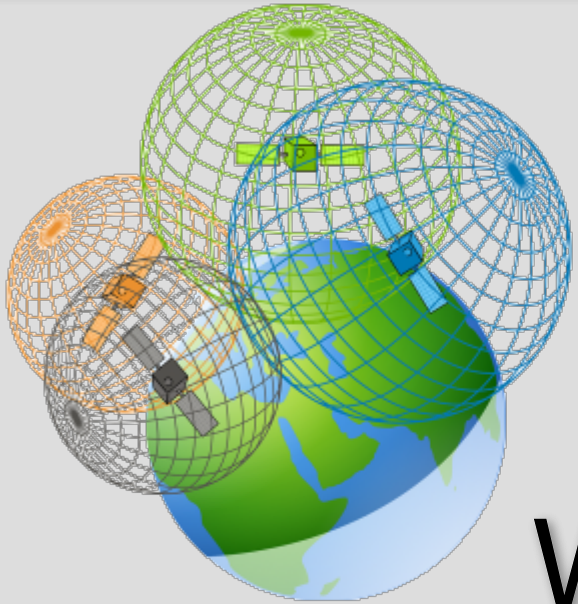
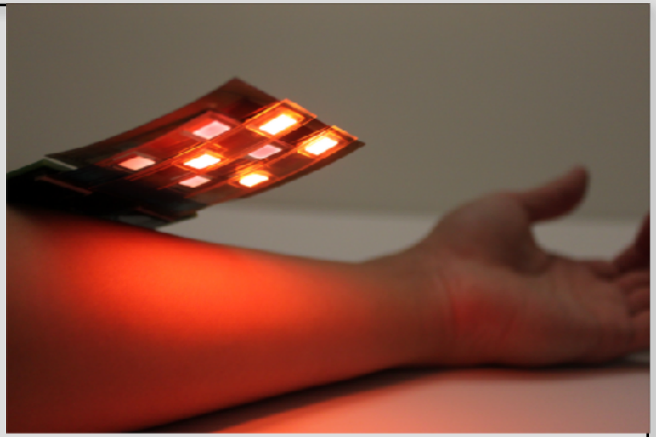




Welcome to EECS 16A!

Designing Information Devices and Systems I

Ana Arias and Miki Lustig

2022

Lecture 12A

Tri-Lateration, Projections

Good morning!

Last time:

- Computing delay with cross-correlation

Today:

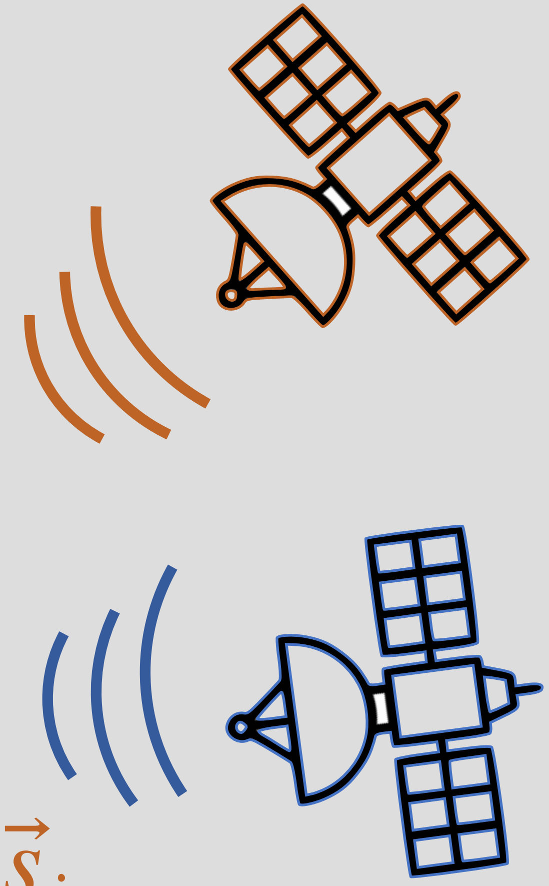
- Finding position with multi-lateration
- Projections
- Least Squares (Maybe)

Localization

- Satellites transmit a unique code
    - Radio signal
- Signal is received and digitized by a receiver

Two problems:

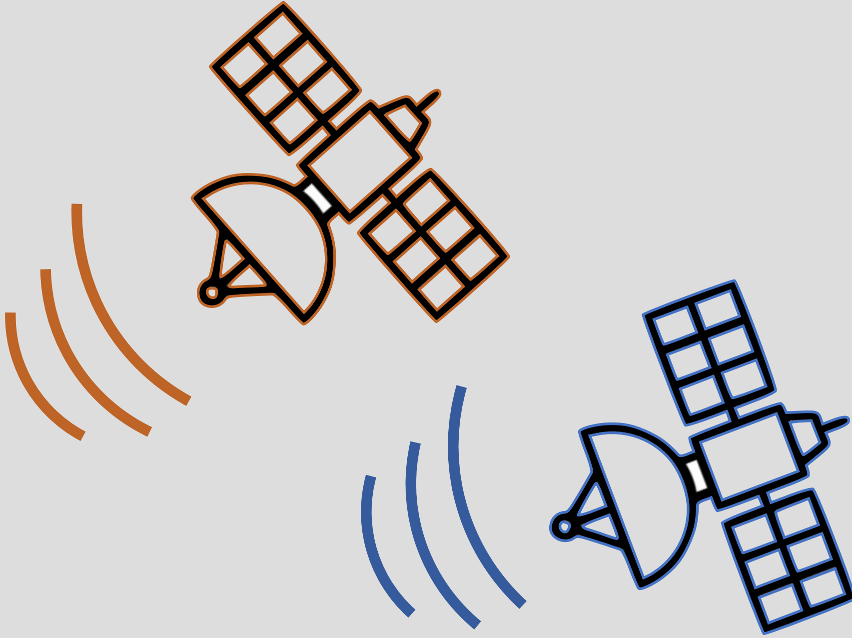
1. Interference
2. Timing



What are good properties for the codes $\vec{s}_i$

Received Signal

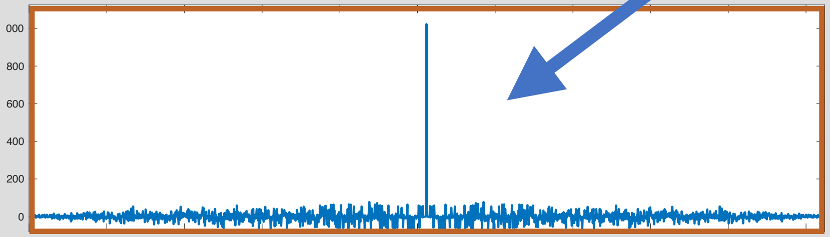
$$r[n] = s_1[n-\tau_1] + s_2[n-\tau_2] + w[n]$$



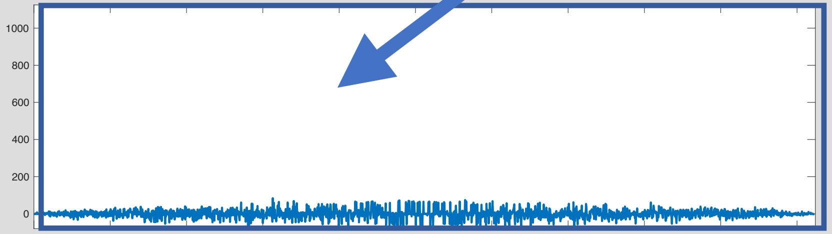
Correlate with $s_1[n]$:

$$\text{corr}_{\vec{r}}(\vec{s}_1)[k] = \langle r[n], s_1[n-k] \rangle$$

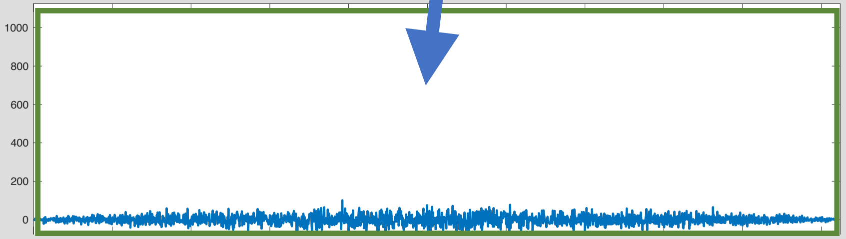
$$= \langle s_1[n-\tau_1], s_1[n-k] \rangle + \langle s_2[n-\tau_2], s_1[n-k] \rangle + \langle w[n], s_1[n-k] \rangle$$



