r}, \vec{s}_1 \rangle = \langle \vec{s}_1 + \vec{s}_2 + \vec{n}, \vec{s}_1 \rangle$$

$$= \underbrace{\langle \vec{s}_1, \vec{s}_1 \rangle}_{\text{Desired}} + \underbrace{\langle \vec{s}_2, \vec{s}_1 \rangle}_{\text{Interference}} + \underbrace{\langle \vec{n}, \vec{s}_1 \rangle}_{\text{Small}}$$



Q: How to design codes that don’t interfere?

A: Make them orthogonal! $\langle \vec{s}_2, \vec{s}_1 \rangle = 0$

Example

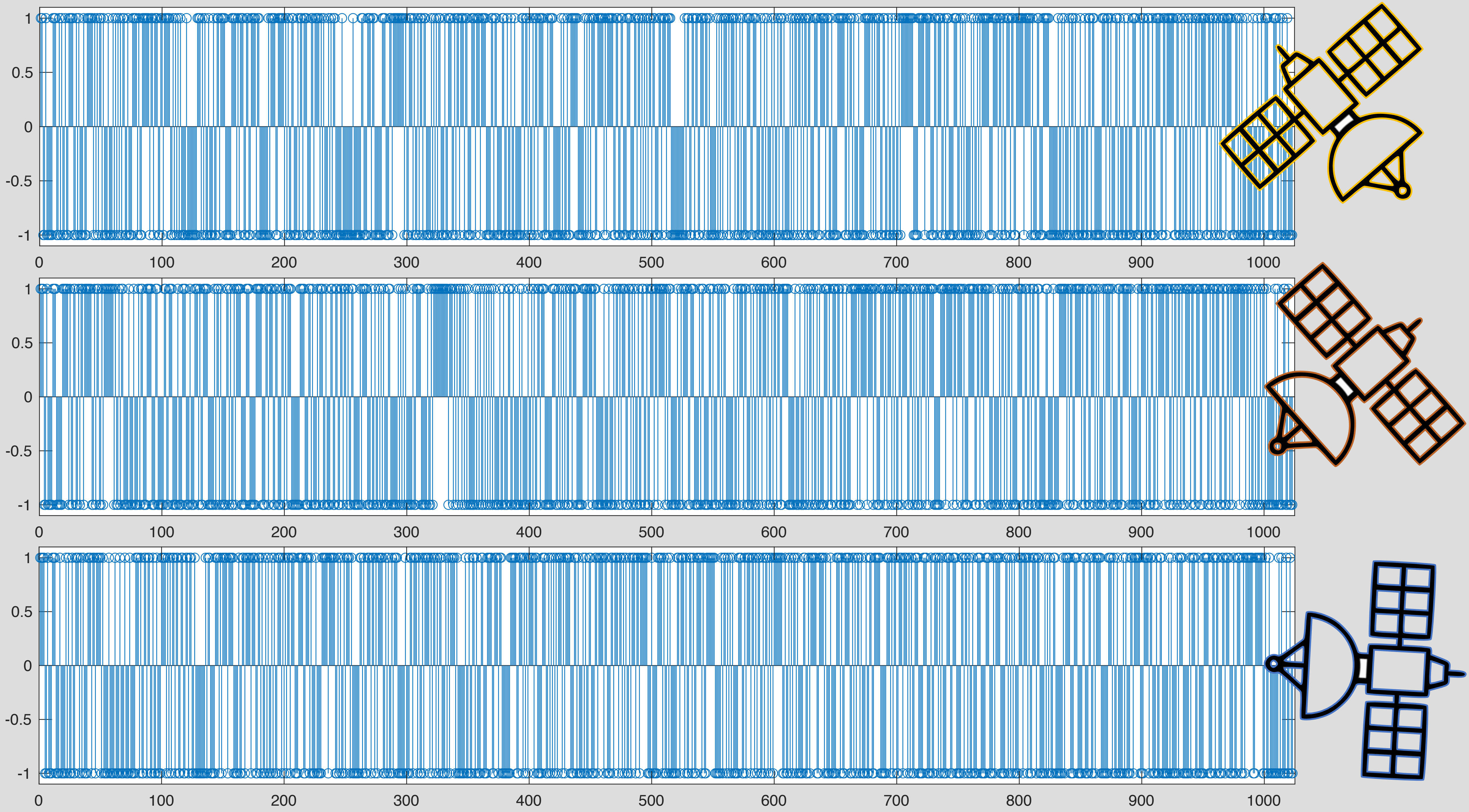
$\vec{r}_1 = [2.01, -0.03]^T$ $\vec{r}_2 = [1.01, -1.03]$

$\vec{s}_1 = [1, 1]^T$

$\vec{s}_2 = [1, -1]^T$

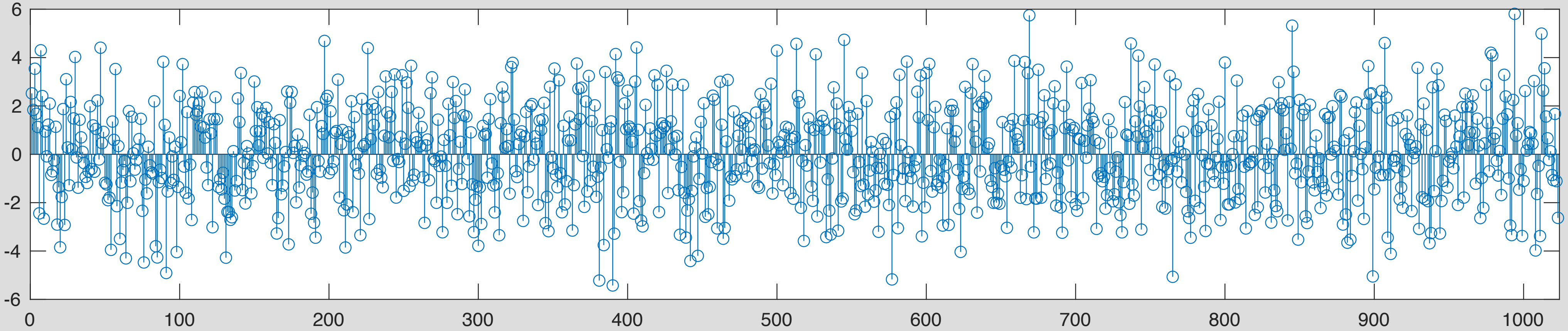
$\vec{n} = [0.01, -0.03]^T$

GPS Gold Codes

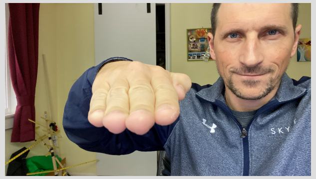


Example:

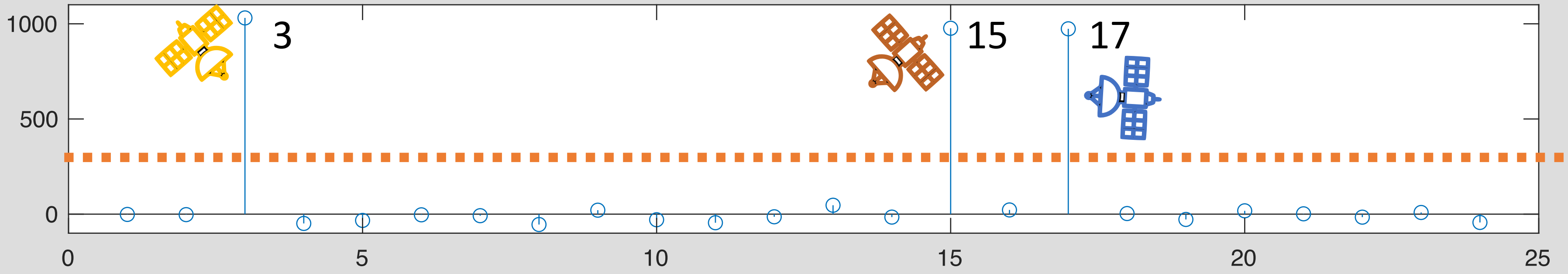
$\vec{r} =$



$$\langle \vec{r}, \vec{s}_i \rangle = \vec{r}^T \vec{s}_i$$

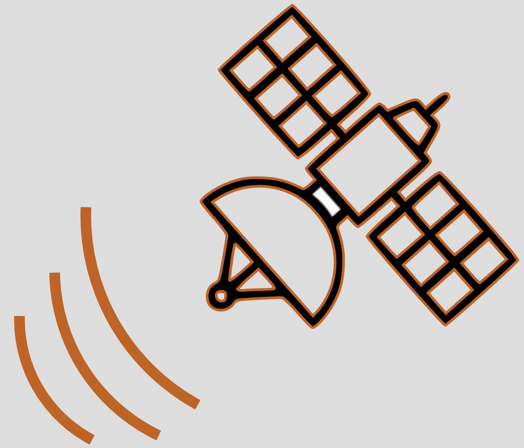


$$\begin{bmatrix} \quad \vec{r}^T \quad \end{bmatrix} \begin{bmatrix} \vec{s}_1 & \vec{s}_2 & \vec{s}_3 & \cdots & \vec{s}_{24} \end{bmatrix} = \begin{bmatrix} \vec{r}^T\vec{s}_1 & \vec{r}^T\vec{s}_2 & \cdots & \vec{r}^T\vec{s}_{24} \end{bmatrix}$$

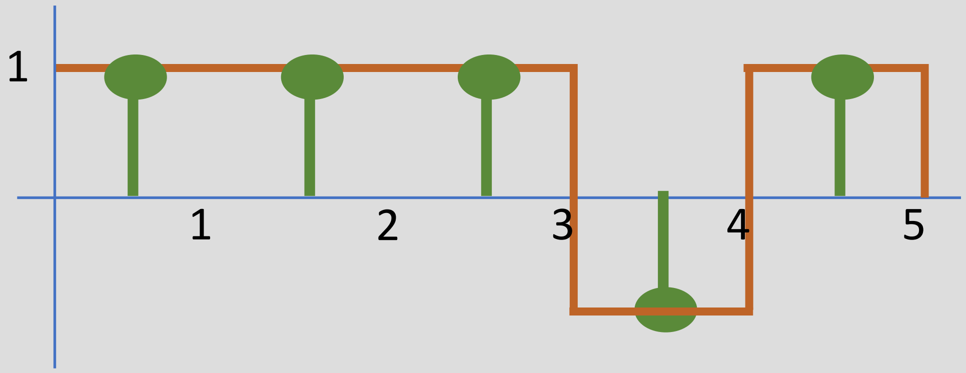


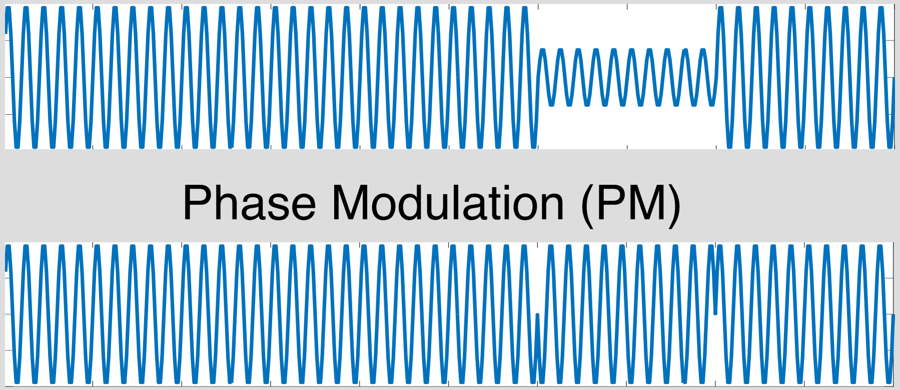
Timing….

- Satellites transmit a (modulated) unique code
    - Radio signal
- Signal is received (demodulated) and digitized by a receiver



$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

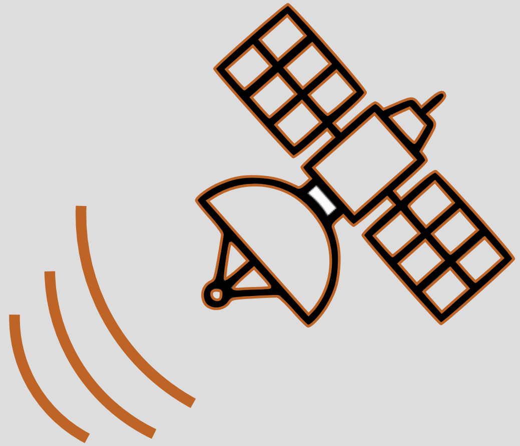


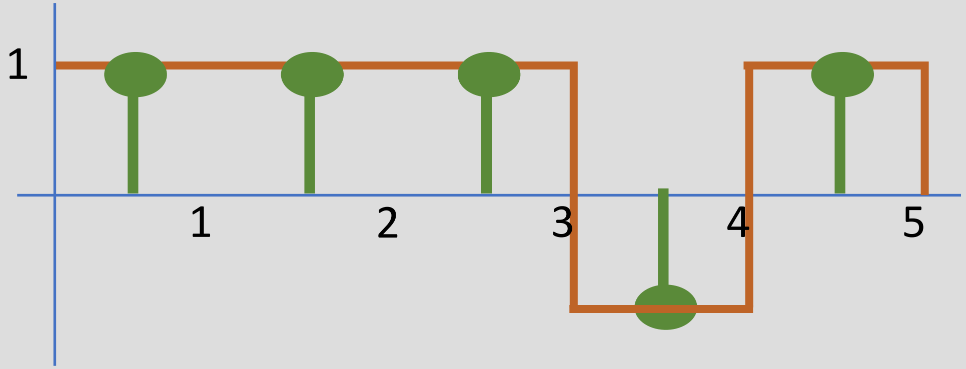


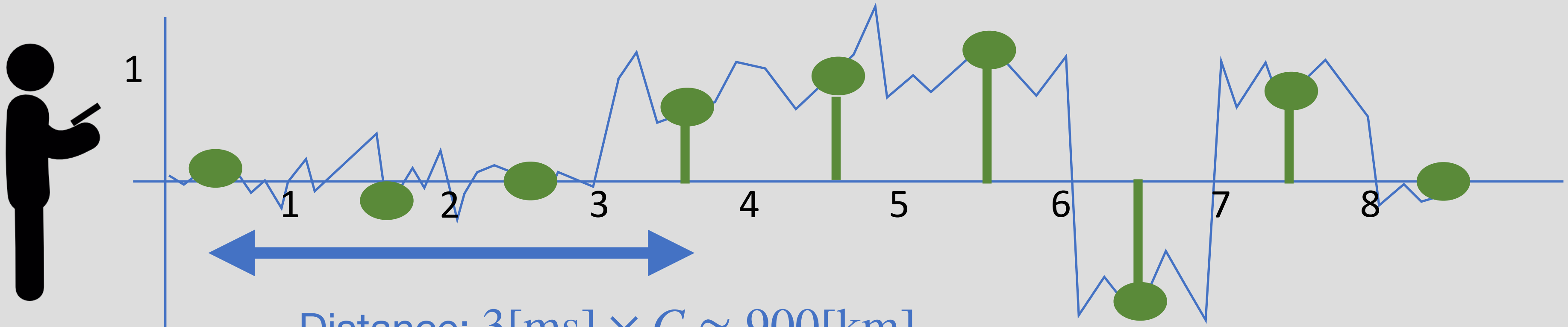
Timing….

- Satellites transmit a unique code
  - Radio signal
- Signal is received and digitized by a receiver

TX: $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$

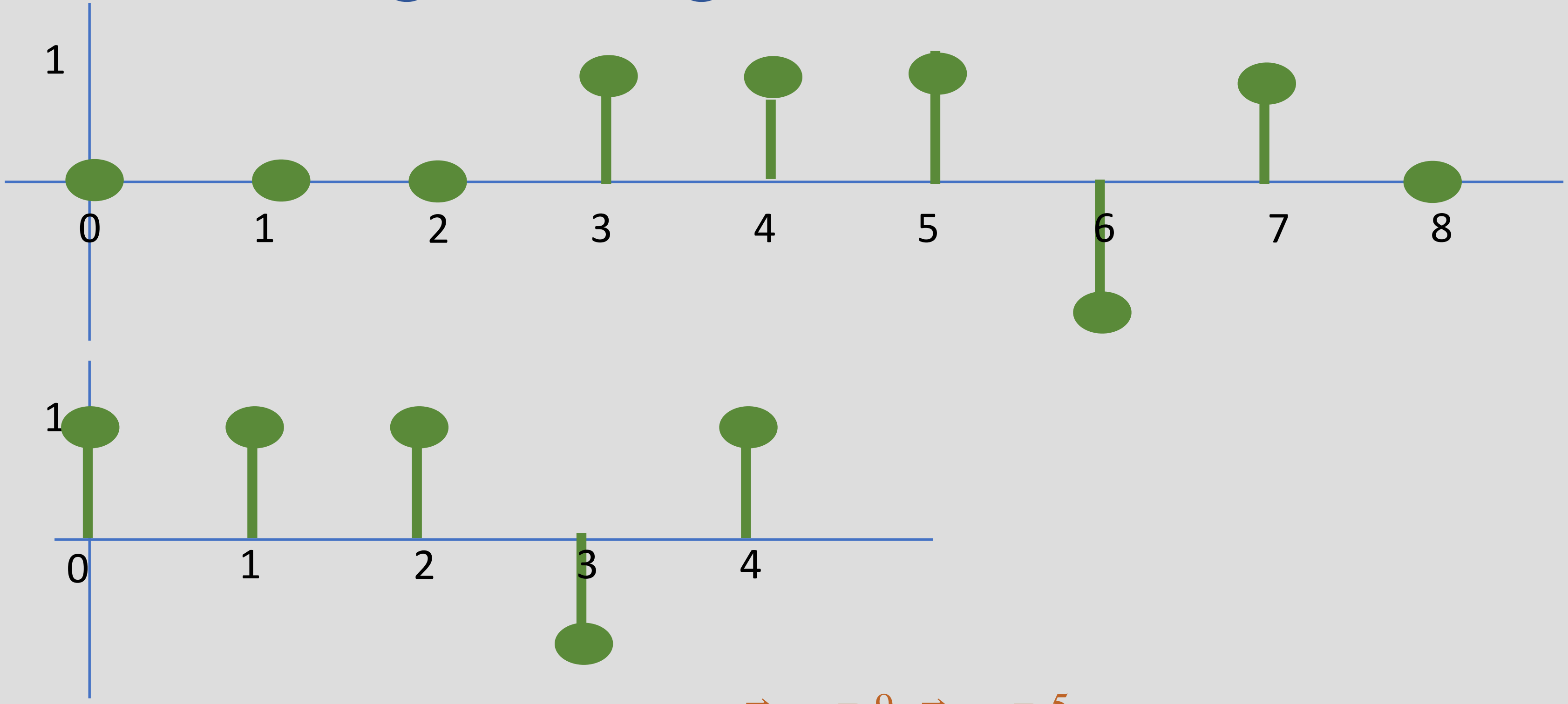






Distance: $3[\mathrm{ms}] \times C \approx 900[\mathrm{km}]$

“Pattern Matching” of Signals



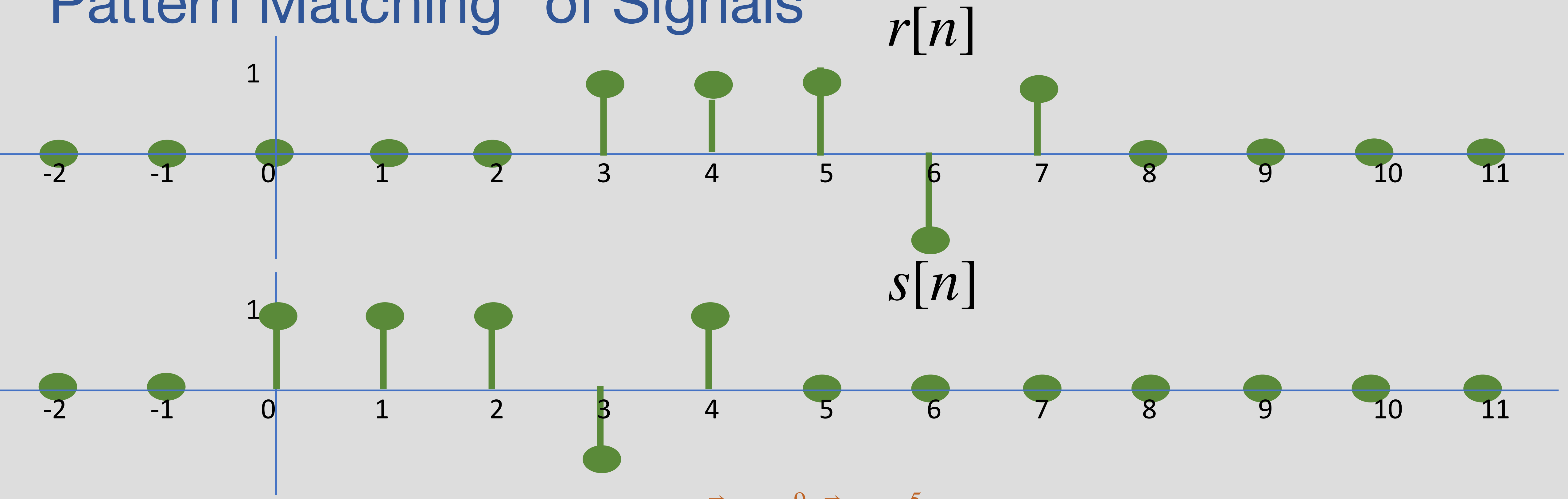
Problem: vectors (signals) not the same length…$\vec{r} \in \mathbb{R}^9$, $\vec{s} \in \mathbb{R}^5$

Solution: Define infinite signals $r[n], s[n]$ by zero-padding

$$\vec{r} = [r_0 \quad r_1 \quad r_2 \quad \cdots \quad r_8]^T \qquad \Rightarrow r[n] = \begin{cases} r_n & 0 \le n \le 8 \\ 0 & \text{elsewhere} \end{cases}$$

$$\vec{s} = [s_0 \quad s_1 \quad s_2 \quad \cdots \quad s_4]^T \qquad \Rightarrow s[n] = \begin{cases} s_n & 0 \le n \le 4 \\ 0 & \text{elsewhere} \end{cases}$$

"Pattern Matching" of Signals



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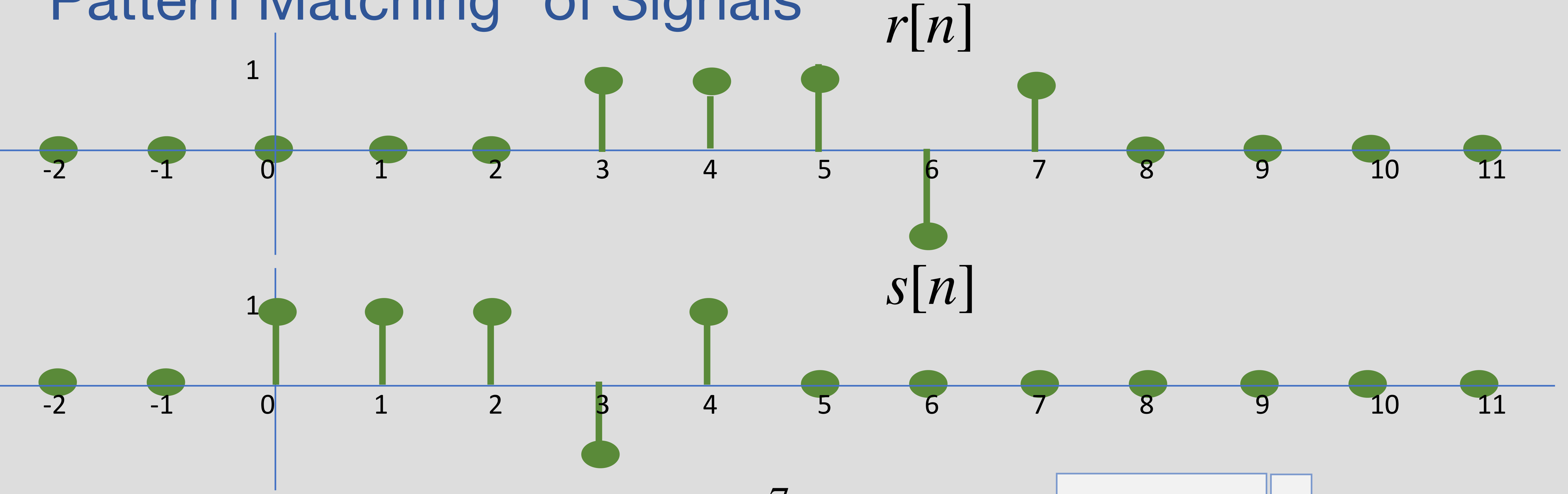
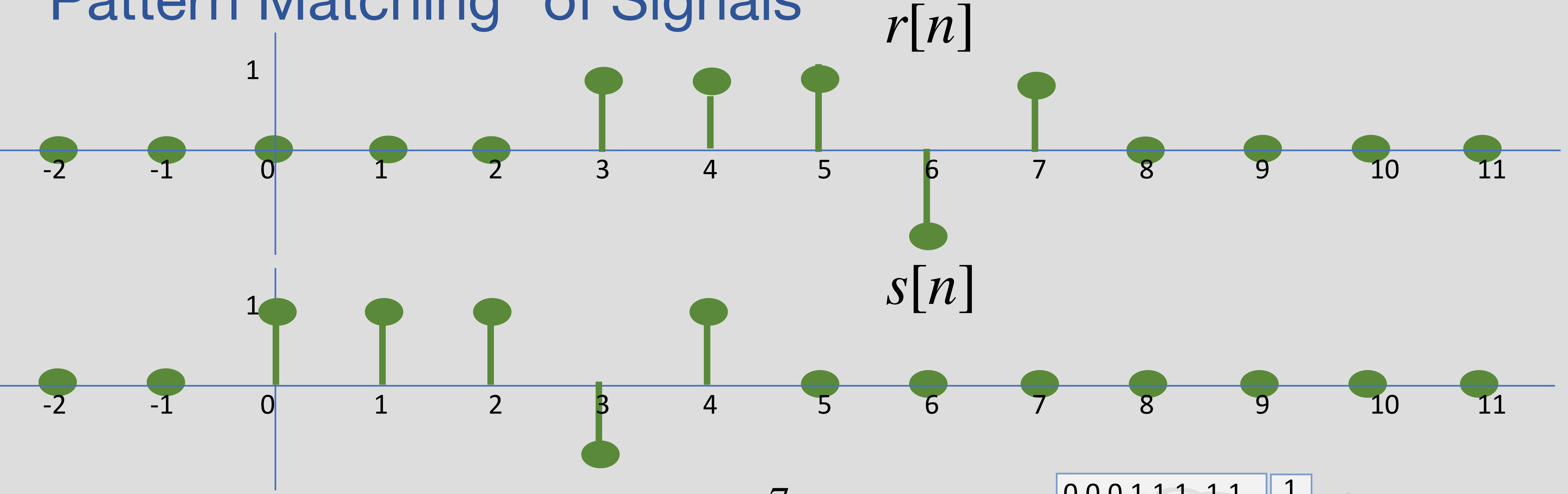
“Pattern Matching” of Signals



$$\langle r[n], s[n] \rangle = \sum_{n=-\infty}^{\infty} r[n]s[n] = \sum_{n=0}^{7} r[n]s[n] =$$

"Pattern Matching" of Signals

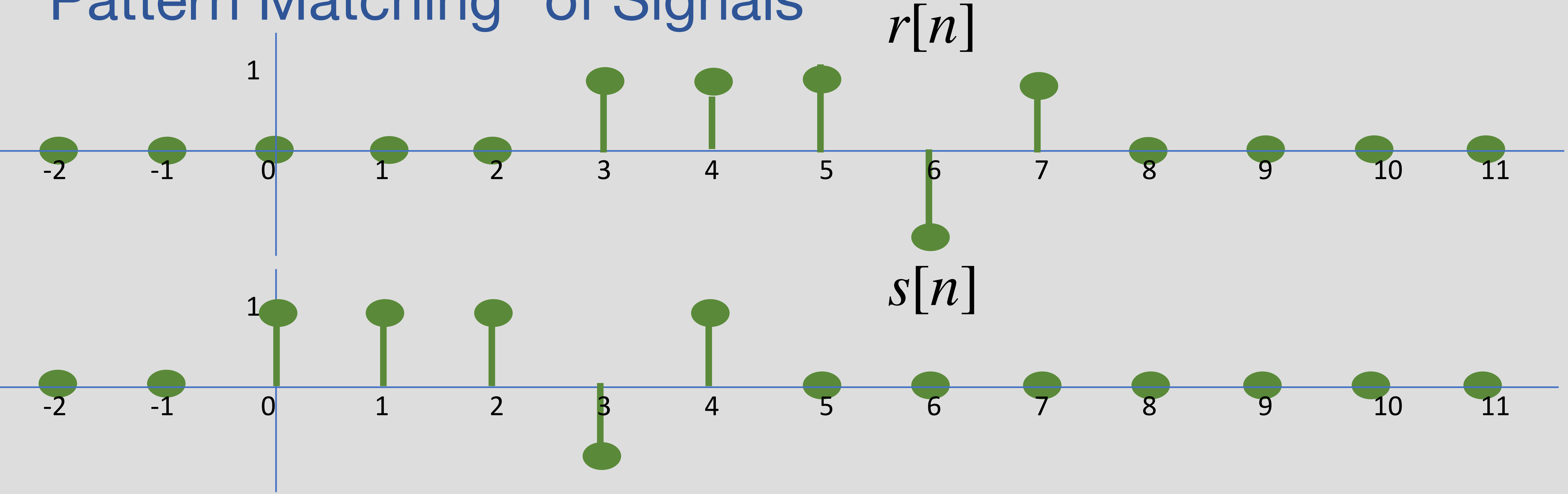


$$\langle r[n], s[n] \rangle = \sum_{n=-\infty}^{\infty} r[n]s[n] = \sum_{n=0}^{7} r[n]s[n] = \boxed{0,0,0,1,1,1,-1,1}\ \boxed{\begin{matrix}1\\1\\1\\-1\\1\\0\\0\\0\end{matrix}} = 0$$

Q: How to match with shifted version?

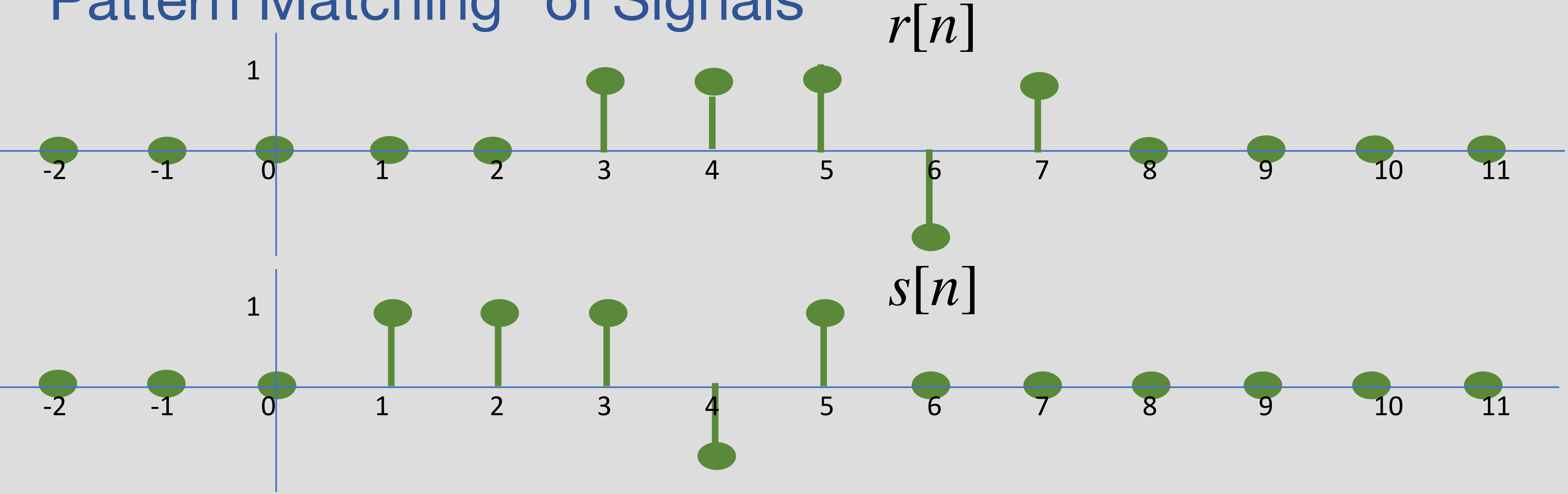
A: compute: $\langle r[n], s[n-1] \rangle = \sum_{n=-\infty}^{\infty} r[n]s[n-1]$