Auto-correlation looks like an impulse



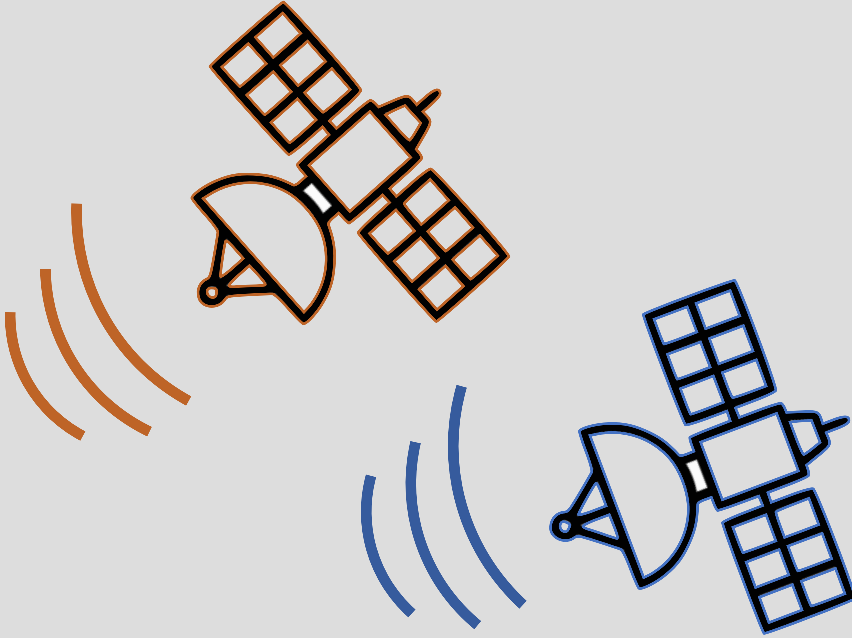
cross-correlation is small

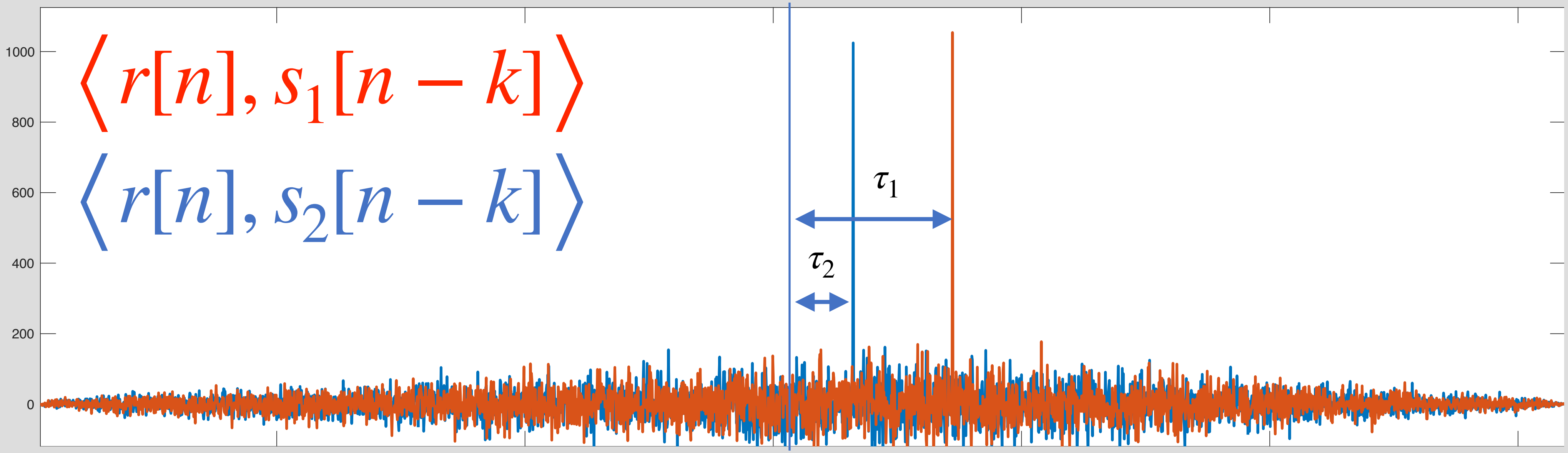


cross-correlation with noise is small
(always true)

Received Signal

$$r[n] = s_1[n-\tau_1] + s_2[n-\tau_2] + w[n]$$





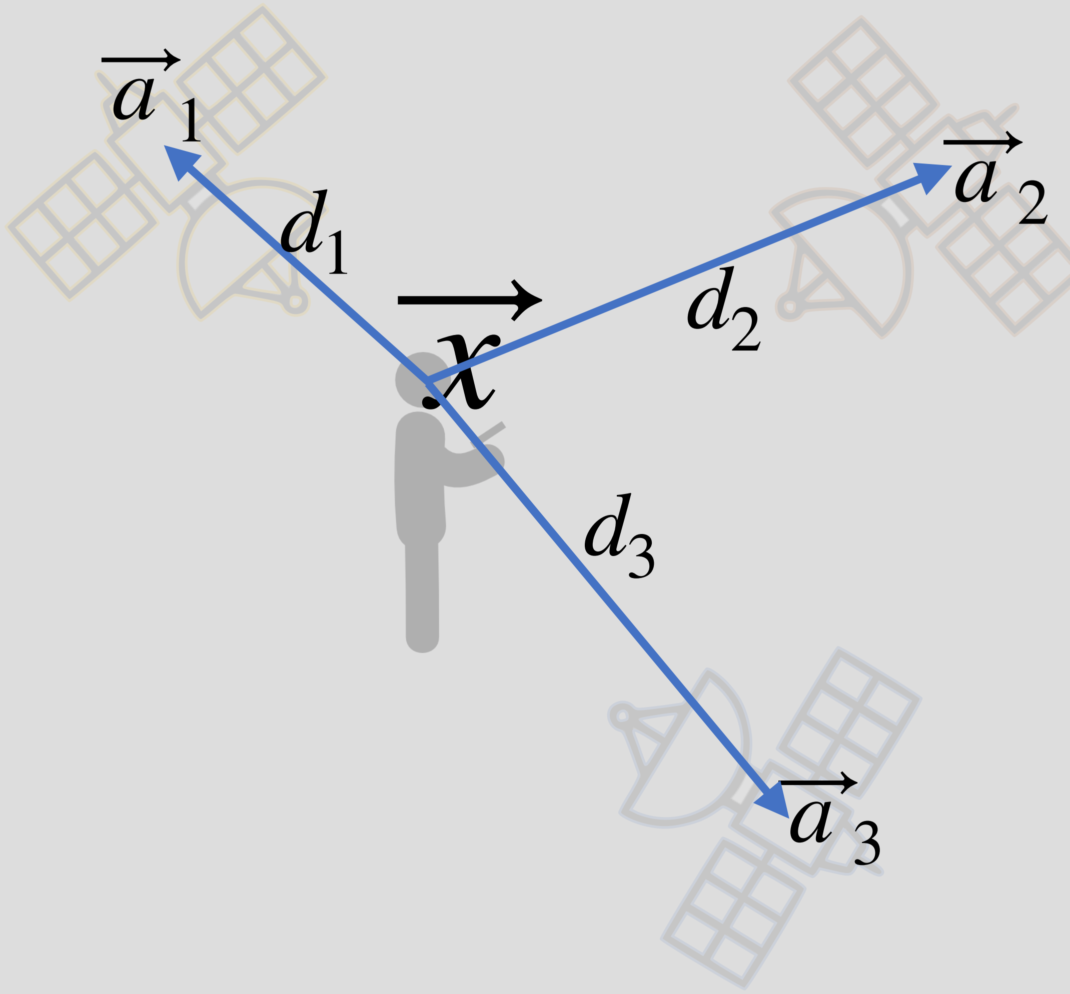
0

Trilateration

(1) $$\|\vec{x} - \vec{a}_1\|^2 = d_1^2$$

(2) $$\|\vec{x} - \vec{a}_2\|^2 = d_2^2$$

(3) $$\|\vec{x} - \vec{a}_3\|^2 = d_3^2$$



$$d_1 = \tau_1 C$$
$$d_2 = \tau_2 C$$
$$d_3 = \tau_3 C$$

Trilateration

(1) $$\|\vec{x} - \vec{a}_1\|^2 = d_1^2$$

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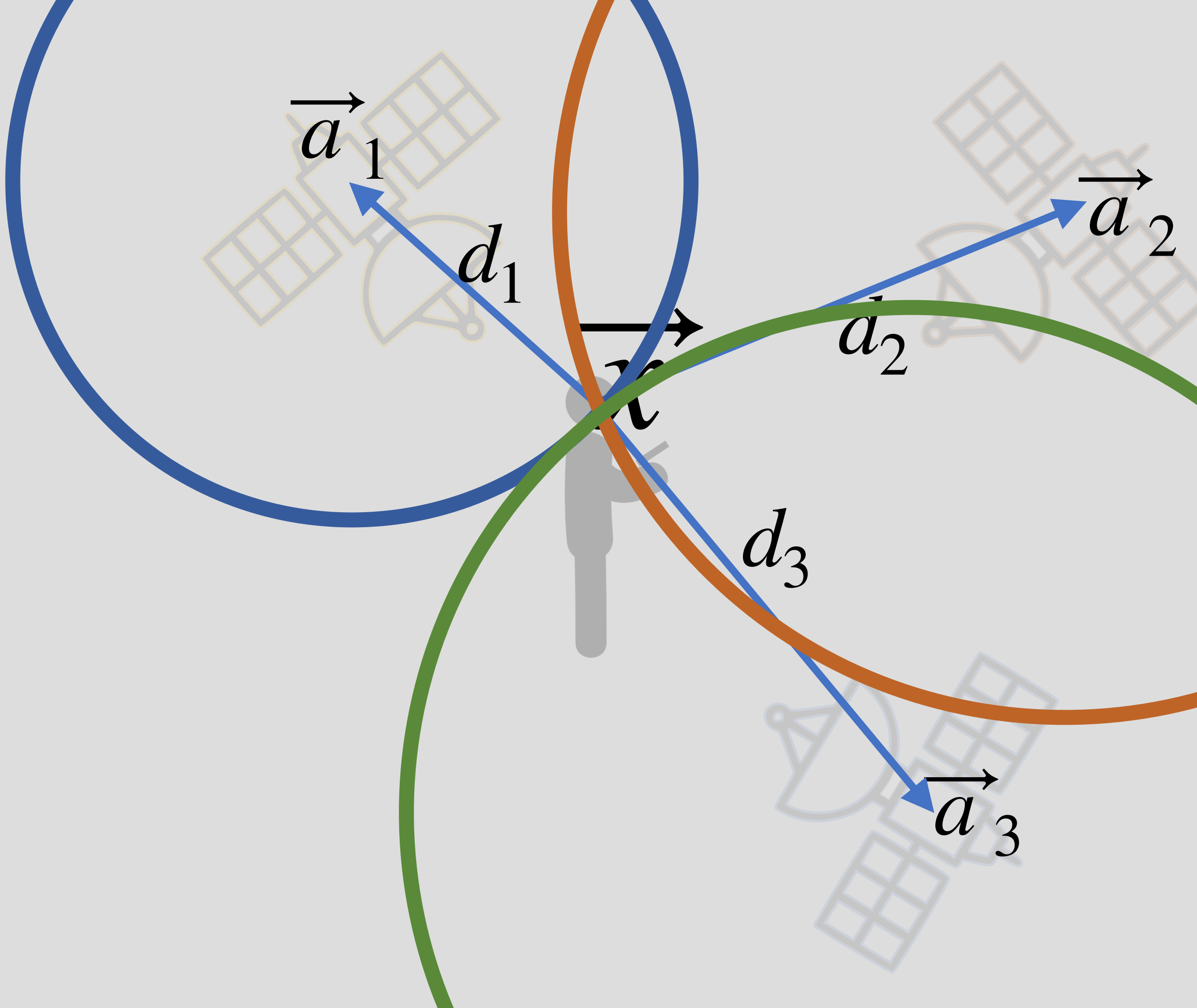