“Pattern Matching” of Signals



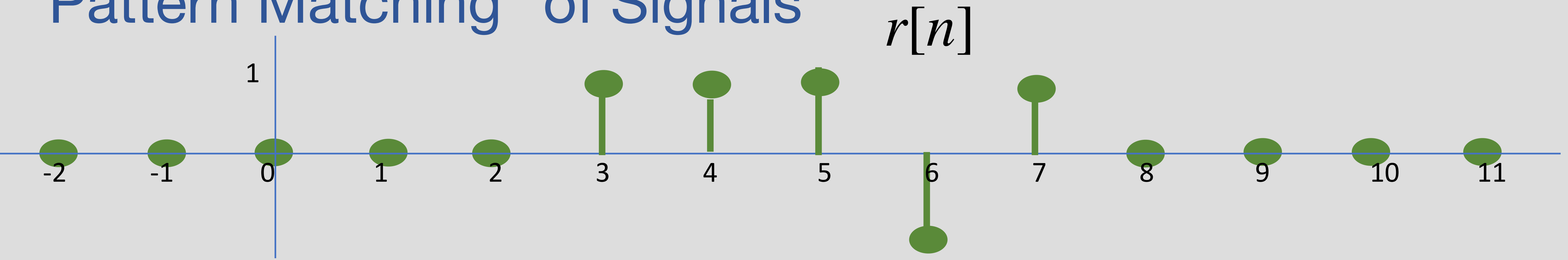
$$\langle r[n], s[n-1] \rangle = \sum_{n=-\infty}^{\infty} r[n]s[n-1]$$

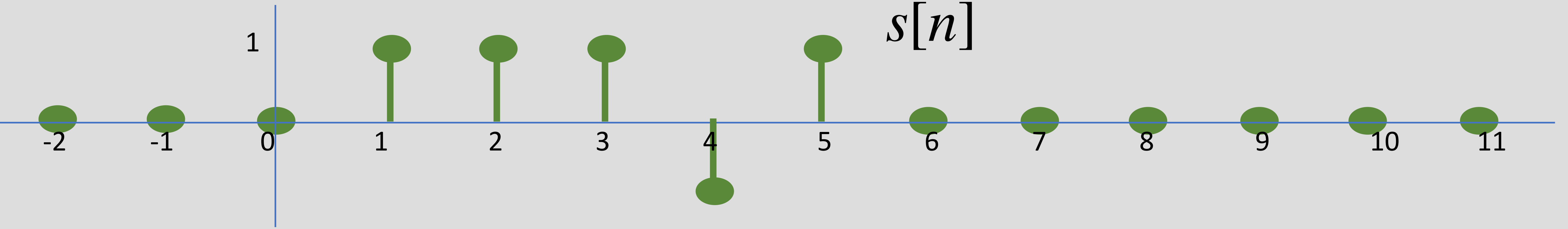
"Pattern Matching" of Signals



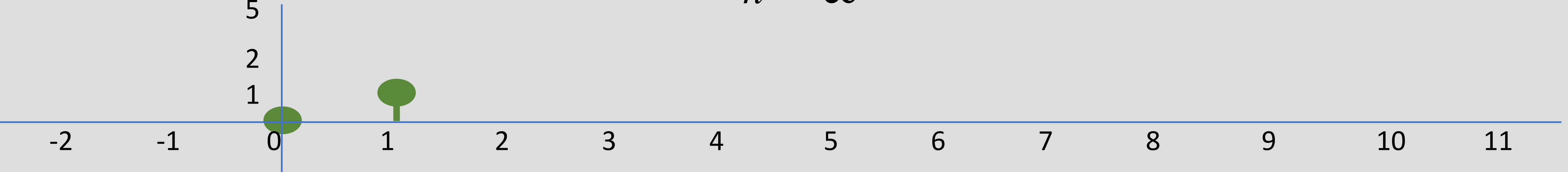
$$\mathrm{corr}_{\vec{r}}(\vec{s})[1] = \langle r[n], s[n-1] \rangle = \sum_{n=-\infty}^{\infty} r[n]s[n-1] = 1$$

“Pattern Matching” of Signals





$$\mathrm{corr}_{\vec{r}}(\vec{s})[1] = \langle r[n], s[n-1]\rangle = \sum_{n=-\infty}^{\infty} r[n]s[n-1] = 1$$



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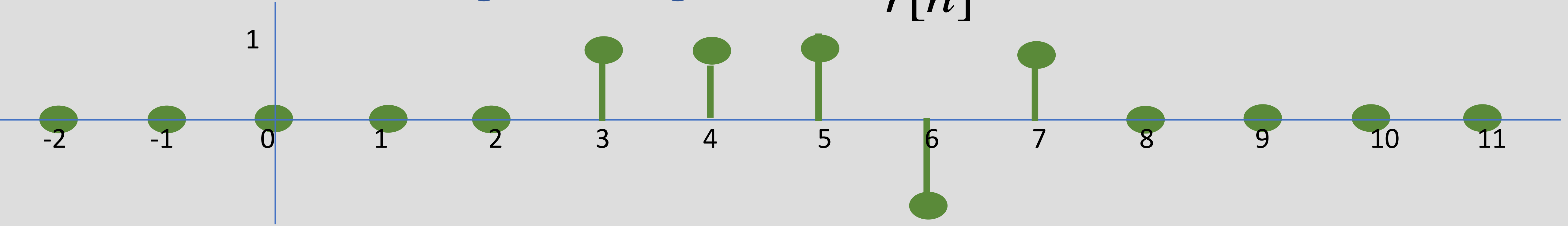
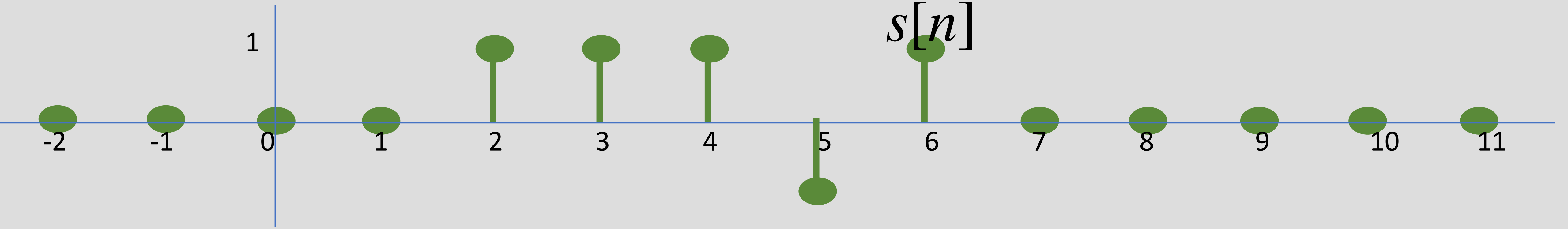
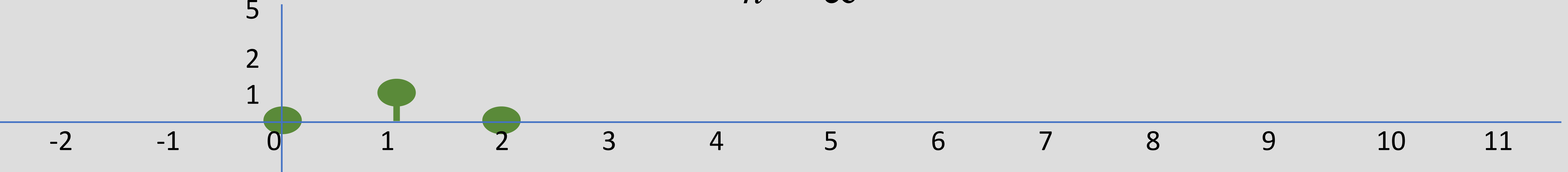
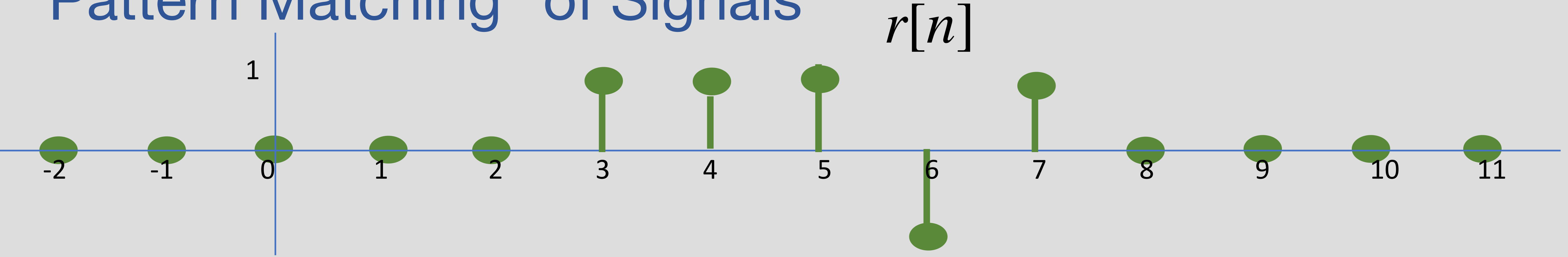


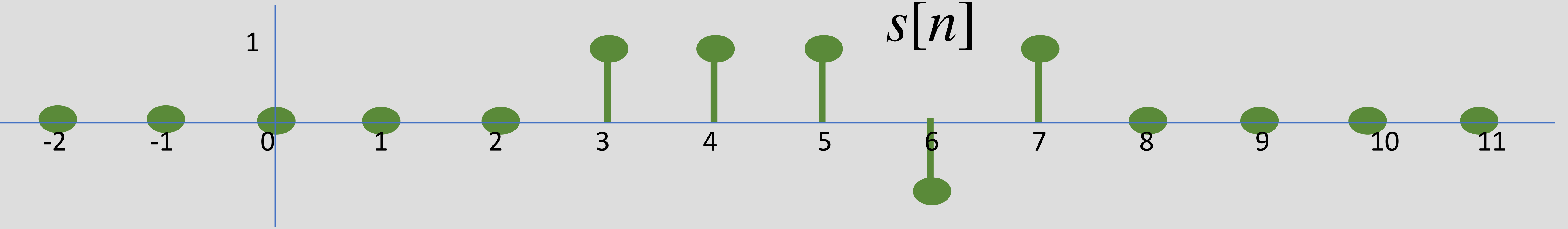


$$\mathrm{corr}_{\vec{r}}(\vec{s})[2] = \langle r[n], s[n-2] \rangle = \sum_{n=-\infty}^{\infty} r[n]s[n-2] = 0$$

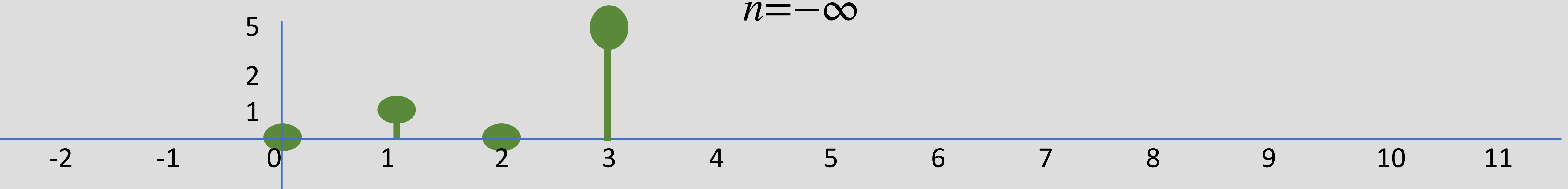


“Pattern Matching” of Signals

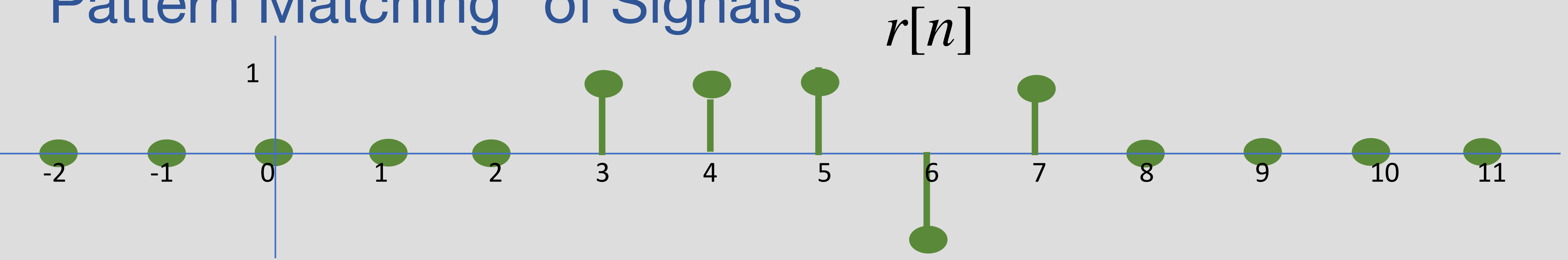


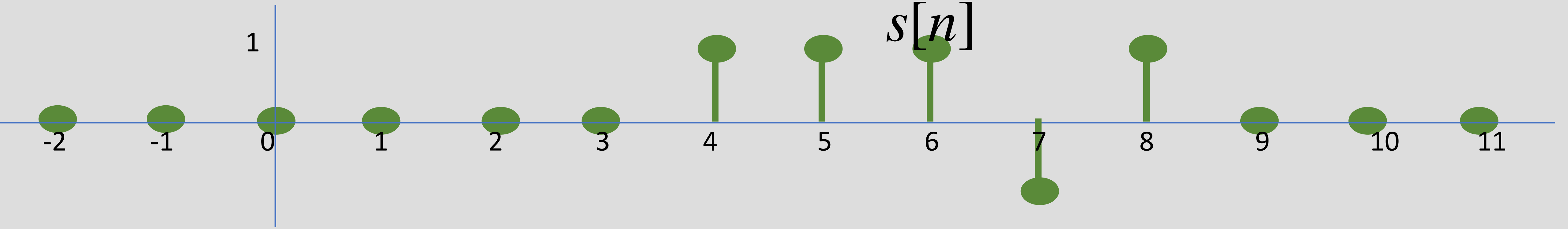


$$\text{corr}_{\vec{r}}(\vec{s})[3] = \langle r[n], s[n-3]\rangle = \sum_{n=-\infty}^{\infty} r[n]s[n-3] = 5$$

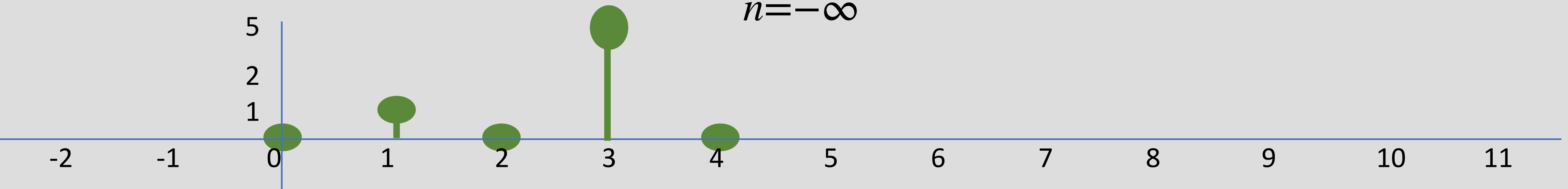


"Pattern Matching" of Signals

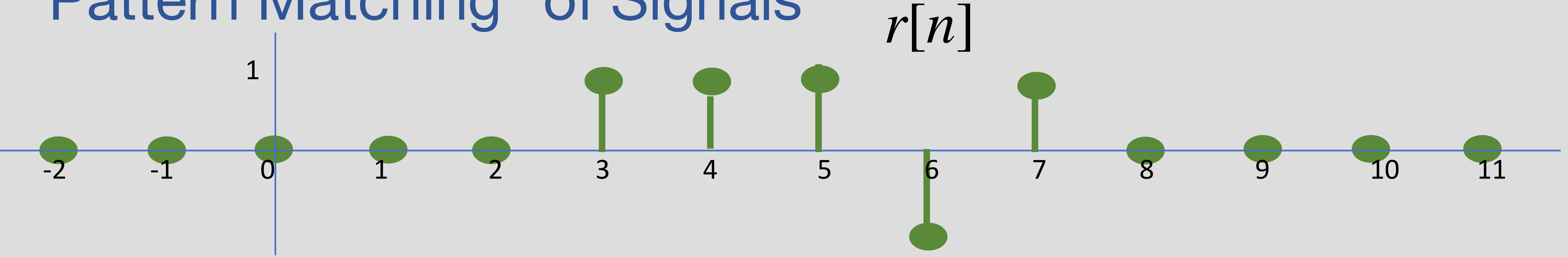


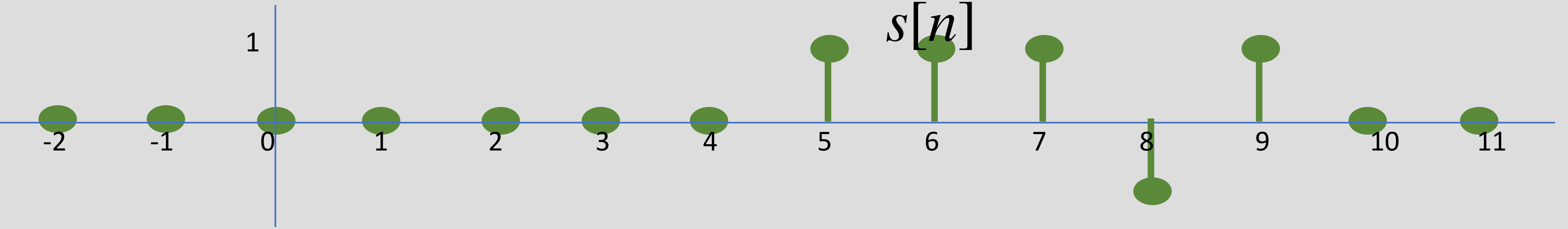


$$\mathrm{corr}_{\vec{r}}(\vec{s})[4] = \langle r[n], s[n-4] \rangle = \sum_{n=-\infty}^{\infty} r[n]s[n-4] = 0$$