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Trilateration

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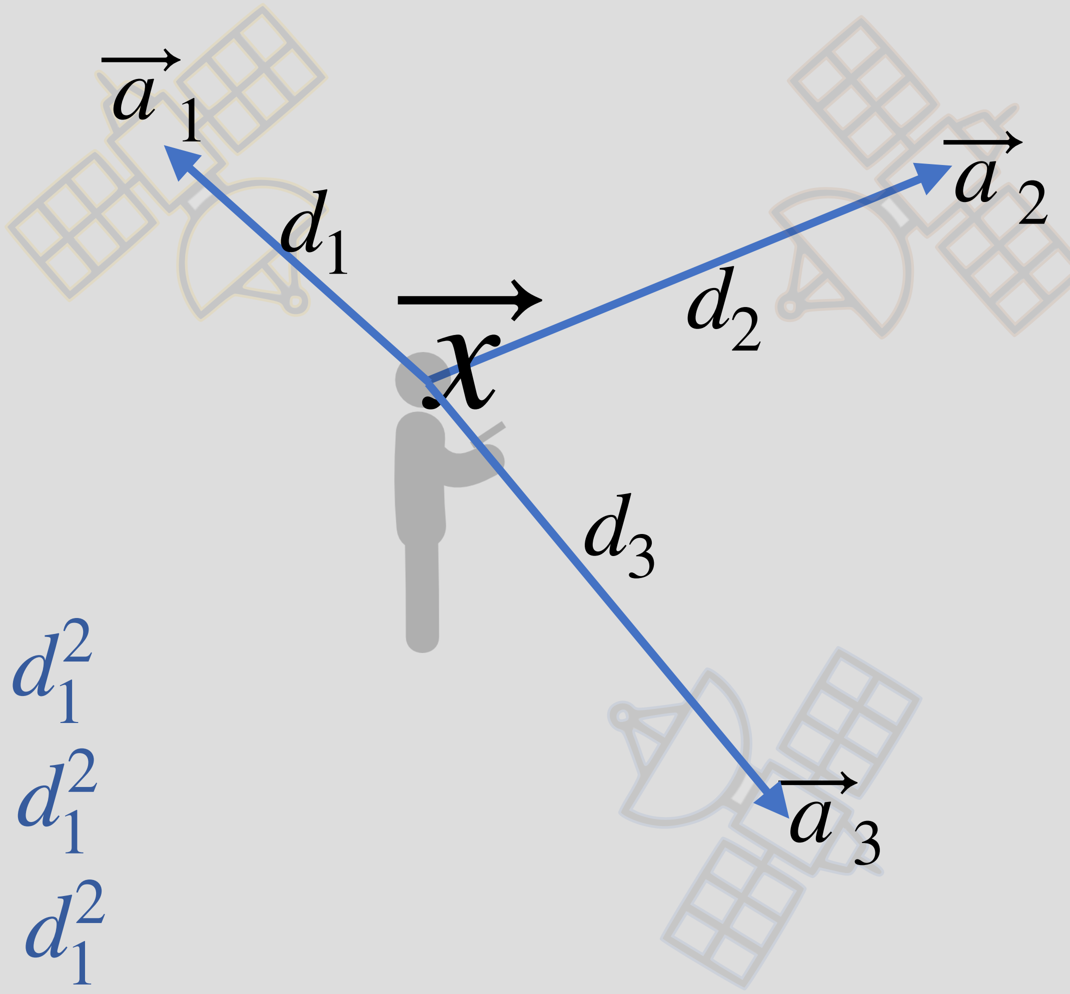

$$\|\vec{x} - \vec{a}_1\|^2 = d_1^2$$

$$(\vec{x} - \vec{a}_1)^T(\vec{x} - \vec{a}_1) = d_1^2$$

$$\vec{x}^T\vec{x} - \vec{a}_1^T\vec{x} - \vec{x}^T\vec{a}_1 + \vec{a}_1^T\vec{a}_1 = d_1^2$$

$$\|\vec{x}\|^2 - 2\vec{a}_1^T\vec{x} + \|\vec{a}_1\|^2 = d_1^2$$

$$\|\vec{x}\|^2 - 2\vec{a}_1^T\vec{x} + \|\vec{a}_1\|^2 = C^2\tau_1^2$$

$$d_1 = \tau_1 C$$

$$d_2 = \tau_2 C$$

$$d_3 = \tau_3 C$$

Trilateration

(1) $\|\vec{x} - \vec{a}_1\|^2 = d_1^2$

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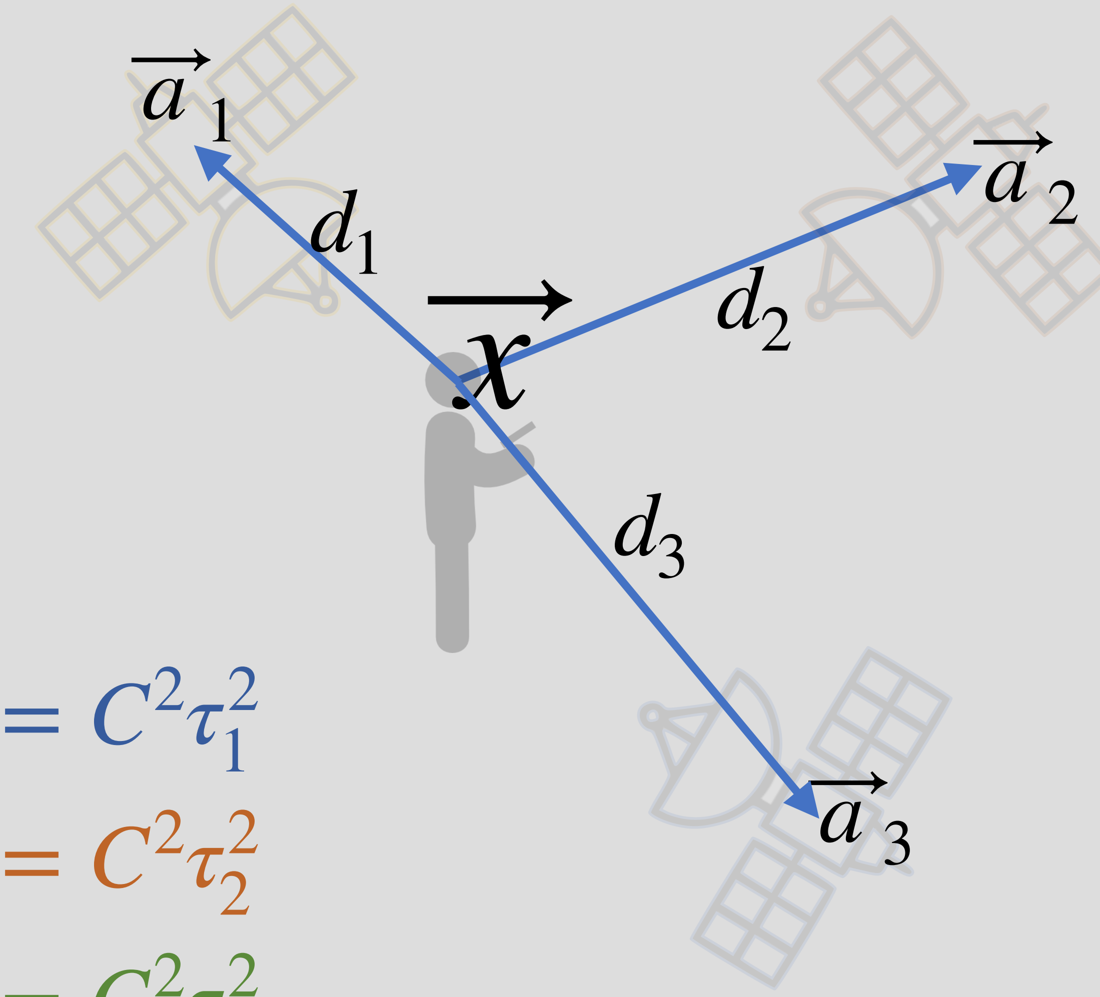

(1) $\|\vec{x}\|^2 - 2\vec{a}_1^T\vec{x} + \|\vec{a}_1\|^2 = C^2\tau_1^2$

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(3) $\|\vec{x}\|^2 - 2\vec{a}_3^T\vec{x} + \|\vec{a}_3\|^2 = C^2\tau_3^2$

$d_1 = \tau_1 C$

$d_2 = \tau_2 C$

$d_3 = \tau_3 C$

Trilateration

(1) $\|\vec{x}\|^2 - 2\vec{a}_1^T\vec{x} + \|\vec{a}_1\|^2 = C^2\tau_1^2$

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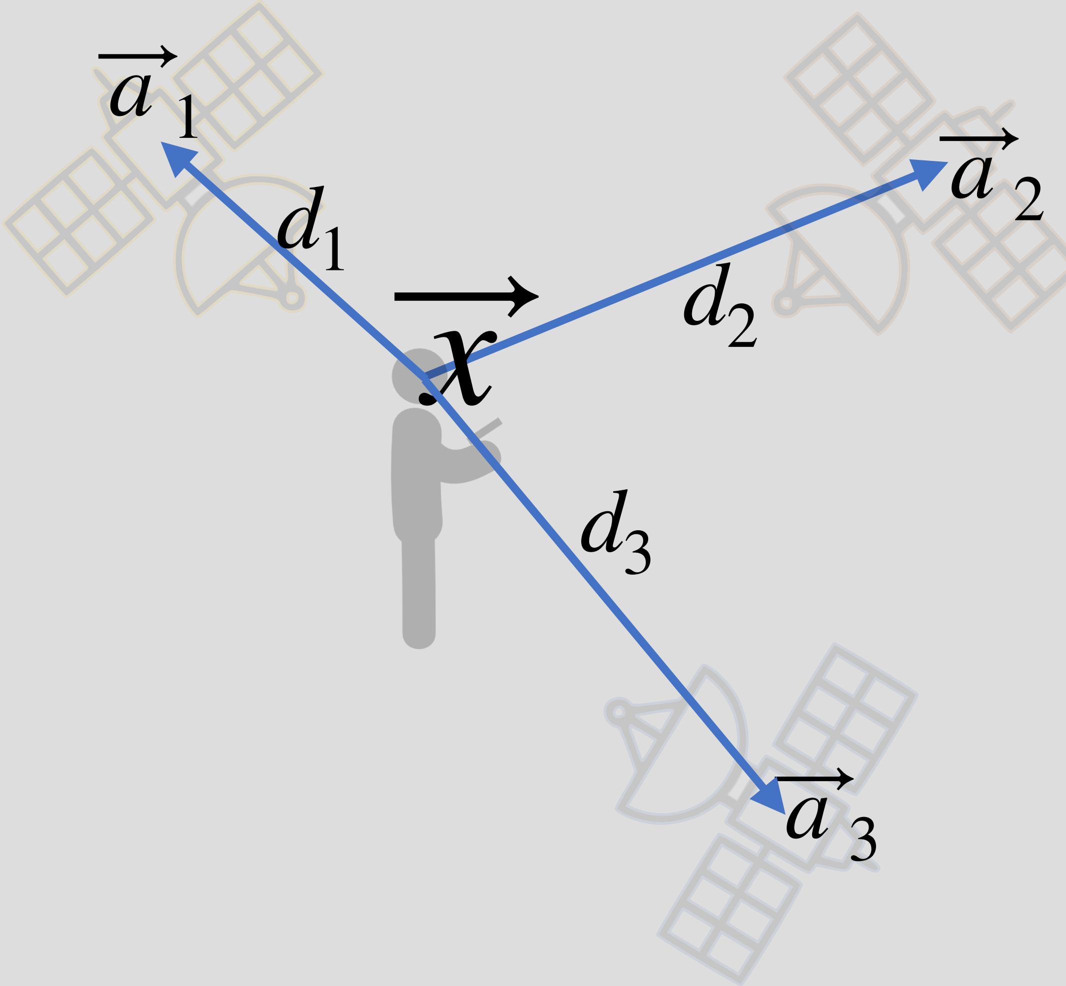
(3) $\|\vec{x}\|^2 - 2\vec{a}_3^T\vec{x} + \|\vec{a}_3\|^2 = C^2\tau_3^2$

(2) - (1)

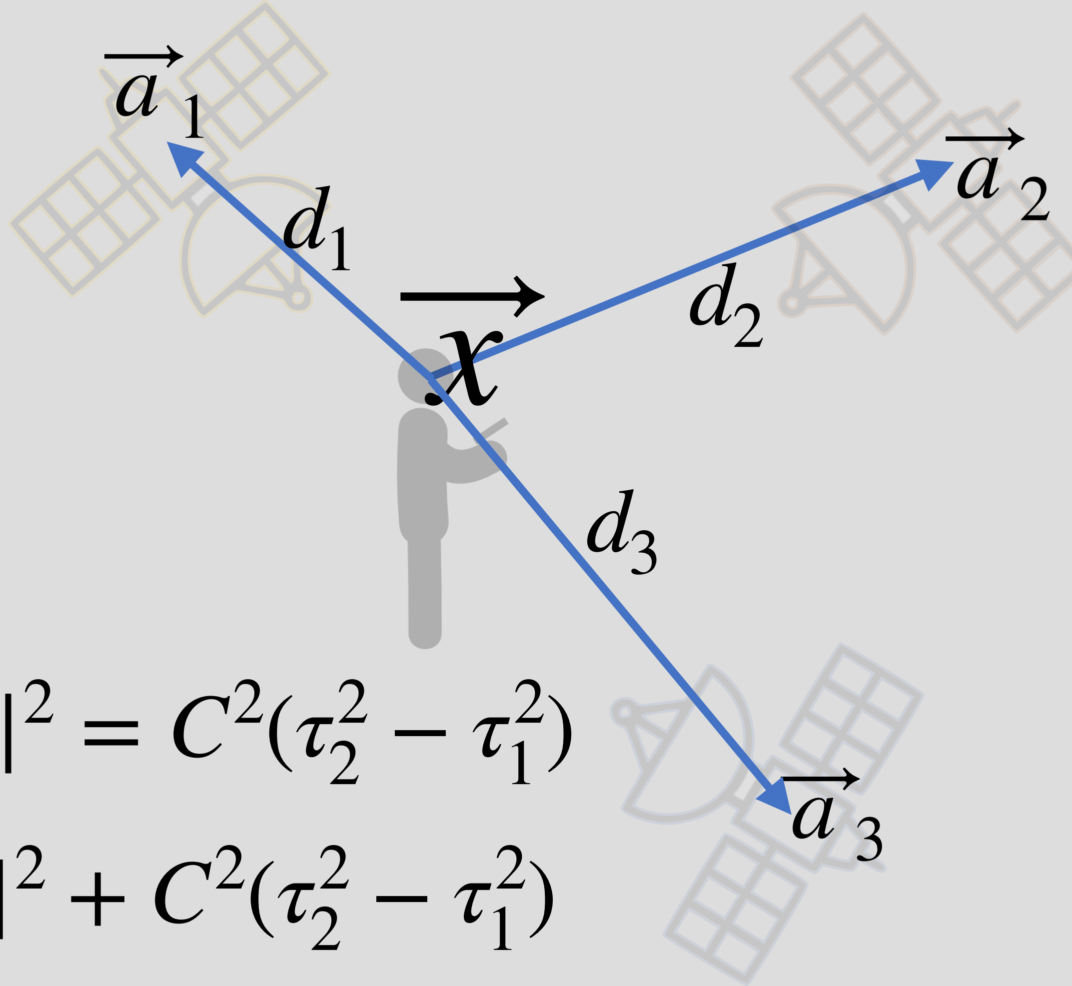


Trilateration

(1) $\|\vec{x}\|^2 - 2\vec{a}_1^T\vec{x} + \|\vec{a}_1\|^2 = C^2\tau_1^2$

(2) $\|\vec{x}\|^2 - 2\vec{a}_2^T\vec{x} + \|\vec{a}_2\|^2 = C^2\tau_2^2$

(3) $\|\vec{x}\|^2 - 2\vec{a}_3^T\vec{x} + \|\vec{a}_3\|^2 = C^2\tau_3^2$



(2) - (1)

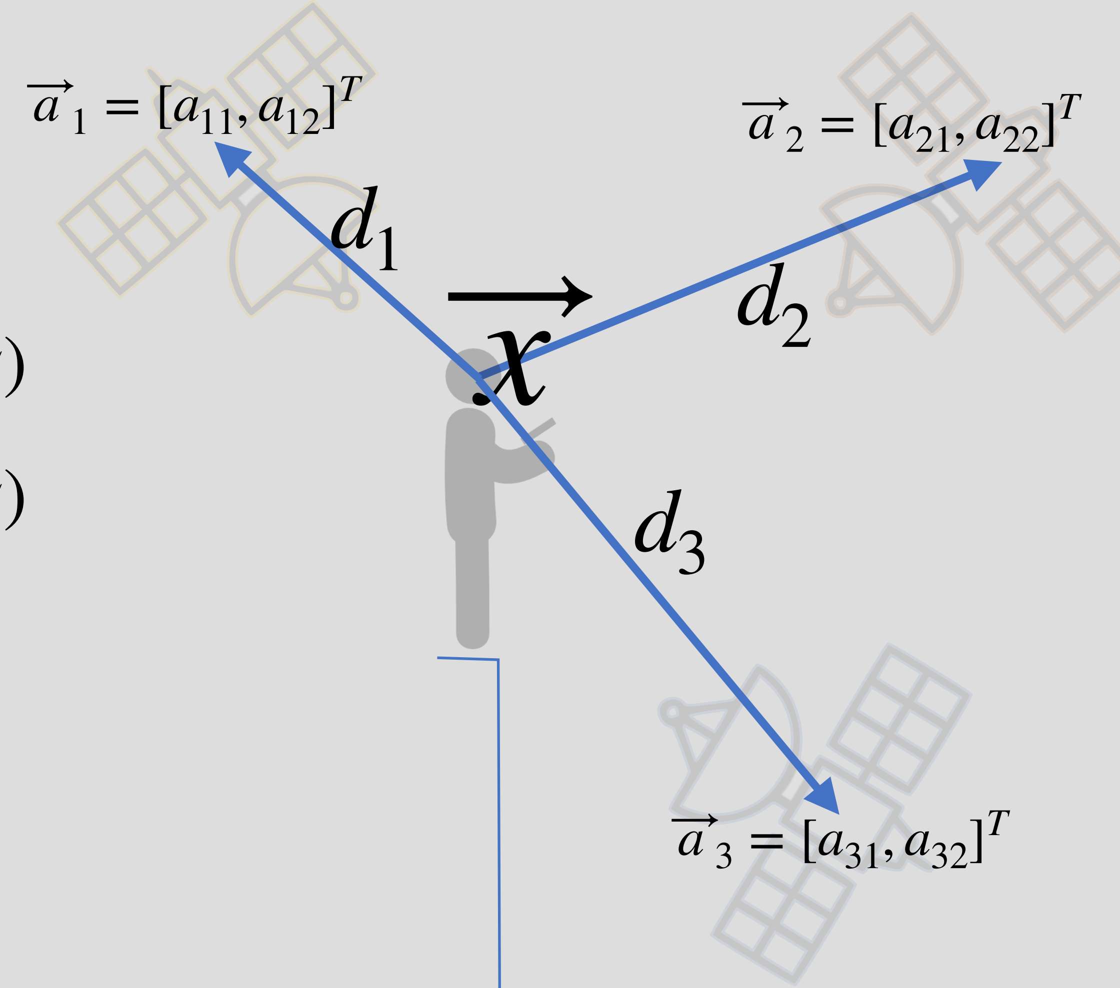
$$-2\vec{a}_2^T\vec{x} + 2\vec{a}_1^T\vec{x} + \|\vec{a}_2\|^2 - \|\vec{a}_1\|^2 = C^2(\tau_2^2 - \tau_1^2)$$

$$2(\vec{a}_1 - \vec{a}_2)^T\vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\tau_2^2 - \tau_1^2)$$

(3) - (1)

$$2(\vec{a}_1 - \vec{a}_3)^T\vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + C^2(\tau_3^2 - \tau_1^2)$$