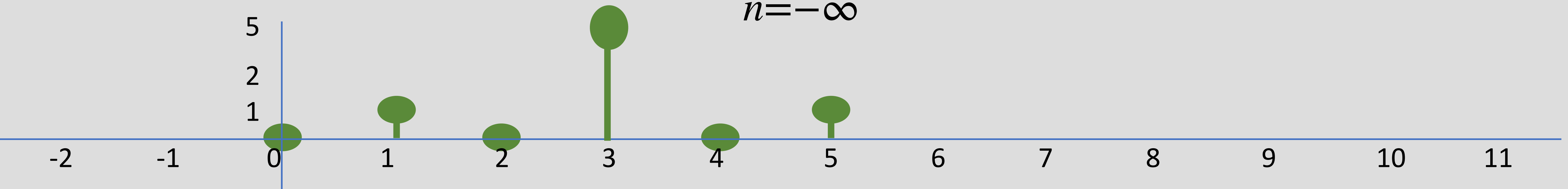


“Pattern Matching” of Signals

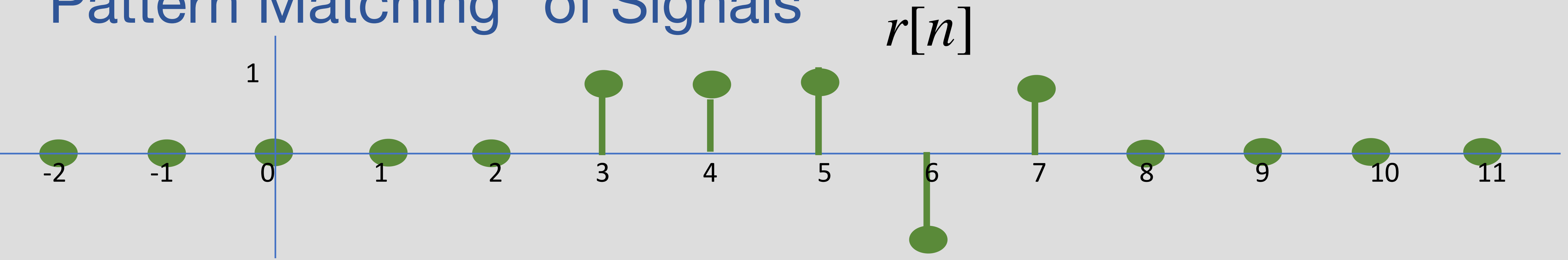


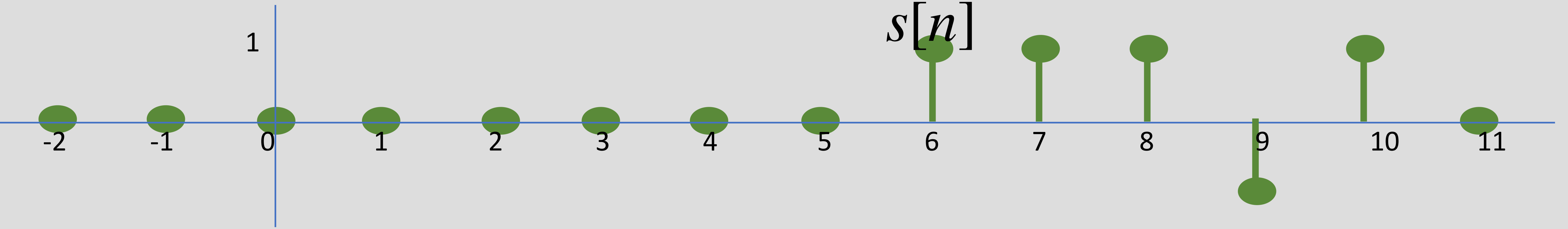


$$\mathrm{corr}_{\vec{r}}(\vec{s})[5] = \langle r[n], s[n-5] \rangle = \sum_{n=-\infty}^{\infty} r[n]s[n-5] = 1$$

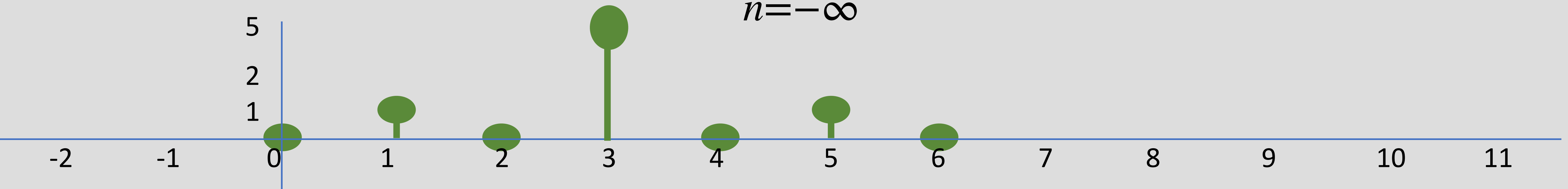


“Pattern Matching” of Signals

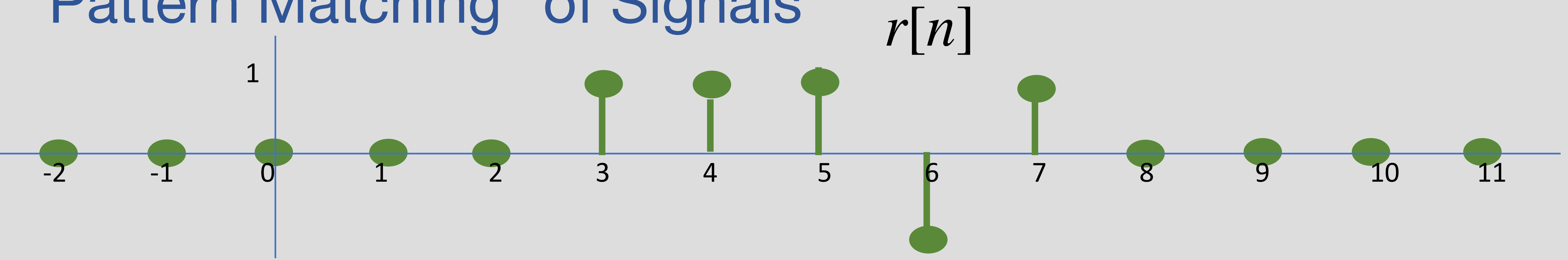


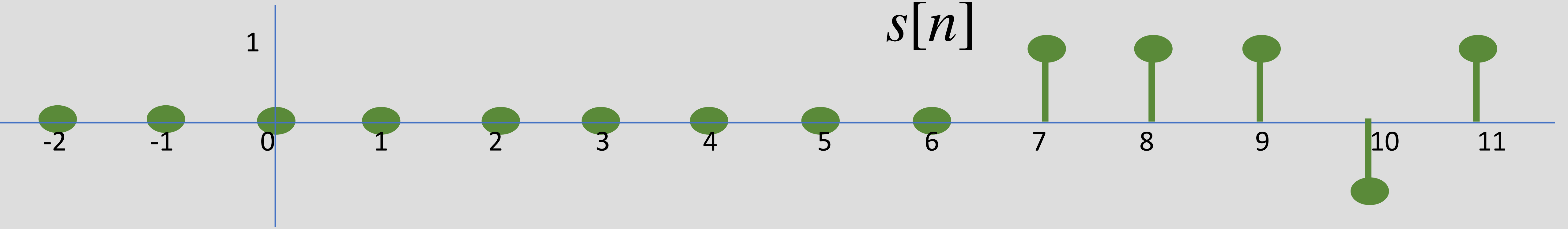


$$\mathrm{corr}_{\vec{r}}(\vec{s})[6] = \langle r[n], s[n-6]\rangle = \sum_{n=-\infty}^{\infty} r[n]s[n-6] = 0$$

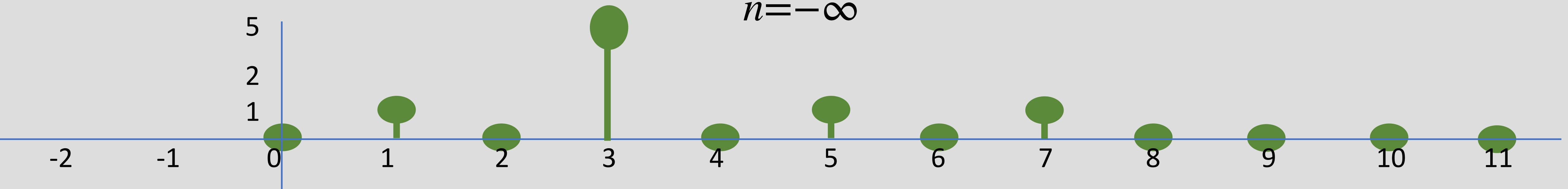


“Pattern Matching” of Signals



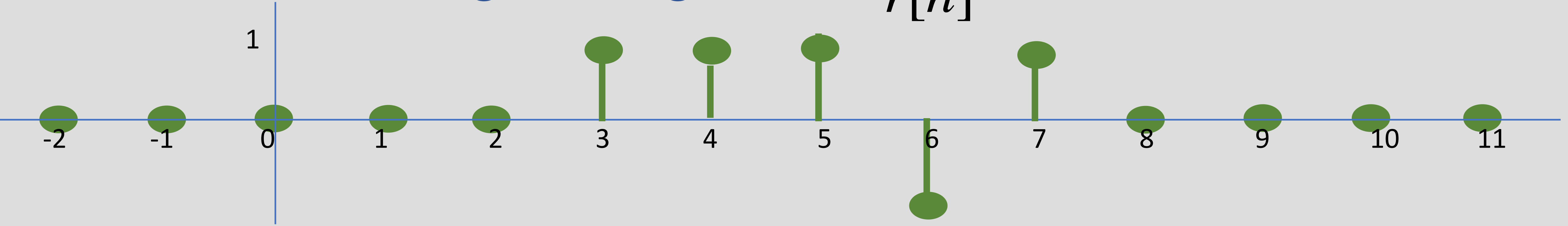


$$\mathrm{corr}_{\vec{r}}(\vec{s})[7] = \langle r[n], s[n-7]\rangle = \sum_{n=-\infty}^{\infty} r[n]s[n-7] = 1$$

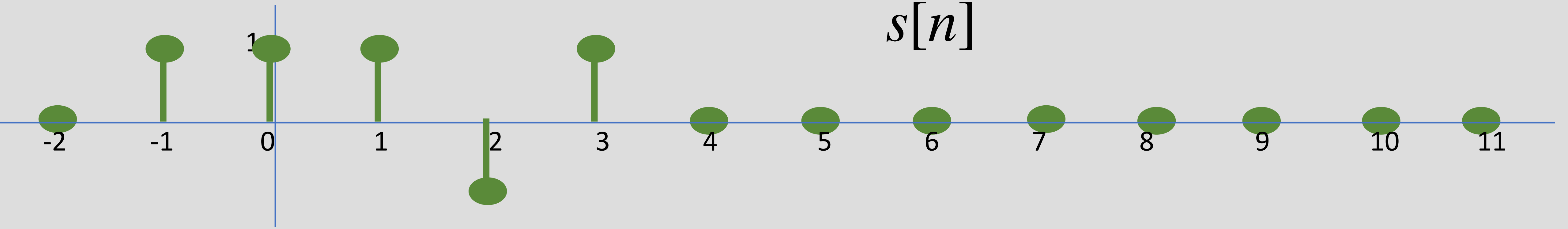


“Pattern Matching” of Signals

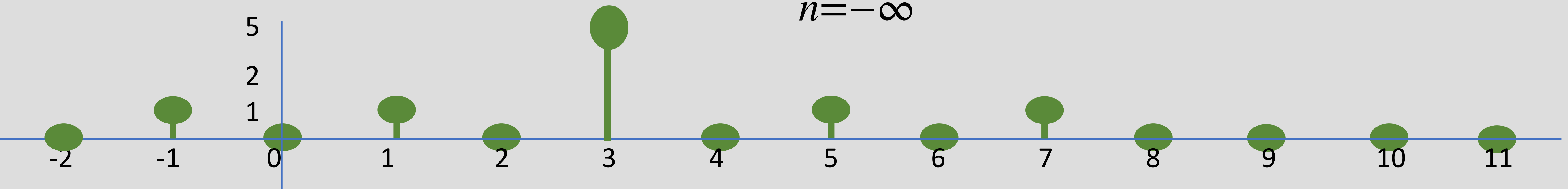
$r[n]$



$s[n]$

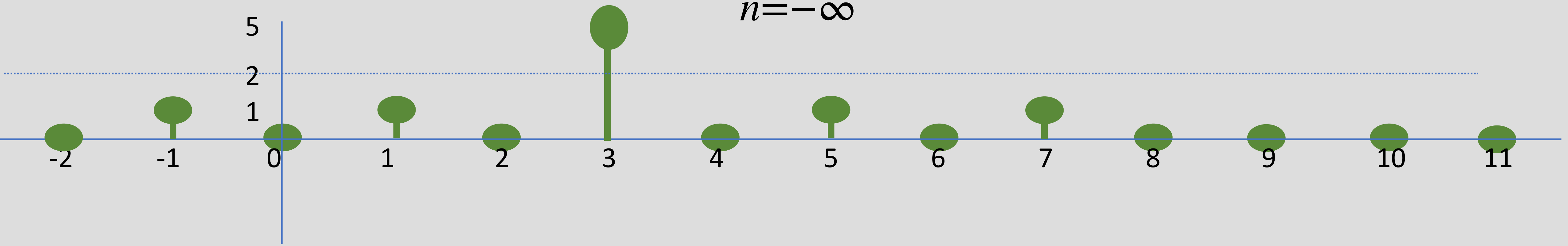


$$\mathrm{corr}_{\vec{r}}(\vec{s})[-1] = \langle r[n], s[n+1] \rangle = \sum_{n=-\infty}^{\infty} r[n]s[n+1] = 1$$



Cross Correlation

$$\mathrm{corr}_{\vec{r}}(\vec{s})[k] = \langle r[n], s[n-k] \rangle = \sum_{n=-\infty}^{\infty} r[n]s[n-k]$$



$$k^* = \underset{k}{\mathrm{argmax}} \ \mathrm{corr}_{\vec{r}}(\vec{s})[k]$$

$$k^* = 3$$

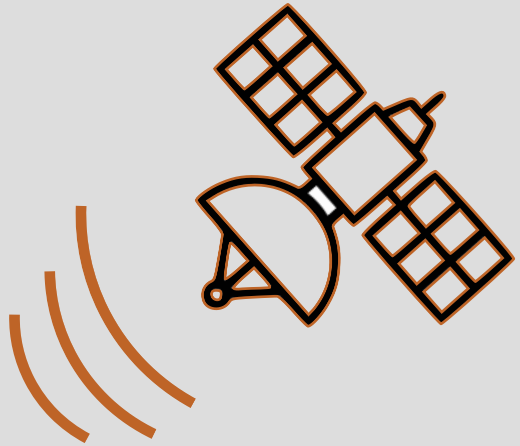
Cross Correlation Properties

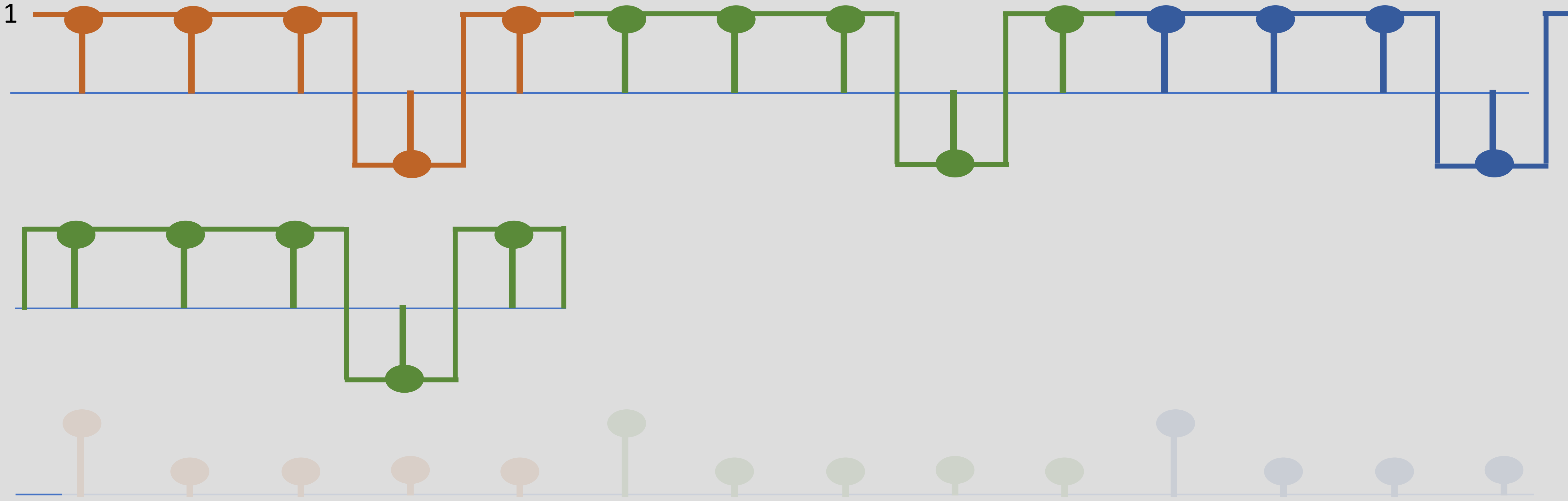
- If $\overrightarrow{x} \in \mathbb{R}^N$, and $\overrightarrow{y} \in \mathbb{R}^M$, then the length of $\mathrm{corr}_{\overrightarrow{x}}(\overrightarrow{y})$ is $N + M - 1$
- $\mathrm{corr}_{\overrightarrow{x}}(\overrightarrow{y}) \neq \mathrm{corr}_{\overrightarrow{y}}(\overrightarrow{x})$
- $\mathrm{corr}_{\overrightarrow{x}}(\overrightarrow{x})$ is called auto-correlation