Trilateration

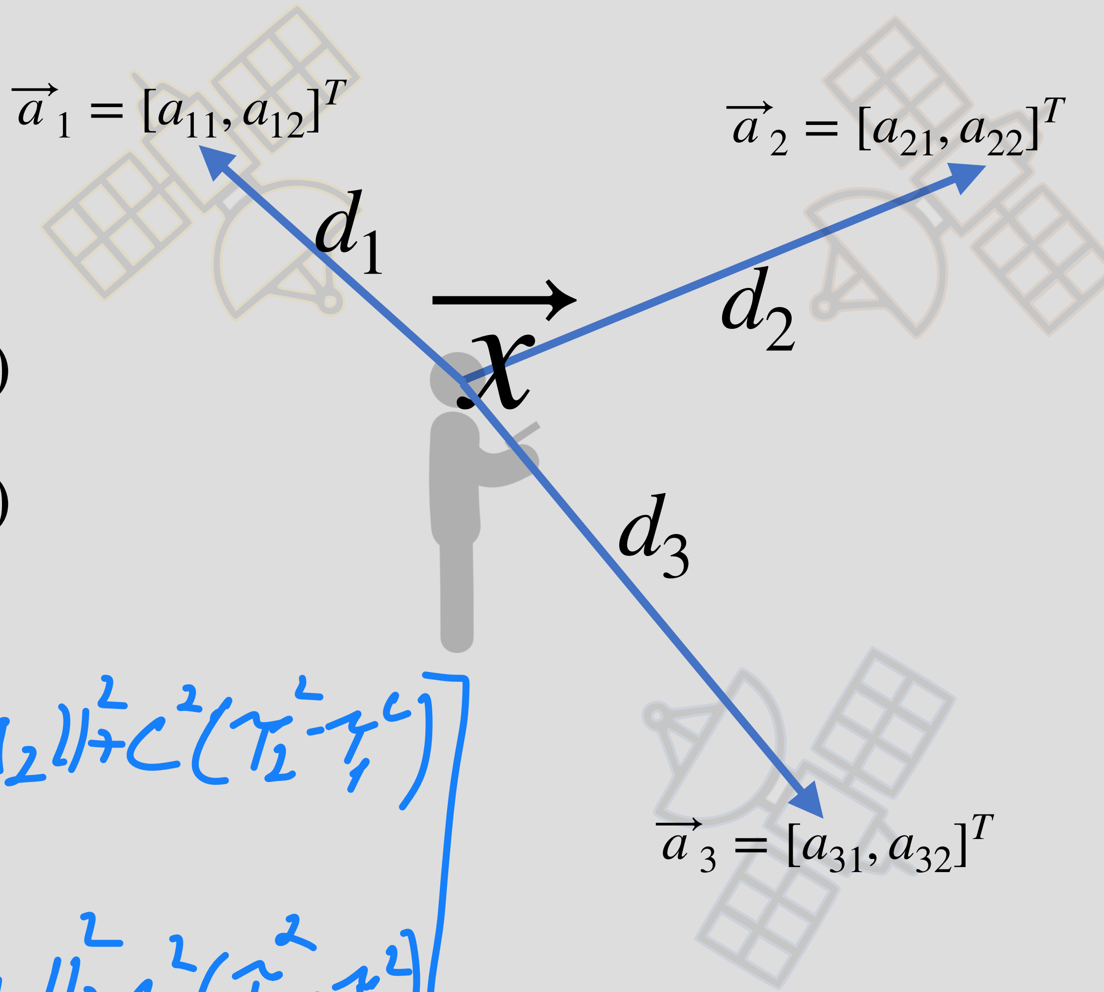


$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\tau_2^2 - \tau_1^2)$$

$$2(\vec{a}_1 - \vec{a}_3)^T \vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + C^2(\tau_3^2 - \tau_1^2)$$

$$=$$

Trilateration



$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\tau_2^2 - \tau_1^2)$$

$$2(\vec{a}_1 - \vec{a}_3)^T \vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + C^2(\tau_3^2 - \tau_1^2)$$

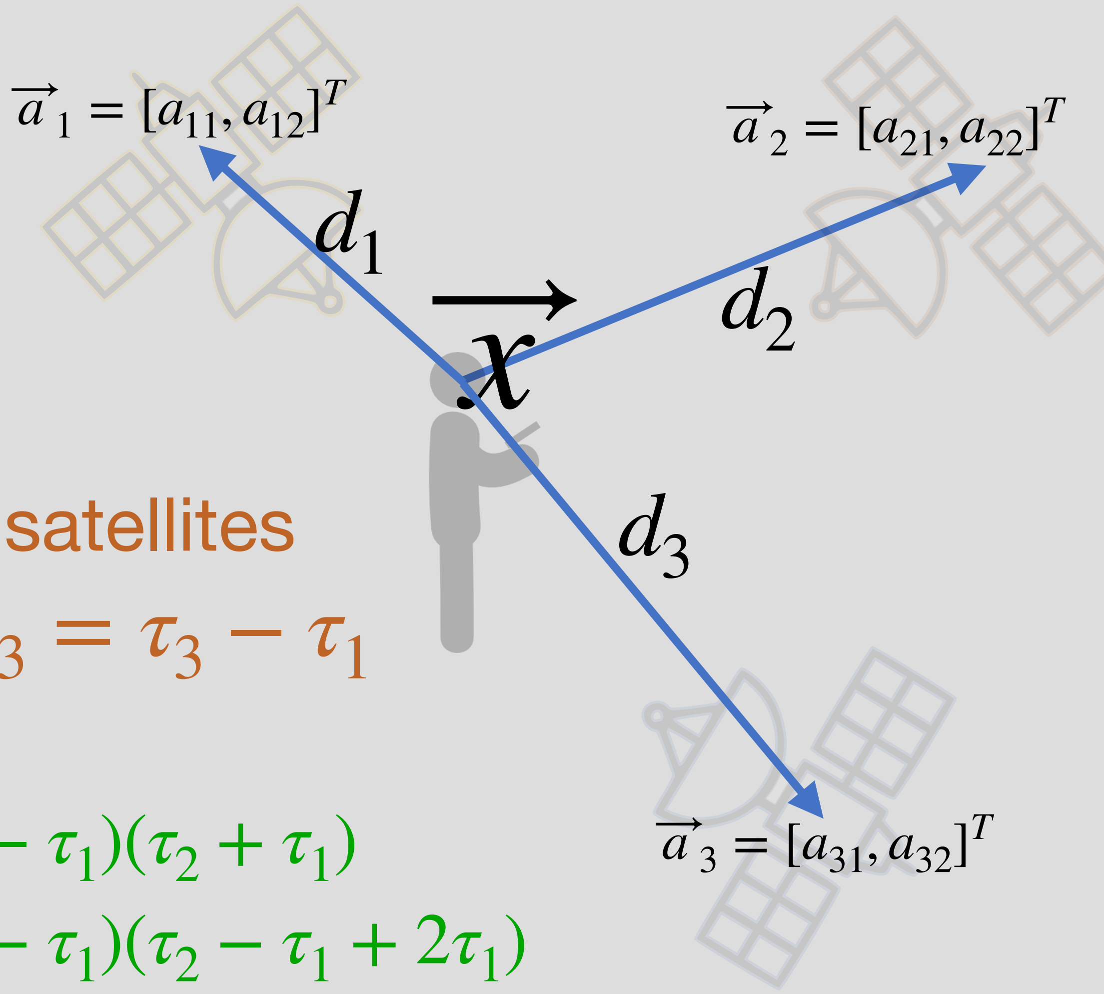
$$2\begin{bmatrix} a_{11} - a_{21} & a_{12} - a_{22} \\ a_{11} - a_{31} & a_{12} - a_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \|a_1\|^2 - \|a_2\|^2 + c^2(\tau_2^2 - \tau_1^2) \\ \|a_1\|^2 - \|a_3\|^2 + c^2(\tau_3^2 - \tau_1^2) \end{bmatrix}$$

Solve via gaussian elimination!

Trilateration

$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\tau_2^2 - \tau_1^2)$$

$$2(\vec{a}_1 - \vec{a}_3)^T \vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + C^2(\tau_3^2 - \tau_1^2)$$



Problem — receiver clock is not synced to satellites

τ_1 is unknown, but $\Delta\tau_2 = \tau_2 - \tau_1$, and $\Delta\tau_3 = \tau_3 - \tau_1$ are known

$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\tau_2 - \tau_1)(\tau_2 + \tau_1)$$

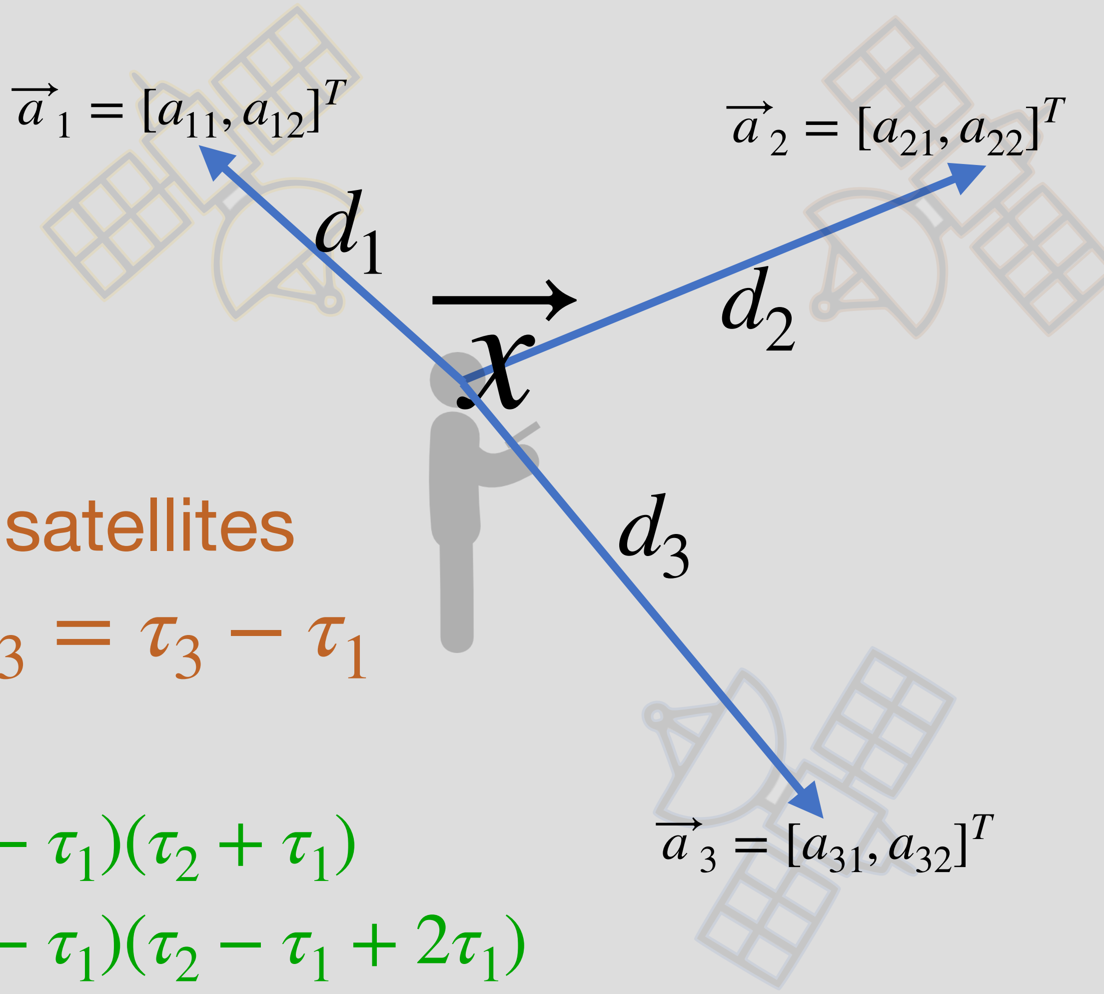
$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\tau_2 - \tau_1)(\tau_2 - \tau_1 + 2\tau_1)$$

$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\Delta\tau_2)(\Delta\tau_2 + 2\tau_1)$$

Trilateration

$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\tau_2^2 - \tau_1^2)$$

$$2(\vec{a}_1 - \vec{a}_3)^T \vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + C^2(\tau_3^2 - \tau_1^2)$$



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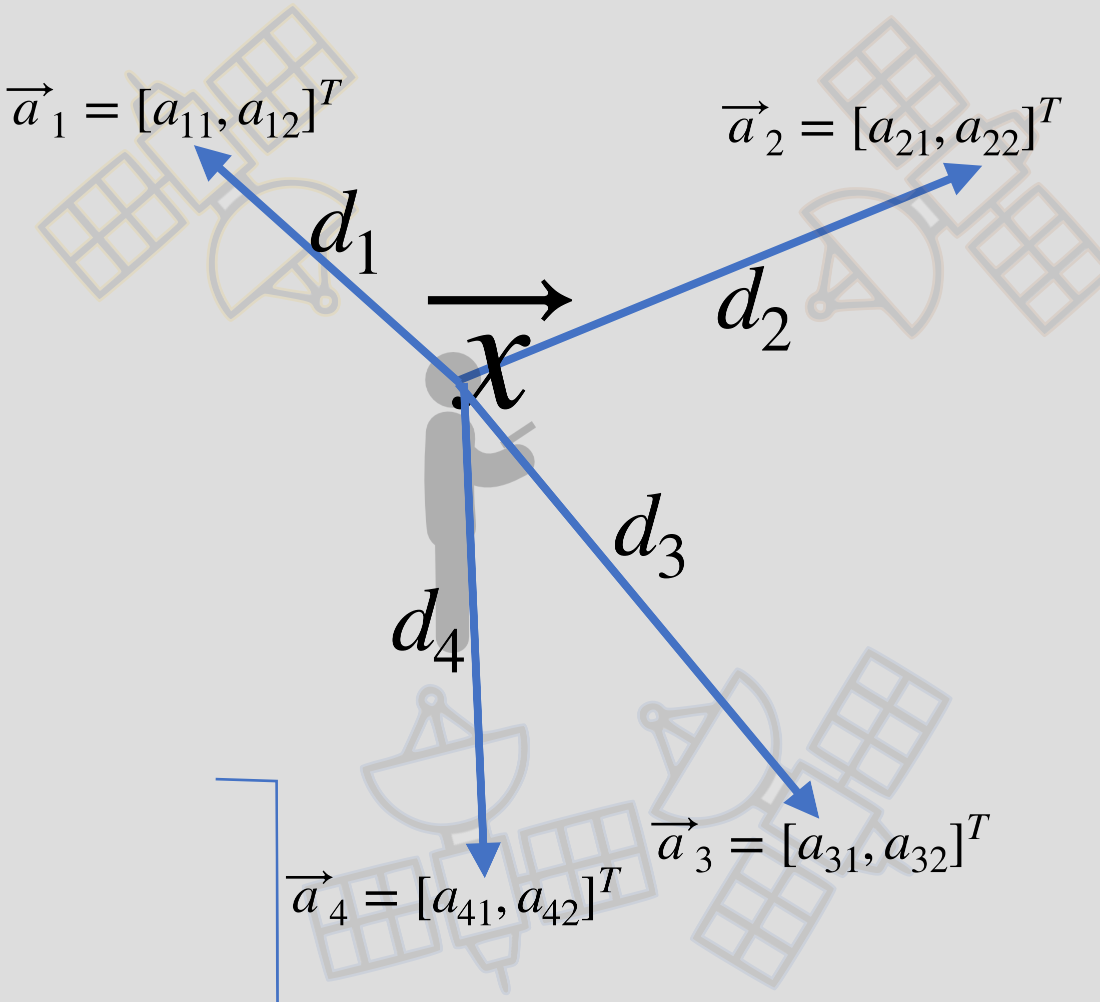
$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\Delta\tau_2)(\Delta\tau_2 + 2\tau_1)$$

$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} - 2C^2\Delta\tau_2\tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\Delta\tau_2)^2$$

Another variable! Need 1 more equation (satellite)

Trilateration

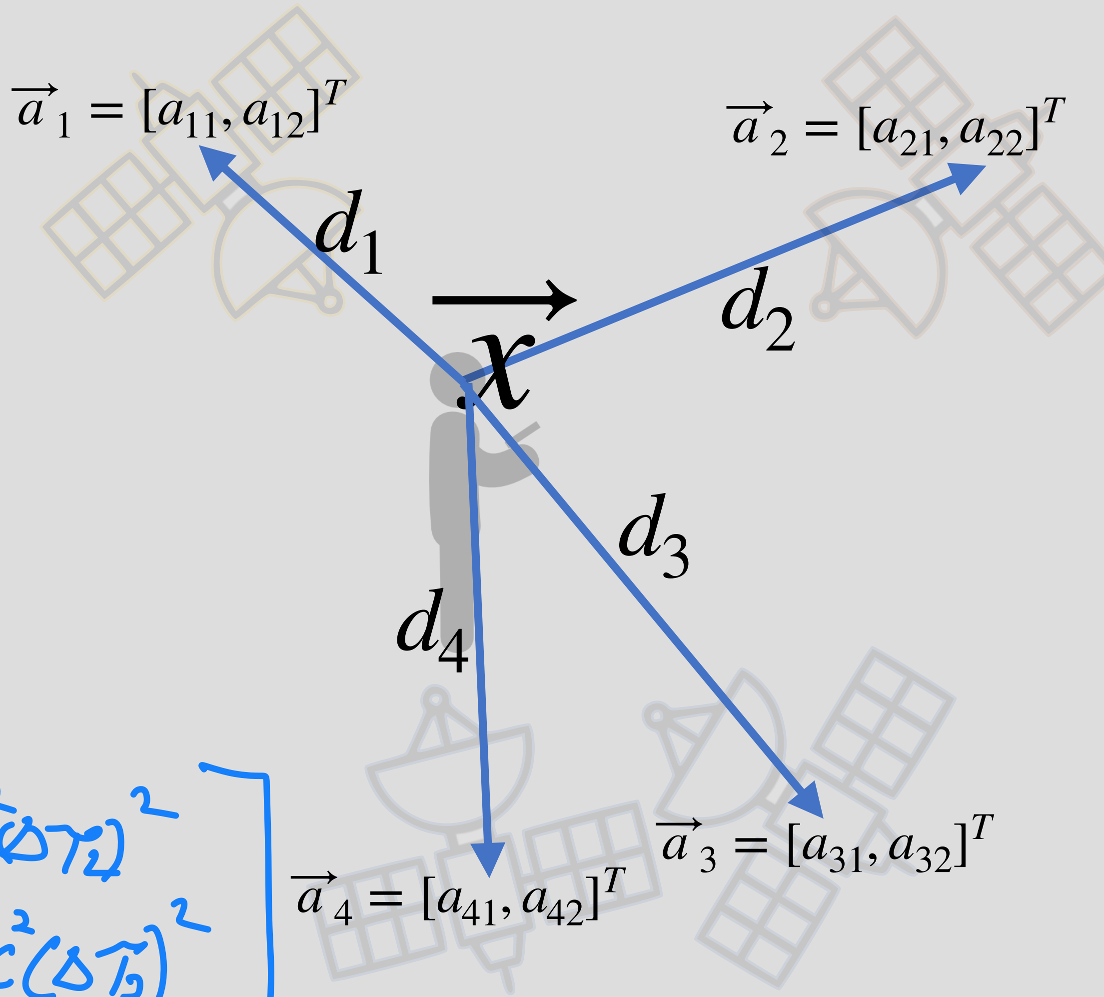
$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} - 2C^2 \Delta\tau_2 \tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\Delta\tau_2)^2$$
$$2(\vec{a}_1 - \vec{a}_3)^T \vec{x} - 2C^2 \Delta\tau_3 \tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + C^2(\Delta\tau_3)^2$$
$$2(\vec{a}_1 - \vec{a}_4)^T \vec{x} - 2C^2 \Delta\tau_4 \tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_4\|^2 + C^2(\Delta\tau_4)^2$$



$$\begin{bmatrix} \quad \end{bmatrix} \begin{bmatrix} \quad \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

Trilateration

$$2(\vec{a}_1 - \vec{a}_2)^T \vec{x} - 2C^2 \Delta\tau_2 \tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\Delta\tau_2)^2$$
$$2(\vec{a}_1 - \vec{a}_3)^T \vec{x} - 2C^2 \Delta\tau_3 \tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + C^2(\Delta\tau_3)^2$$
$$2(\vec{a}_1 - \vec{a}_4)^T \vec{x} - 2C^2 \Delta\tau_4 \tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_4\|^2 + C^2(\Delta\tau_4)^2$$