Periodic Signals

- Satellites repeat the codes over and over
  - Cross correlation is "periodically expanded" instead of zero-padded
  - Result is periodic

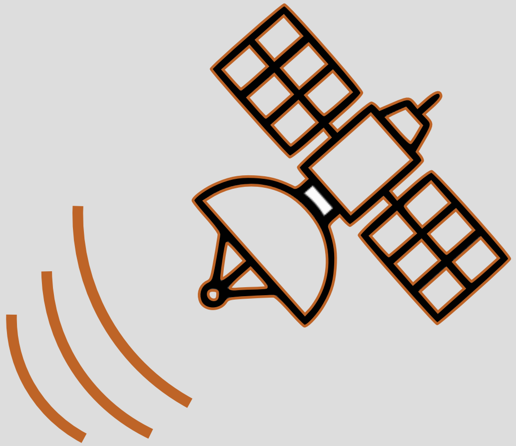
TX: $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$





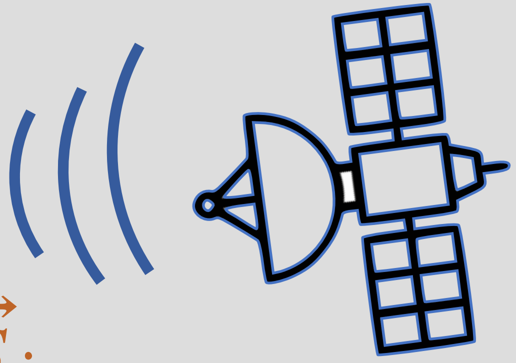
Localization

- Satellites transmit a unique code
  - Radio signal
- Signal is received and digitized by a receiver



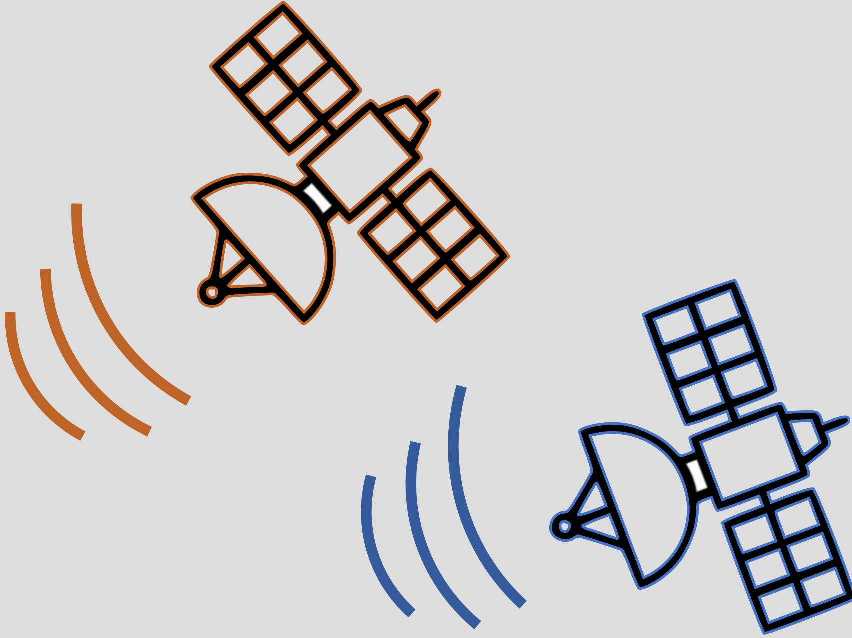
Two problems:

1. Interference
2. Timing



What are good properties for the codes $\vec{s}_i$

Received Signal

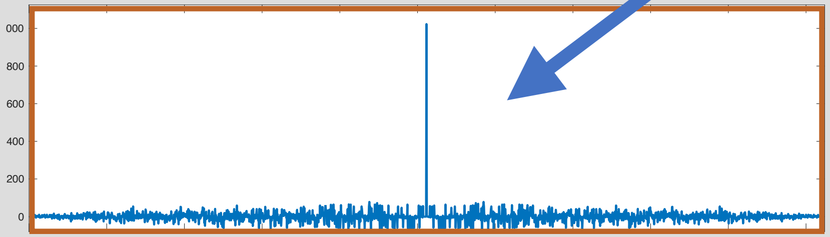


$$r[n] = s_1[n-\tau_1] + s_2[n-\tau_2] + w[n]$$

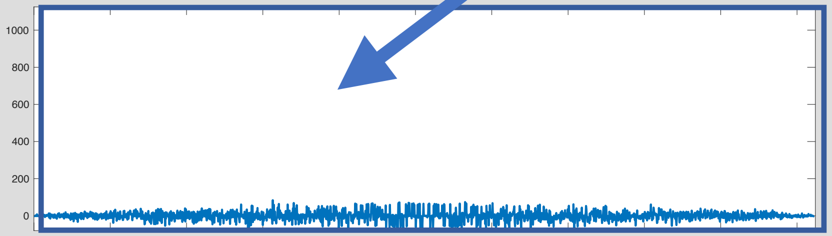
Correlate with $s_1[n]$:

$$\mathrm{corr}_{\vec{r}}(\vec{s}_1)[k] = \langle r[n], s_1[n-k] \rangle$$

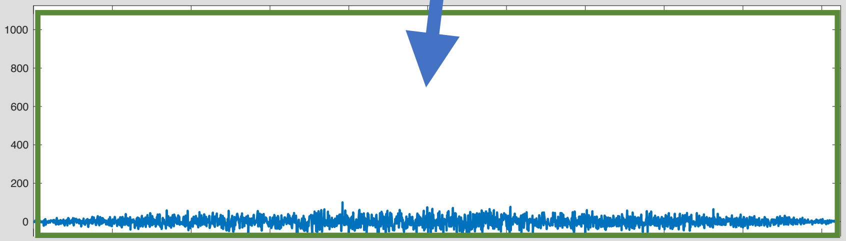
$$= \langle s_1[n-\tau_1], s_1[n-k] \rangle + \langle s_2[n-\tau_2], s_1[n-k] \rangle + \langle w[n], s_1[n-k] \rangle$$



Auto-correlation looks like an impulse



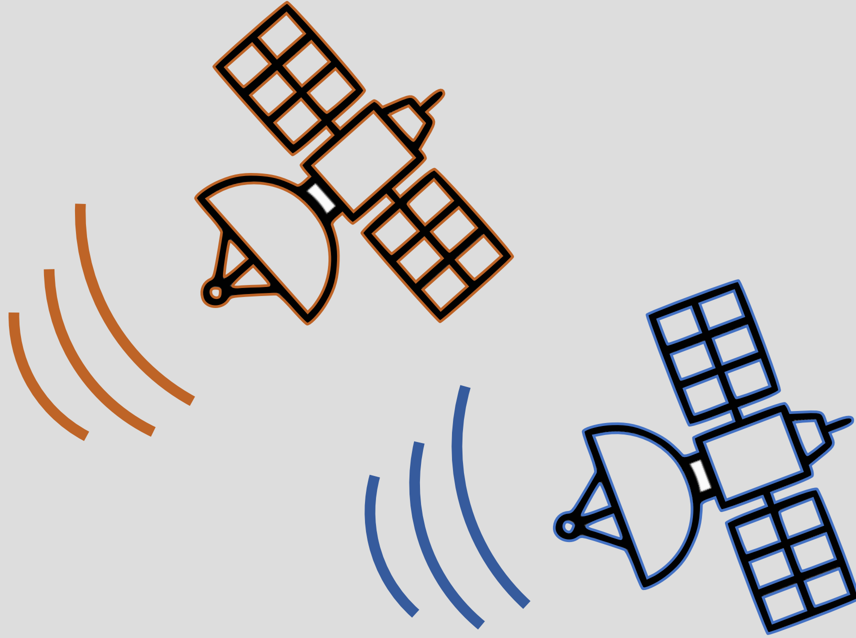
cross-correlation is small

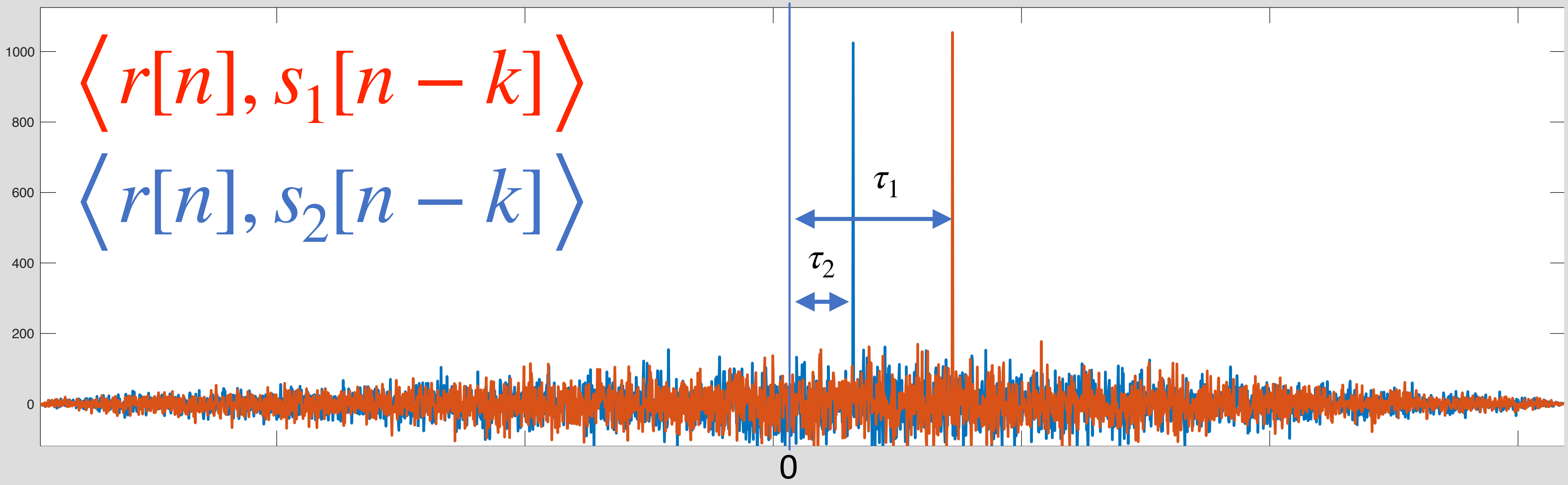


cross-correlation with noise is small
(always true)