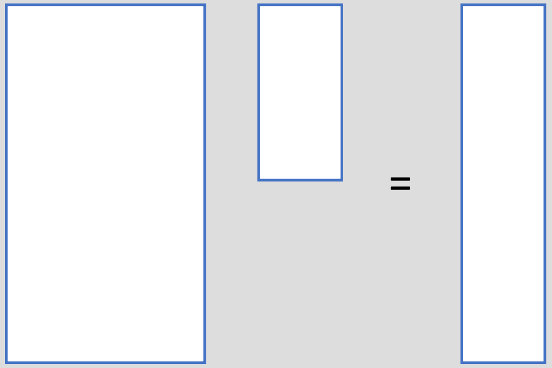


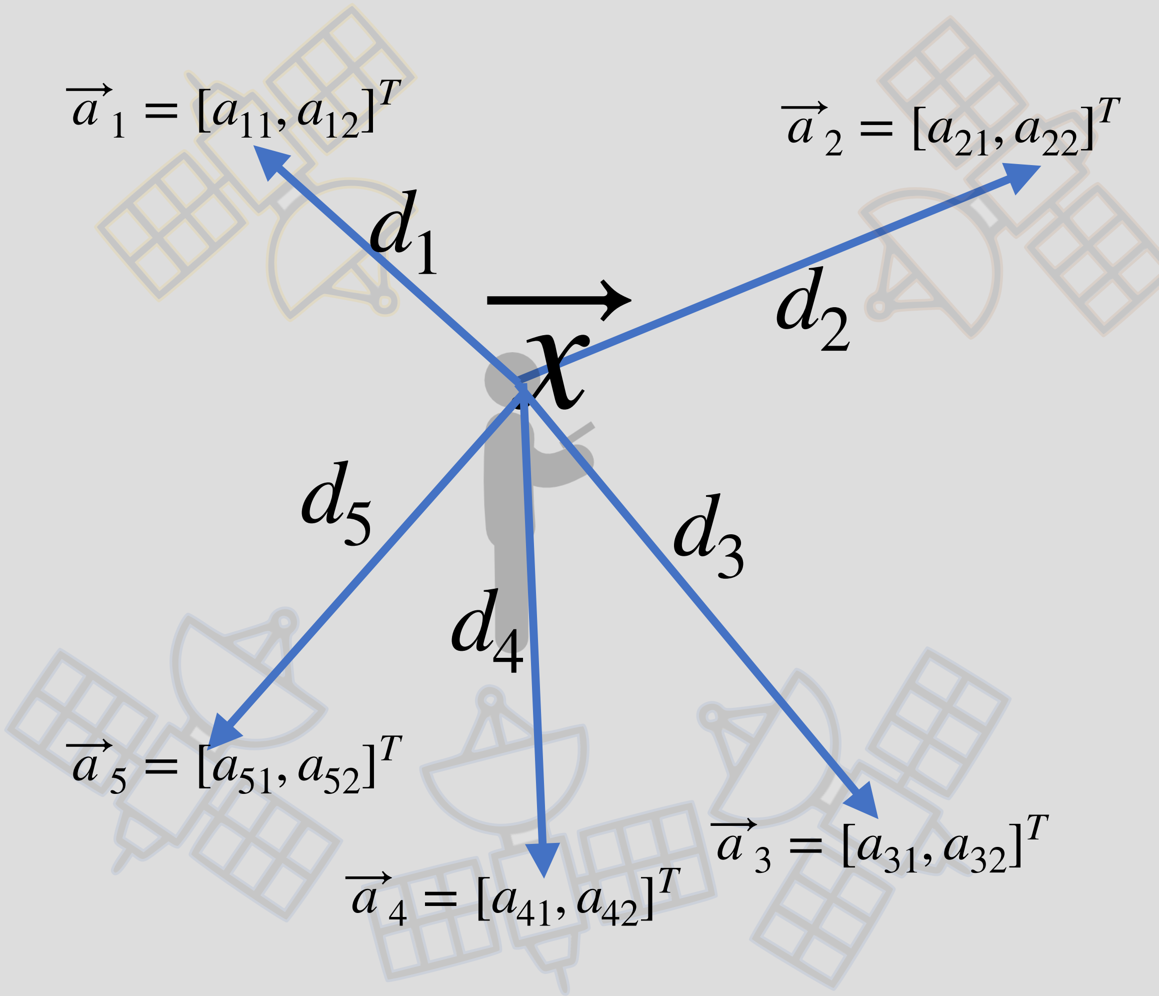
$$2\begin{bmatrix} a_{11}-a_{21} & a_{12}-a_{22} & -C^2\Delta\tau_2 \\ a_{11}-a_{31} & a_{12}-a_{32} & -C^2\Delta\tau_3 \\ a_{11}-a_{41} & a_{12}-a_{42} & -C^2\Delta\tau_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \tau_1 \end{bmatrix} = \begin{bmatrix} \|a_1\|^2 - \|a_2\|^2 + C^2(\Delta\tau_2)^2 \\ \|a_1\|^2 - \|a_3\|^2 + C^2(\Delta\tau_3)^2 \\ \|a_1\|^2 - \|a_4\|^2 + C^2(\Delta\tau_4)^2 \end{bmatrix}$$

Multi-Lateration

$$2(\vec{a}_1 - \vec{a}_2)^T\vec{x} - 2C^2\Delta\tau_2\tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\Delta\tau_2)^2$$
$$2(\vec{a}_1 - \vec{a}_3)^T\vec{x} - 2C^2\Delta\tau_3\tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + C^2(\Delta\tau_3)^2$$
$$2(\vec{a}_1 - \vec{a}_4)^T\vec{x} - 2C^2\Delta\tau_4\tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_4\|^2 + C^2(\Delta\tau_4)^2$$
$$2(\vec{a}_1 - \vec{a}_5)^T\vec{x} - 2C^2\Delta\tau_5\tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_5\|^2 + C^2(\Delta\tau_5)^2$$

More equations than unknowns





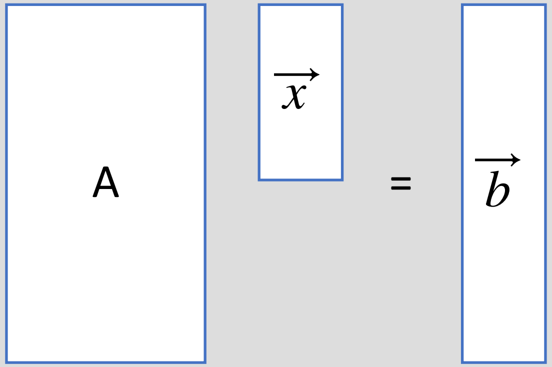
Over-determined — may not have a solution!

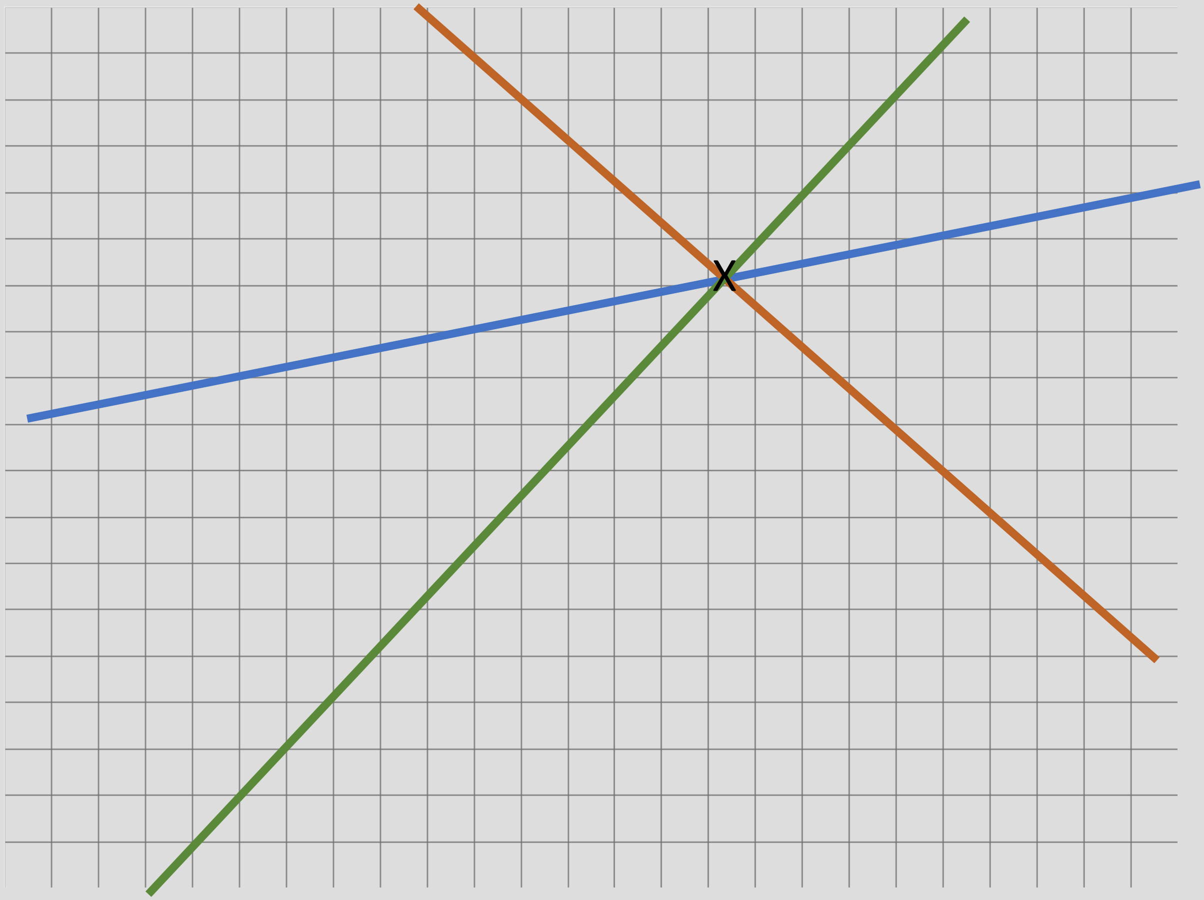
Overdetermined Linear Equations

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

$$a_{31}x_1 + a_{32}x_2 = b_3$$





Q: When is there a solution?

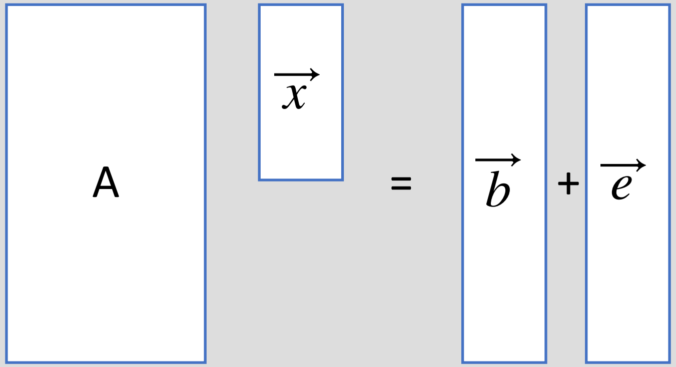
A: When $\overrightarrow{b} \in \text{Span}\{\text{cols of } A\}$