Received Signal

$$r[n] = s_1[n - \tau_1] + s_2[n - \tau_2] + w[n]$$



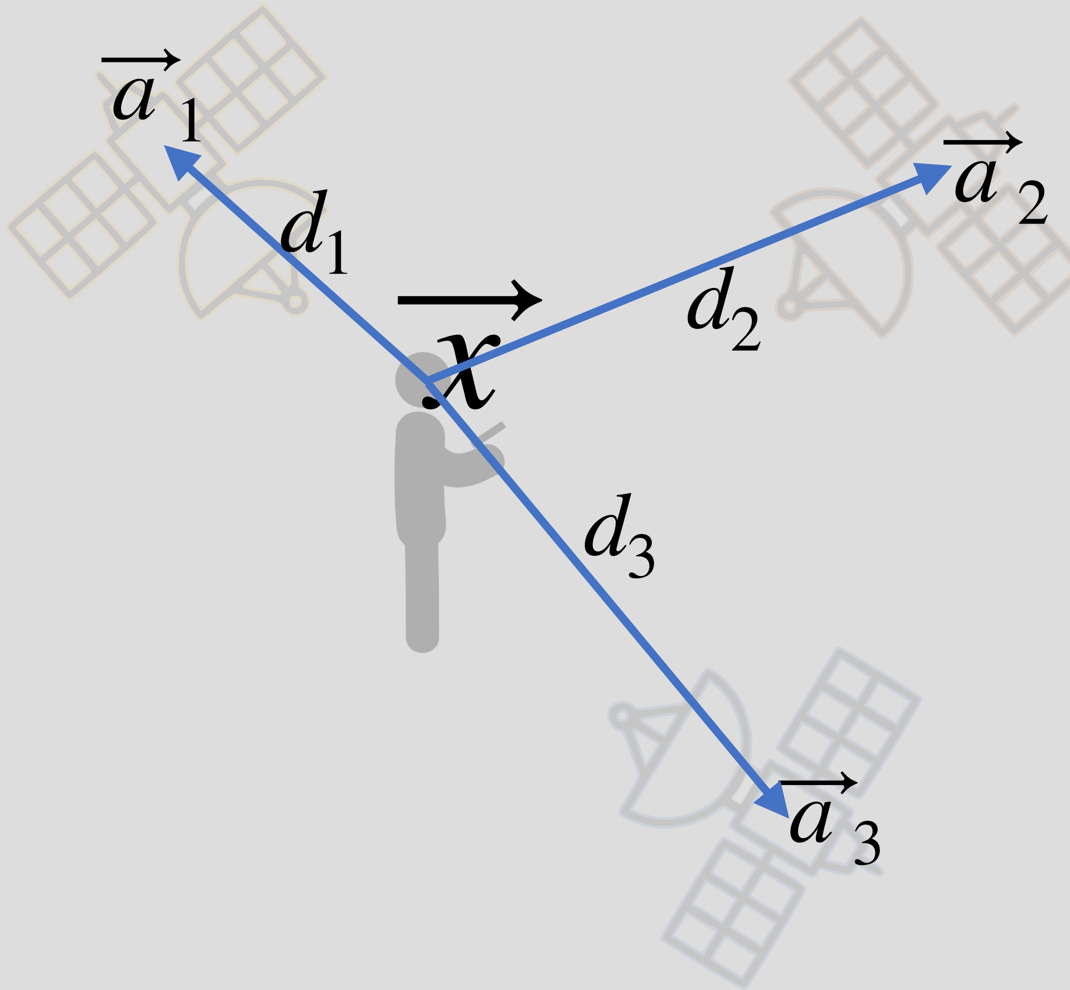


Trilateration

(1) $$\|\vec{x} - \vec{a}_1\|^2 = d_1^2$$

(2) $$\|\vec{x} - \vec{a}_2\|^2 = d_2^2$$

(3) $$\|\vec{x} - \vec{a}_3\|^2 = d_3^2$$



$$d_1 = \tau_1 C$$
$$d_2 = \tau_2 C$$
$$d_3 = \tau_3 C$$

Trilateration

(1) $$\|\vec{x} - \vec{a}_1\|^2 = d_1^2$$

(2) $$\|\vec{x} - \vec{a}_2\|^2 = d_2^2$$

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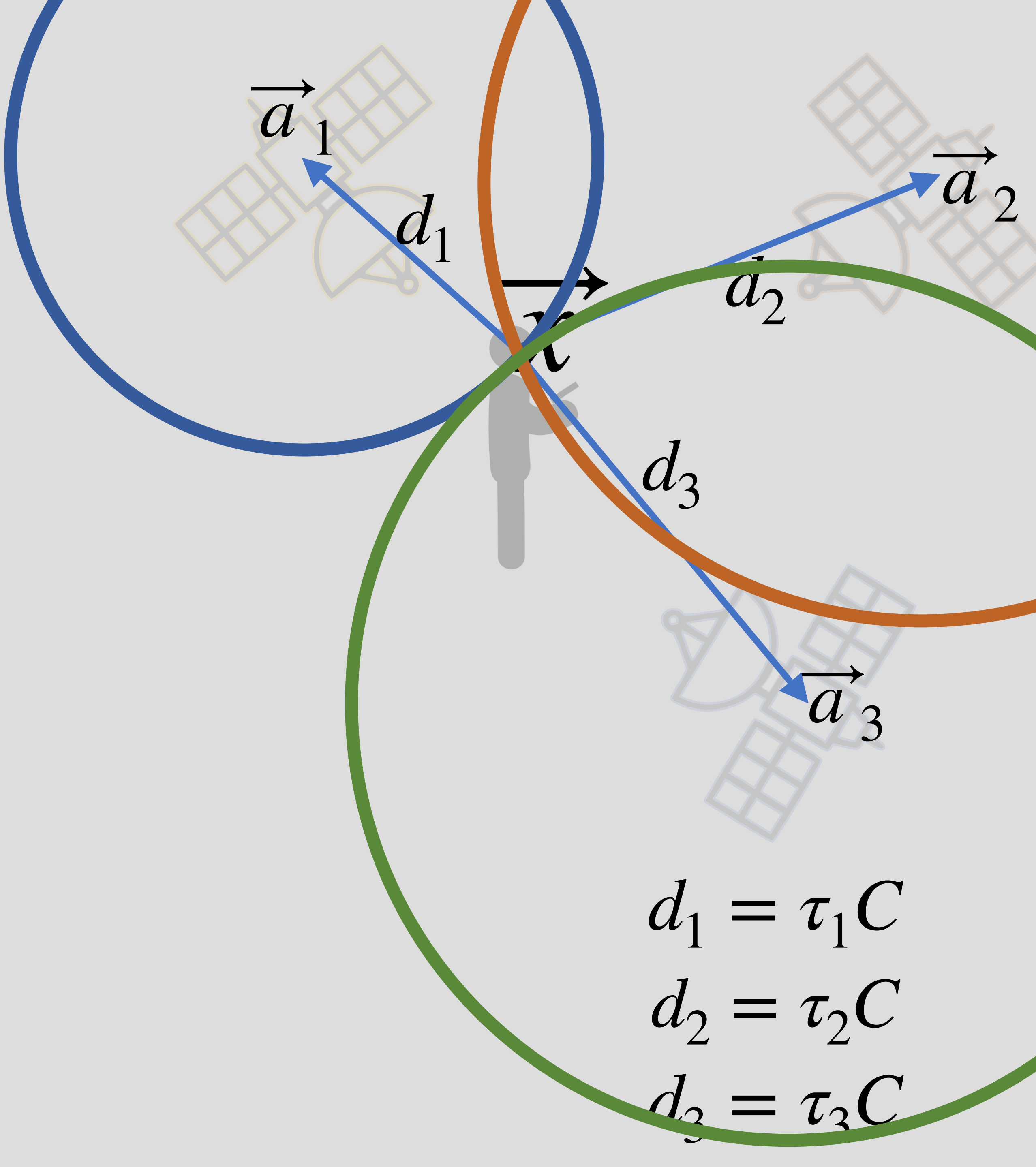
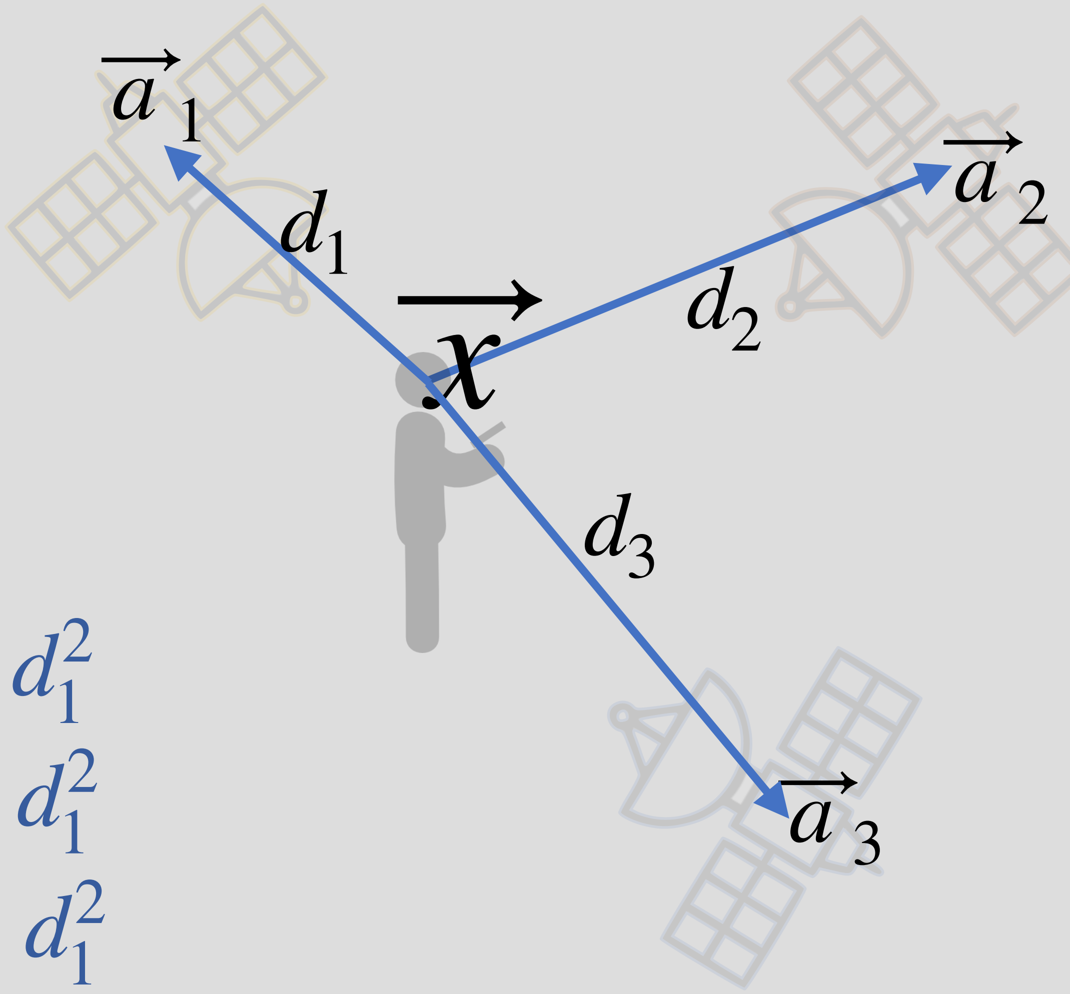


Trilateration

(1) $\|\vec{x} - \vec{a}_1\|^2 = d_1^2$

(2) $\|\vec{x} - \vec{a}_2\|^2 = d_2^2$

(3) $\|\vec{x} - \vec{a}_3\|^2 = d_3^2$



$$\|\vec{x} - \vec{a}_1\|^2 = d_1^2$$

$$(\vec{x} - \vec{a}_1)^T(\vec{x} - \vec{a}_1) = d_1^2$$

$$\vec{x}^T\vec{x} - \vec{a}_1^T\vec{x} - \vec{x}^T\vec{a}_1 + \vec{a}_1^T\vec{a}_1 = d_1^2$$

$$\|\vec{x}\|^2 - 2\vec{a}_1^T\vec{x} + \|\vec{a}_1\|^2 = d_1^2$$

$$\|\vec{x}\|^2 - 2\vec{a}_1^T\vec{x} + \|\vec{a}_1\|^2 = C^2\tau_1^2$$

$$d_1 = \tau_1 C$$

$$d_2 = \tau_2 C$$

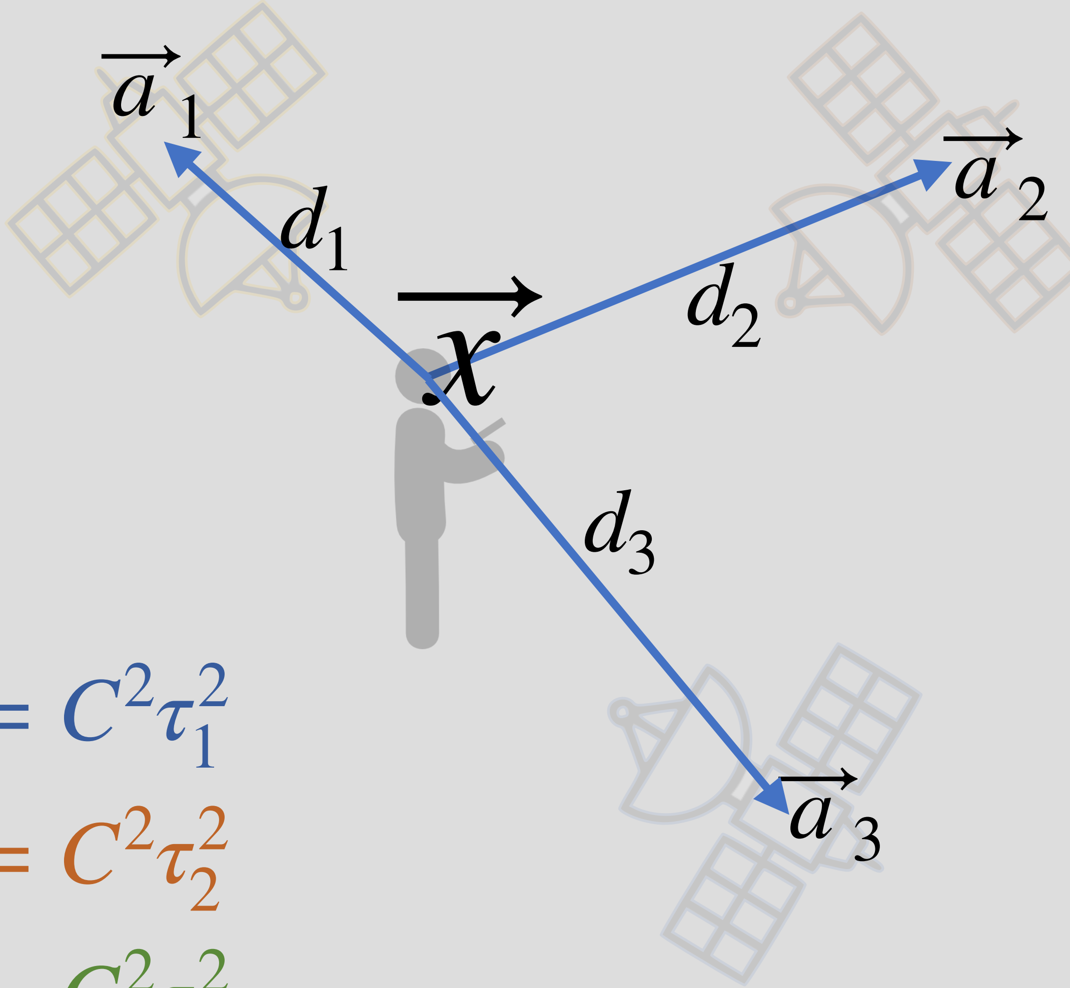
$$d_3 = \tau_3 C$$

Trilateration

(1) $\|\vec{x} - \vec{a}_1\|^2 = d_1^2$

(2) $\|\vec{x} - \vec{a}_2\|^2 = d_2^2$

(3) $\|\vec{x} - \vec{a}_3\|^2 = d_3^2$



(1) $\|\vec{x}\|^2 - 2\vec{a}_1^T\vec{x} + \|\vec{a}_1\|^2 = C^2\tau_1^2$

(2) $\|\vec{x}\|^2 - 2\vec{a}_2^T\vec{x} + \|\vec{a}_2\|^2 = C^2\tau_2^2$

(3) $\|\vec{x}\|^2 - 2\vec{a}_3^T\vec{x} + \|\vec{a}_3\|^2 = C^2\tau_3^2$

$d_1 = \tau_1 C$

$d_2 = \tau_2 C$

$d_3 = \tau_3 C$

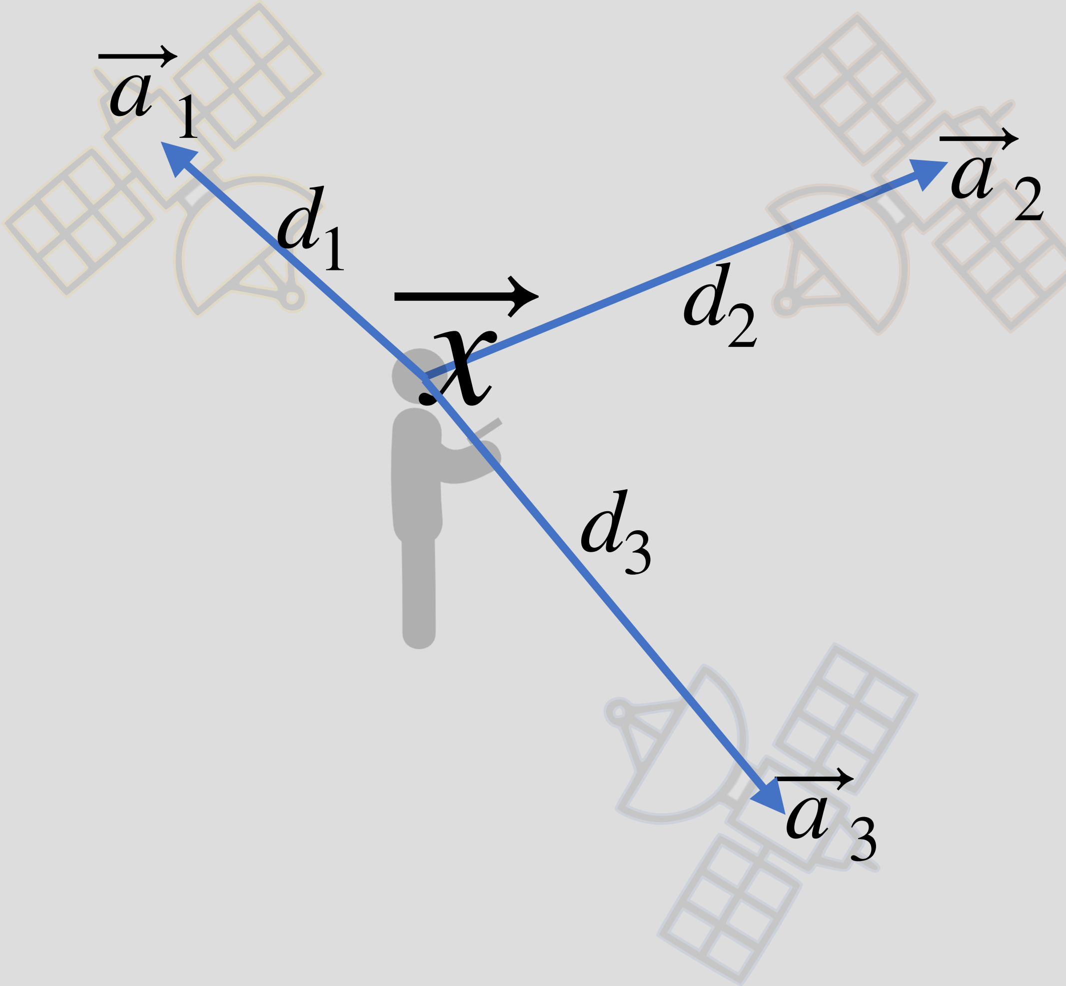
Trilateration

(1) $\|\vec{x}\|^2 - 2\vec{a}_1^T\vec{x} + \|\vec{a}_1\|^2 = C^2\tau_1^2$

(2) $\|\vec{x}\|^2 - 2\vec{a}_2^T\vec{x} + \|\vec{a}_2\|^2 = C^2\tau_2^2$

(3) $\|\vec{x}\|^2 - 2\vec{a}_3^T\vec{x} + \|\vec{a}_3\|^2 = C^2\tau_3^2$

(2) - (1)

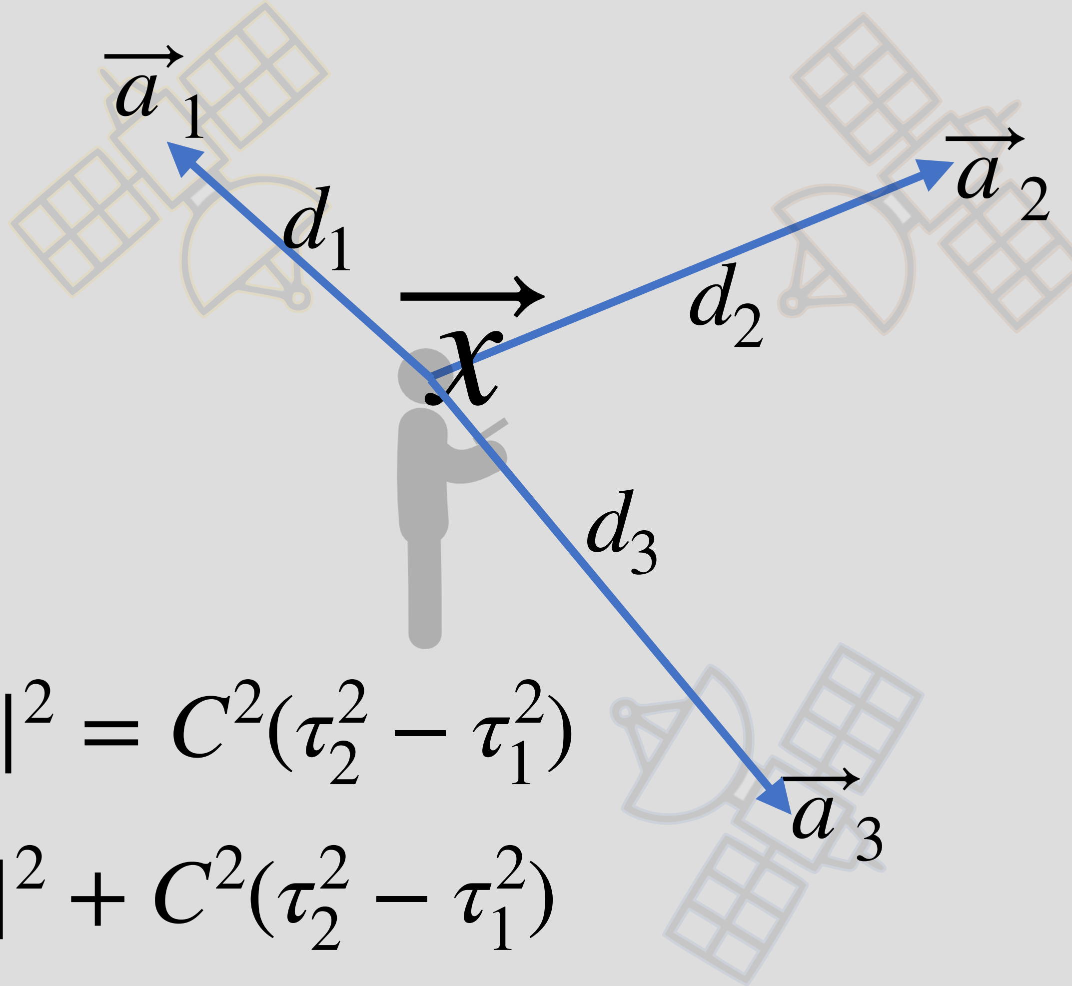


Trilateration

(1) $\|\vec{x}\|^2 - 2\vec{a}_1^T\vec{x} + \|\vec{a}_1\|^2 = C^2\tau_1^2$

(2) $\|\vec{x}\|^2 - 2\vec{a}_2^T\vec{x} + \|\vec{a}_2\|^2 = C^2\tau_2^2$

(3) $\|\vec{x}\|^2 - 2\vec{a}_3^T\vec{x} + \|\vec{a}_3\|^2 = C^2\tau_3^2$



(2) - (1)

$$-2\vec{a}_2^T\vec{x} + 2\vec{a}_1^T\vec{x} + \|\vec{a}_2\|^2 - \|\vec{a}_1\|^2 = C^2(\tau_2^2 - \tau_1^2)$$

$$2(\vec{a}_1 - \vec{a}_2)^T\vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\tau_2^2 - \tau_1^2)$$

(3) - (1)

$$2(\vec{a}_1 - \vec{a}_3)^T\vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + C^2(\tau_3^2 - \tau_1^2)$$

Trilateration

$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\tau_2^2 - \tau_1^2)$$

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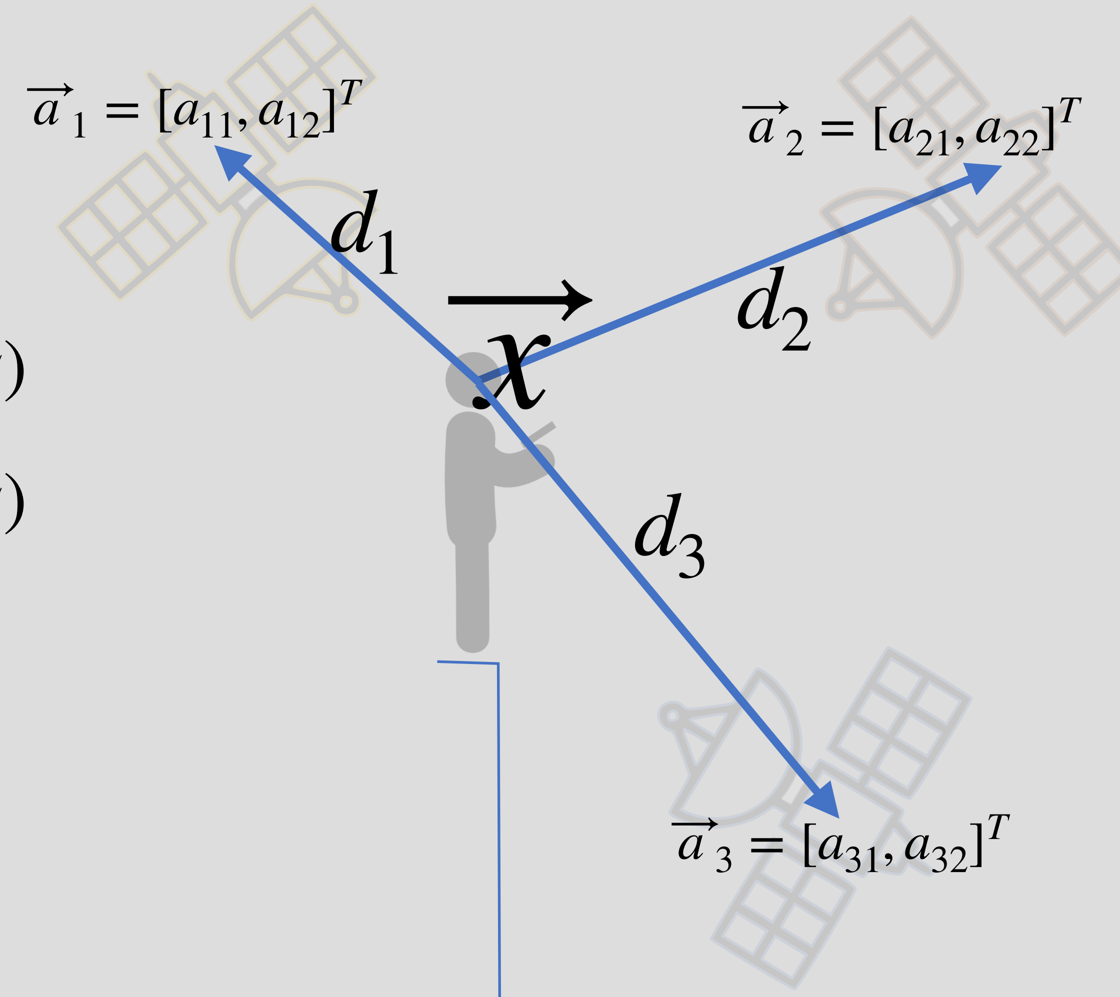

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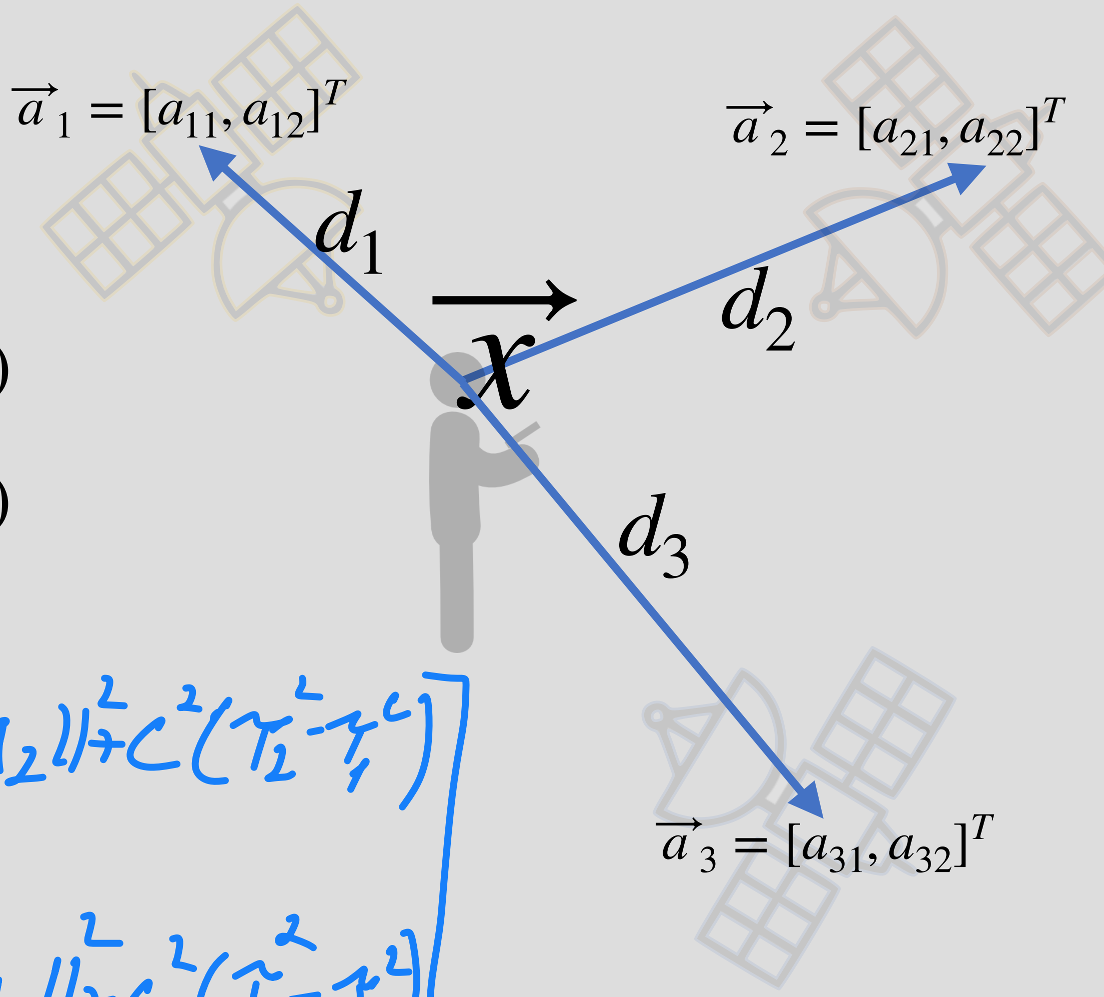
Trilateration



$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\tau_2^2 - \tau_1^2)$$

$$2(\vec{a}_1 - \vec{a}_3)^T \vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + C^2(\tau_3^2 - \tau_1^2)$$

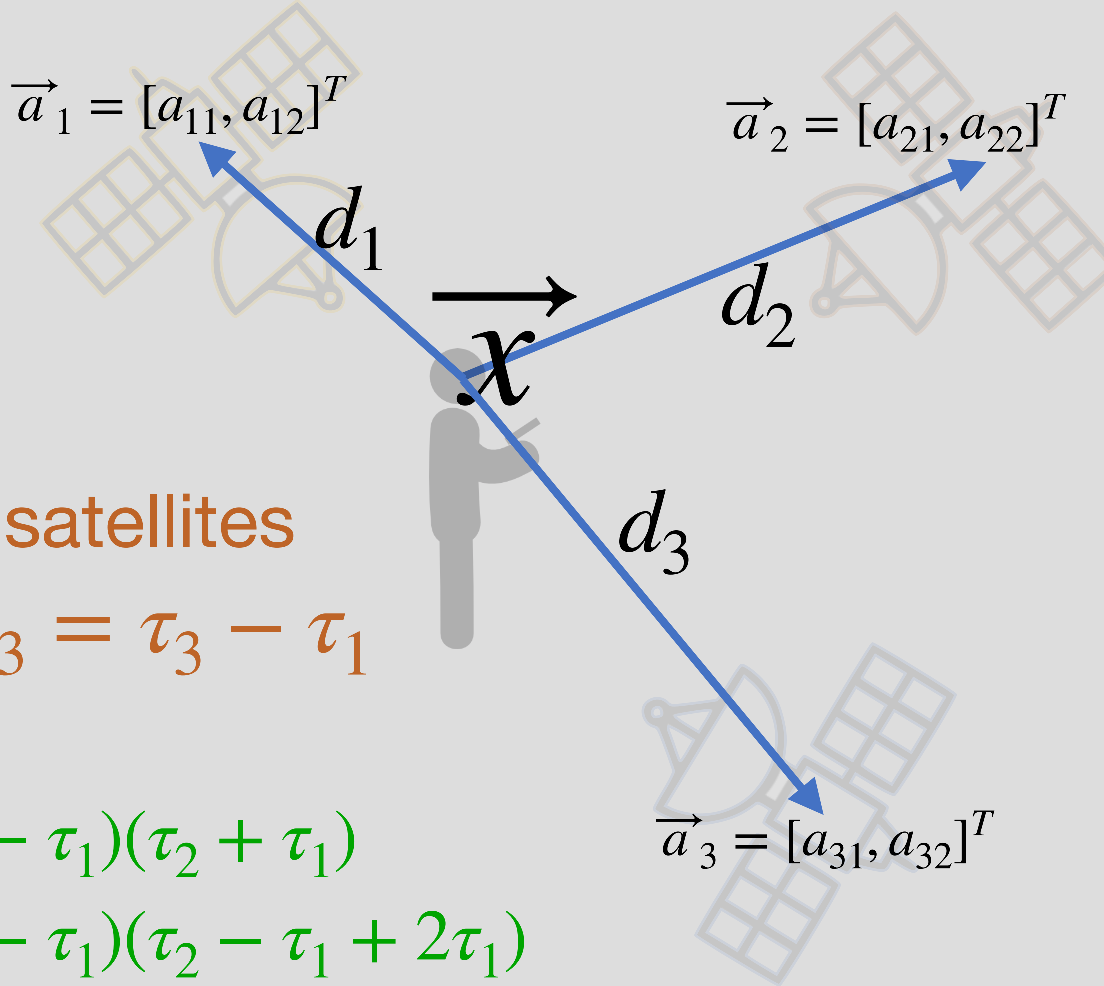
$$2\begin{bmatrix} a_{11}-a_{21} & a_{12}-a_{22} \\ a_{11}-a_{31} & a_{12}-a_{32} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \|a_1\|^2-\|a_2\|^2+c^2(\tau_2^2-\tau_1^2) \\ \|a_1\|^2-\|a_3\|^2+c^2(\tau_3^2-\tau_1^2) \end{bmatrix}$$

Solve via gaussian elimination!

Trilateration

$$2(\vec{a}_1 - \vec{a}_2)^T\vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\tau_2^2 - \tau_1^2)$$

$$2(\vec{a}_1 - \vec{a}_3)^T\vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + C^2(\tau_3^2 - \tau_1^2)$$



Problem — receiver clock is not synced to satellites

τ_1 is unknown, but $\Delta\tau_2 = \tau_2 - \tau_1$, and $\Delta\tau_3 = \tau_3 - \tau_1$ are known

$$2(\vec{a}_1 - \vec{a}_2)^T\vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\tau_2 - \tau_1)(\tau_2 + \tau_1)$$

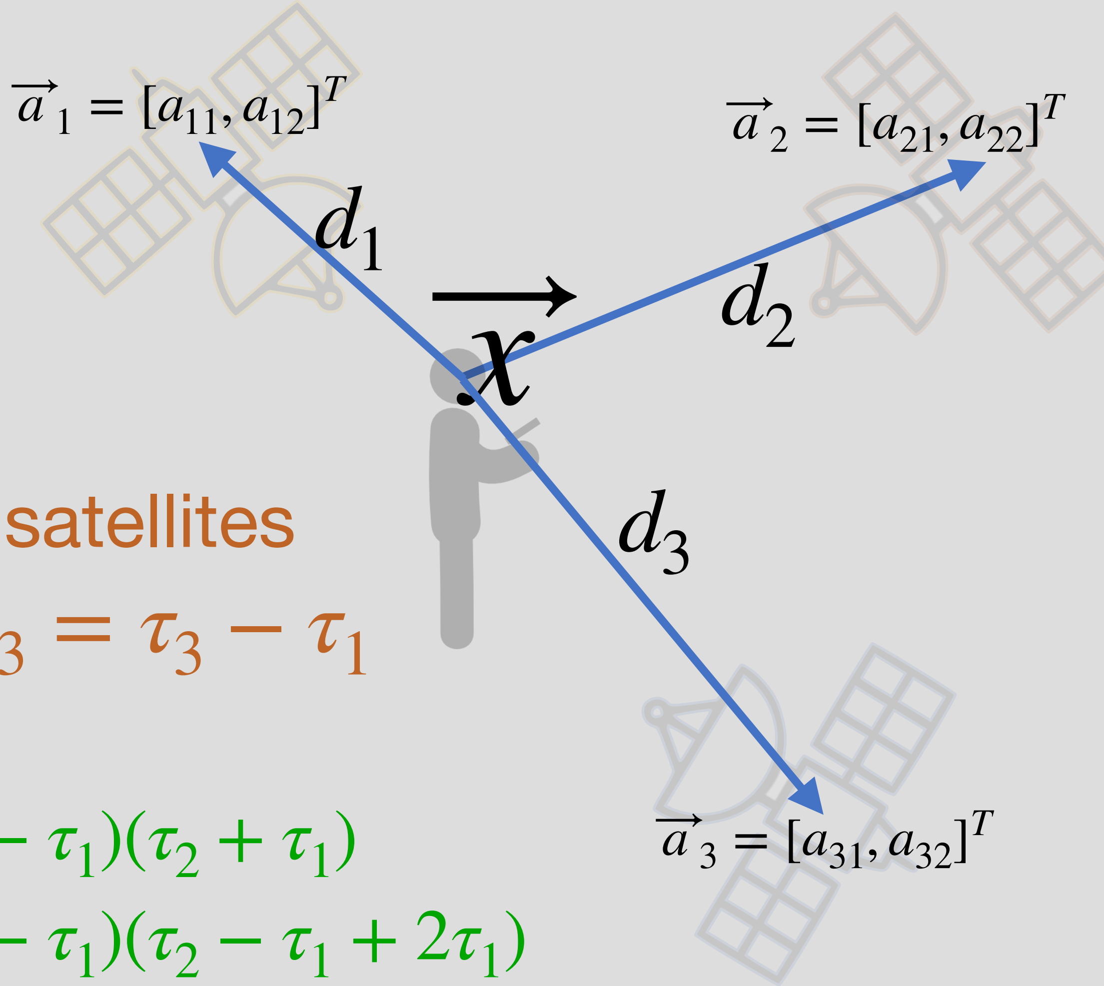
$$2(\vec{a}_1 - \vec{a}_2)^T\vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\tau_2 - \tau_1)(\tau_2 - \tau_1 + 2\tau_1)$$

$$2(\vec{a}_1 - \vec{a}_2)^T\vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\Delta\tau_2)(\Delta\tau_2 + 2\tau_1)$$

Trilateration

$$2(\vec{a}_1 - \vec{a}_2)^T\vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\tau_2^2 - \tau_1^2)$$

$$2(\vec{a}_1 - \vec{a}_3)^T\vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + C^2(\tau_3^2 - \tau_1^2)$$



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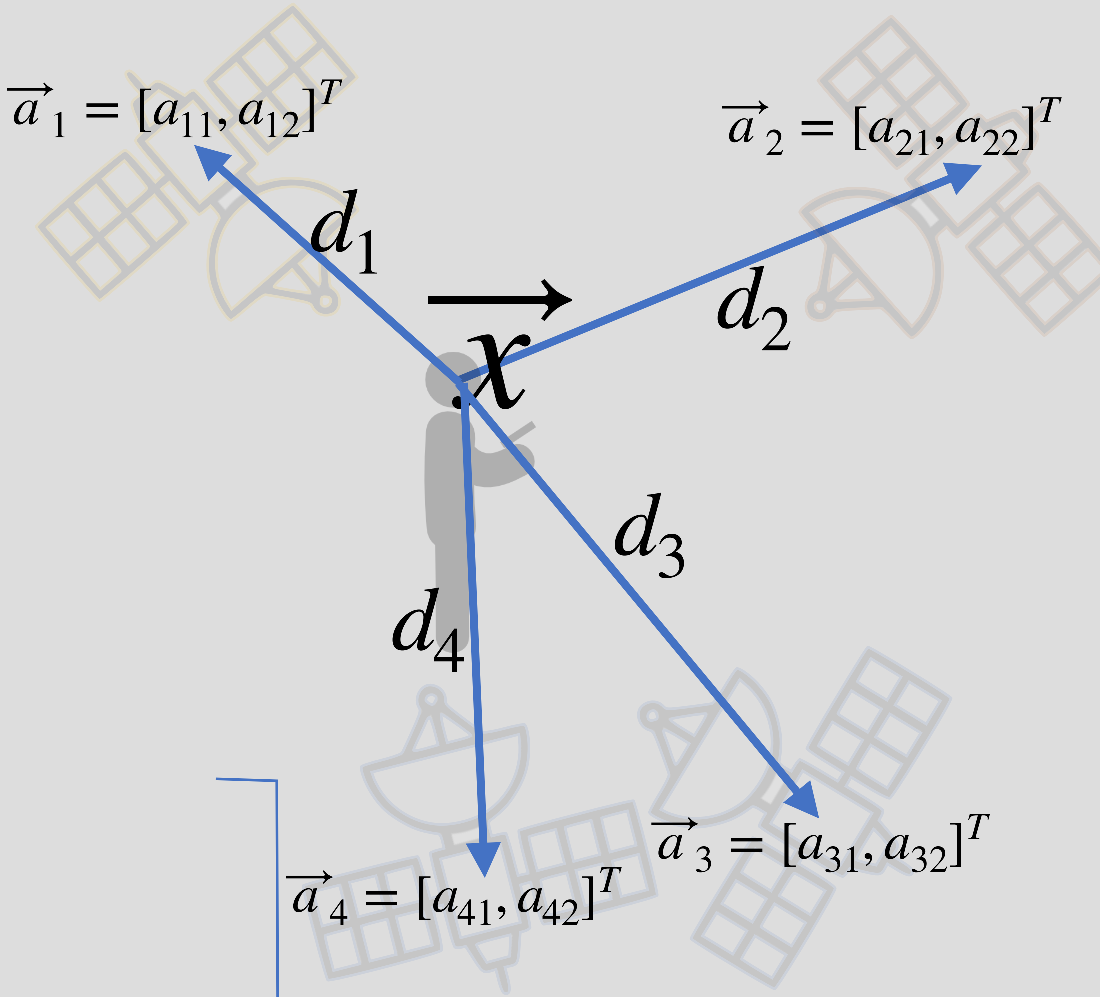
$$2(\vec{a}_1 - \vec{a}_2)^T\vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\Delta\tau_2)(\Delta\tau_2 + 2\tau_1)$$

$$2(\vec{a}_1 - \vec{a}_2)^T\vec{x} - 2C^2\Delta\tau_2\tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\Delta\tau_2)^2$$

Another variable! Need 1 more equation (satellite)

Trilateration

$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} - 2C^2 \Delta\tau_2 \tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\Delta\tau_2)^2$$
$$2(\vec{a}_1 - \vec{a}_3)^T \vec{x} - 2C^2 \Delta\tau_3 \tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + C^2(\Delta\tau_3)^2$$
$$2(\vec{a}_1 - \vec{a}_4)^T \vec{x} - 2C^2 \Delta\tau_4 \tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_4\|^2 + C^2(\Delta\tau_4)^2$$



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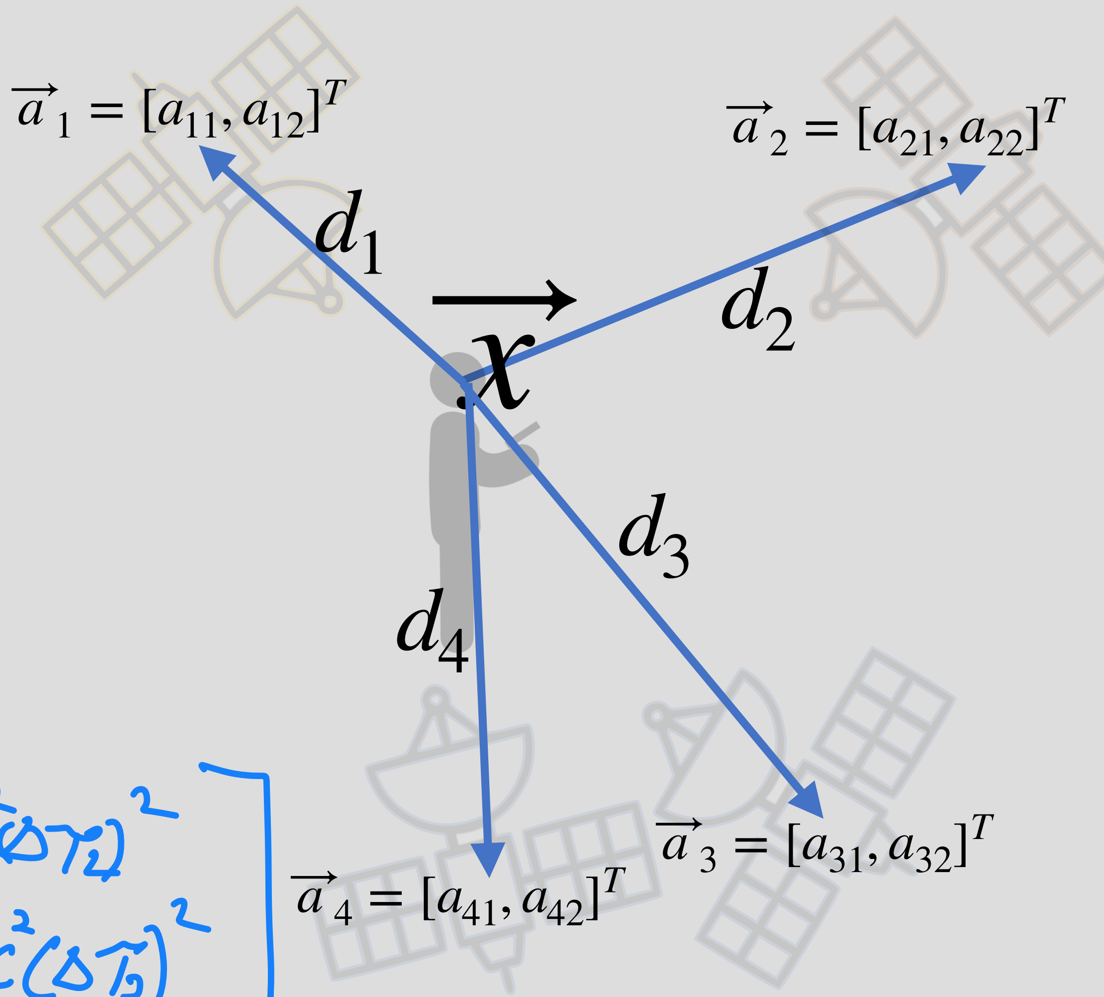
Trilateration



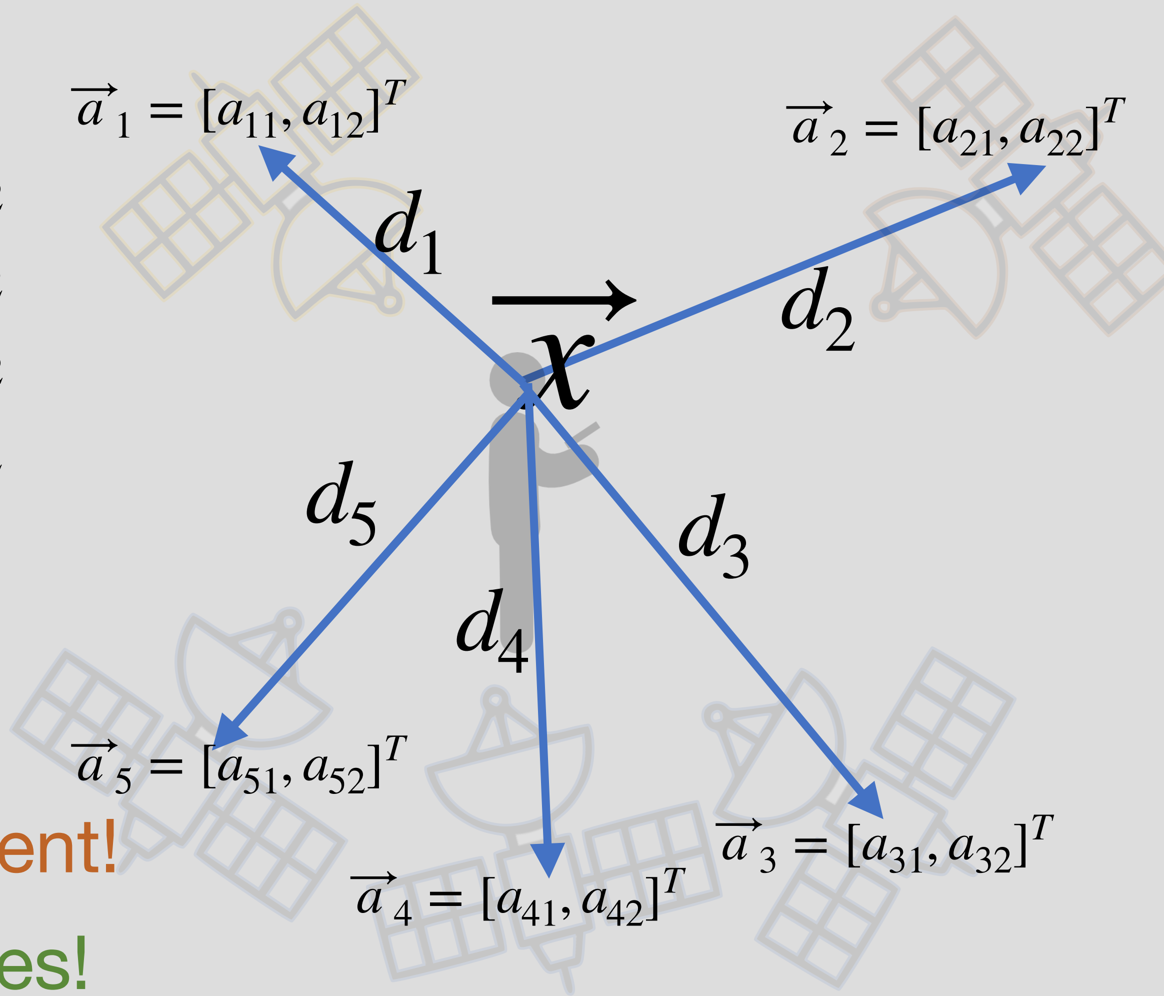
$$2(\vec{a}_1 - \vec{a}_2)^T\vec{x} - 2C^2\Delta\tau_2\tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\Delta\tau_2)^2$$
$$2(\vec{a}_1 - \vec{a}_3)^T\vec{x} - 2C^2\Delta\tau_3\tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + C^2(\Delta\tau_3)^2$$
$$2(\vec{a}_1 - \vec{a}_4)^T\vec{x} - 2C^2\Delta\tau_4\tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_4\|^2 + C^2(\Delta\tau_4)^2$$

$$2\begin{bmatrix} a_{11}-a_{21} & a_{12}-a_{22} & -C^2\Delta\tau_2 \\ a_{11}-a_{31} & a_{12}-a_{32} & -C^2\Delta\tau_3 \\ a_{11}-a_{41} & a_{12}-a_{42} & -C^2\Delta\tau_4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \tau_1 \end{bmatrix} = \begin{bmatrix} \|a_1\|^2 - \|a_2\|^2 + C^2(\Delta\tau_2)^2 \\ \|a_1\|^2 - \|a_3\|^2 + C^2(\Delta\tau_3)^2 \\ \|a_1\|^2 - \|a_4\|^2 + C^2(\Delta\tau_4)^2 \end{bmatrix}$$

Multi-Lateration

$$2(\vec{a}_1 - \vec{a}_2)^T\vec{x} - 2C^2\Delta\tau_2\tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\Delta\tau_2)^2$$
$$2(\vec{a}_1 - \vec{a}_3)^T\vec{x} - 2C^2\Delta\tau_3\tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + C^2(\Delta\tau_3)^2$$
$$2(\vec{a}_1 - \vec{a}_4)^T\vec{x} - 2C^2\Delta\tau_4\tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_4\|^2 + C^2(\Delta\tau_4)^2$$
$$2(\vec{a}_1 - \vec{a}_5)^T\vec{x} - 2C^2\Delta\tau_5\tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_5\|^2 + C^2(\Delta\tau_5)^2$$

More equations than unknowns



Q: With noise, equations will be inconsistent!

A: Find closest solution with Least-Squares!