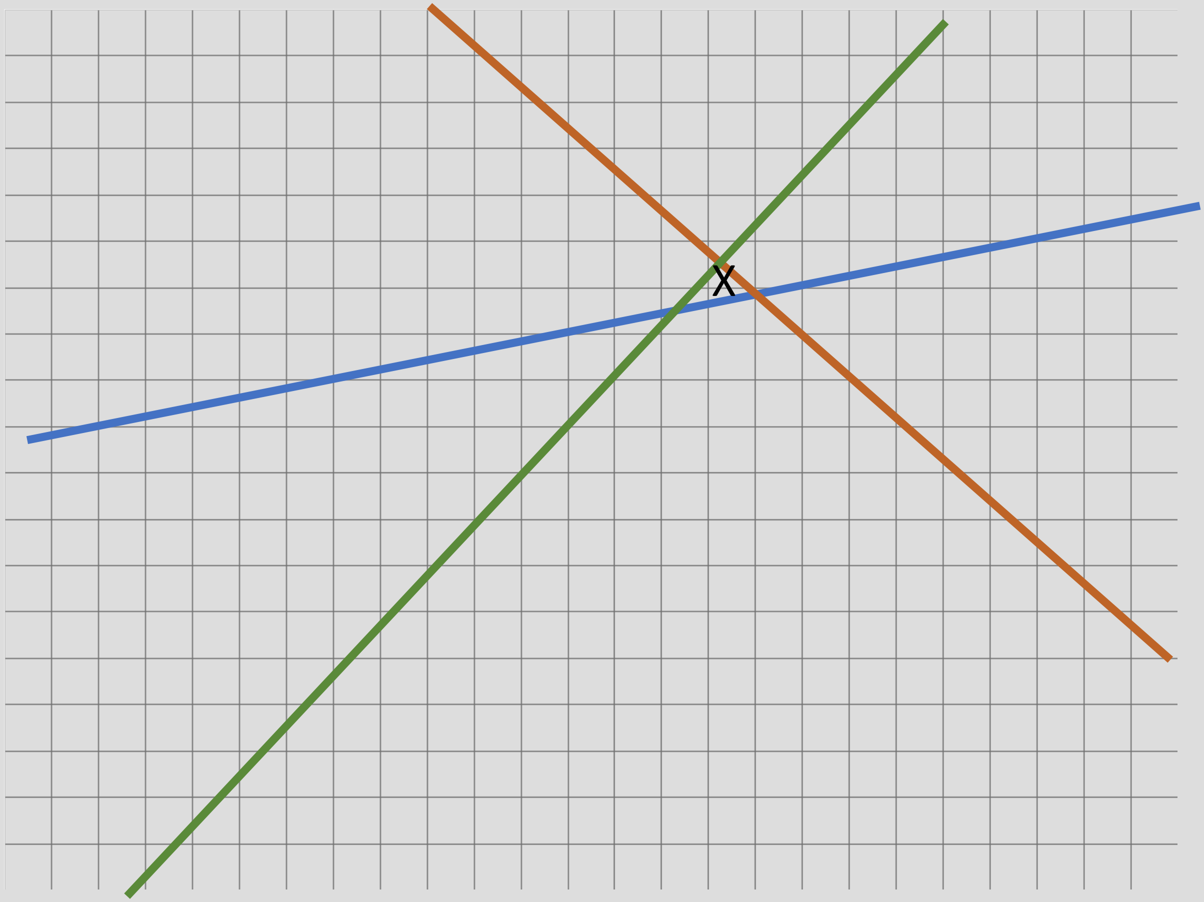
Inconsistent Linear Equations

$$a_{11}x_1 + a_{12}x_2 = b_1 + e_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2 + e_2$$

$$a_{31}x_1 + a_{32}x_2 = b_3 + e_3$$





Q: With noise, equations will be inconsistent! - no solution.

Towards the Least Squares Algorithm

Fact:

We have measurements: $\overrightarrow{b}$

We have a model that : $A\overrightarrow{x} = \overrightarrow{b}$

Problem:

But $A\overrightarrow{x} = \overrightarrow{b}$ does not have a solution!

What to do?

Want to find $\hat{x}$, such that $A\hat{x}$ is the closest to $\overrightarrow{b}$

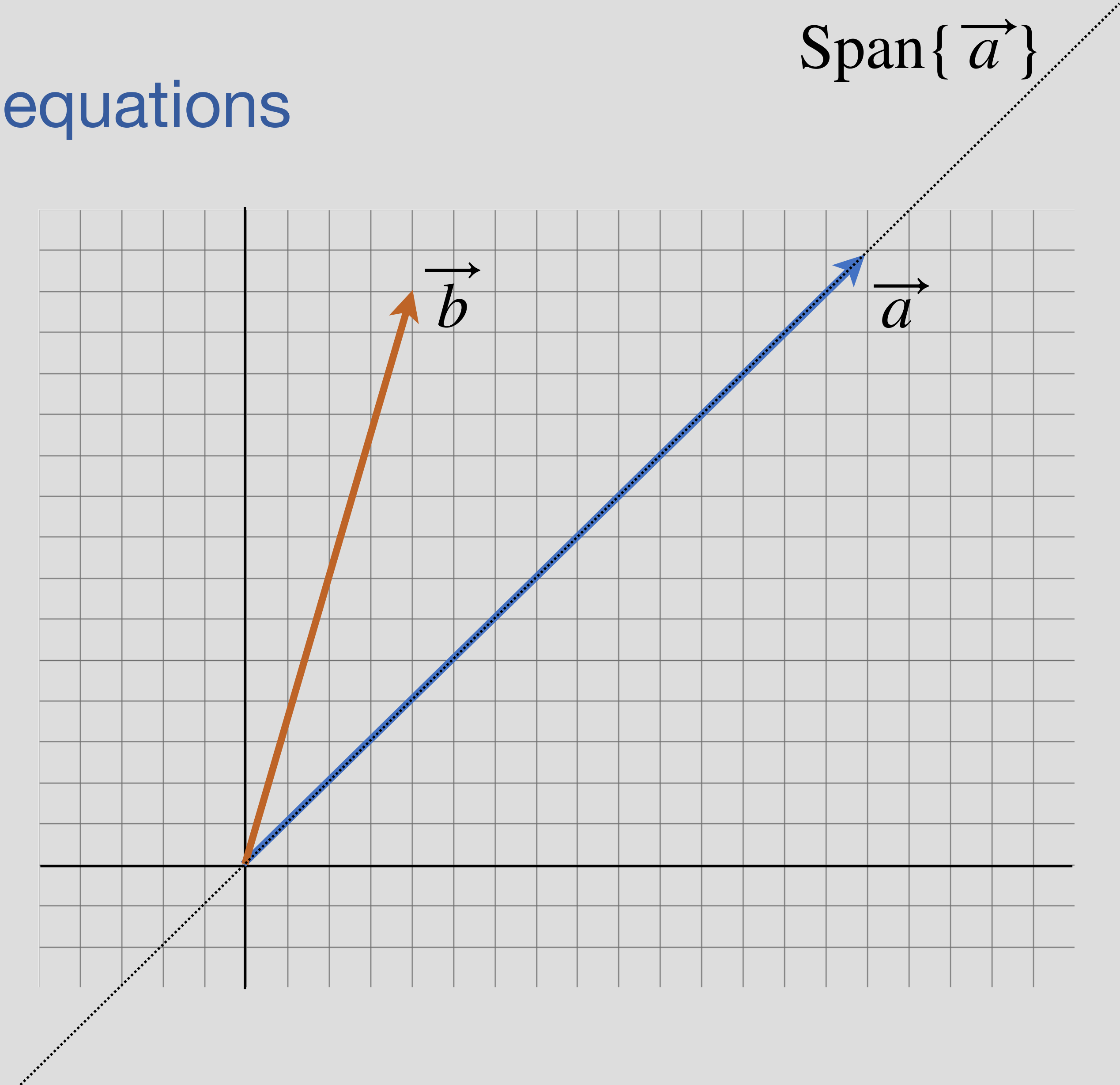
Example: a scalar problem

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$, one unknown, two equations

Solution:

find $\hat{x}$ that has the smallest error

$$\| \vec{e} \| = \| \vec{a}\hat{x} - \vec{b} \|$$



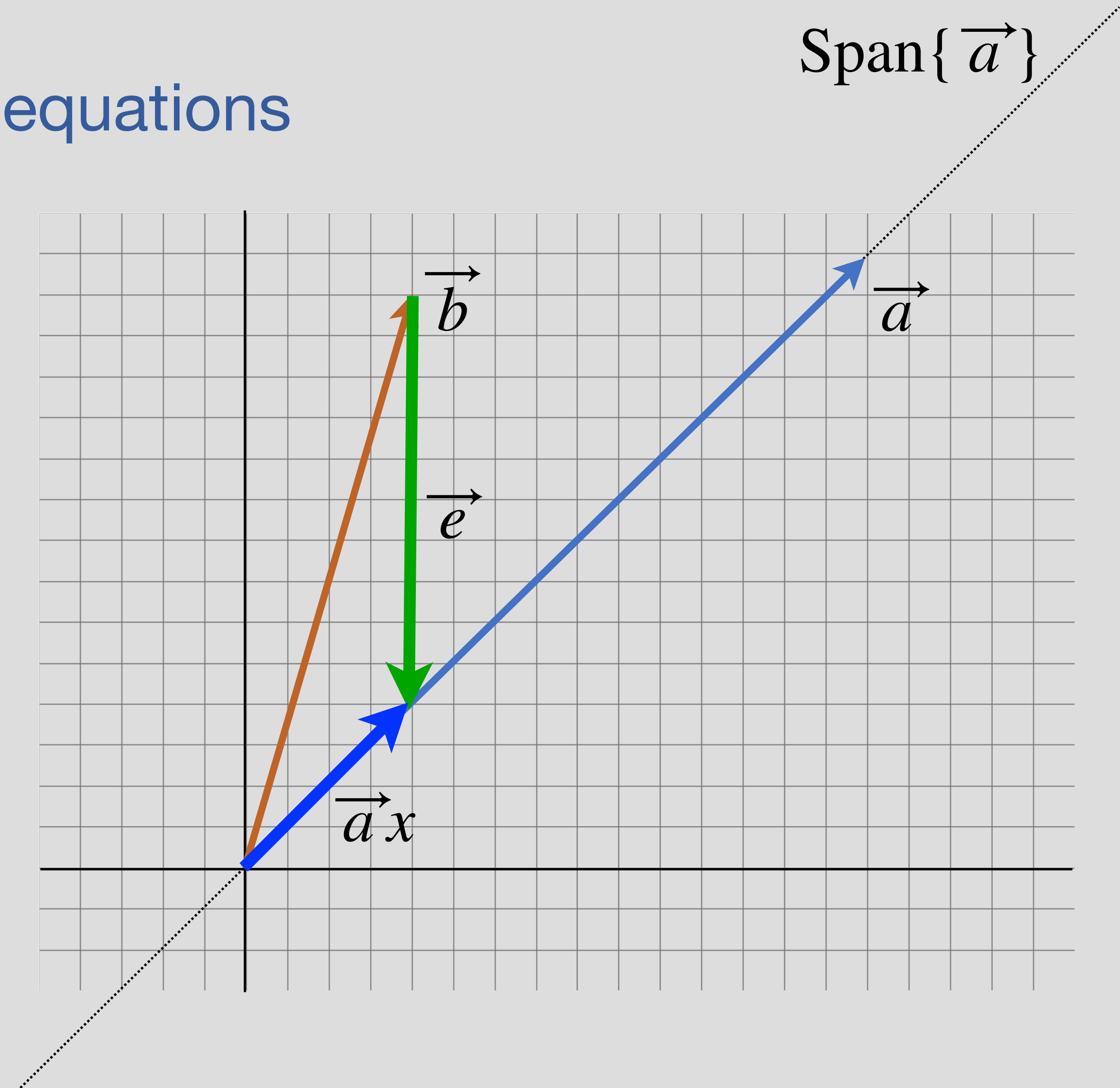
Example: a scalar problem

$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, one unknown, two equations

Solution:

find $\hat{x}$ that has the smallest error

$\| \vec{e} \| = \| \vec{a}\hat{x} - \vec{b} \| \leq \| \vec{a} x - \vec{b} \|$



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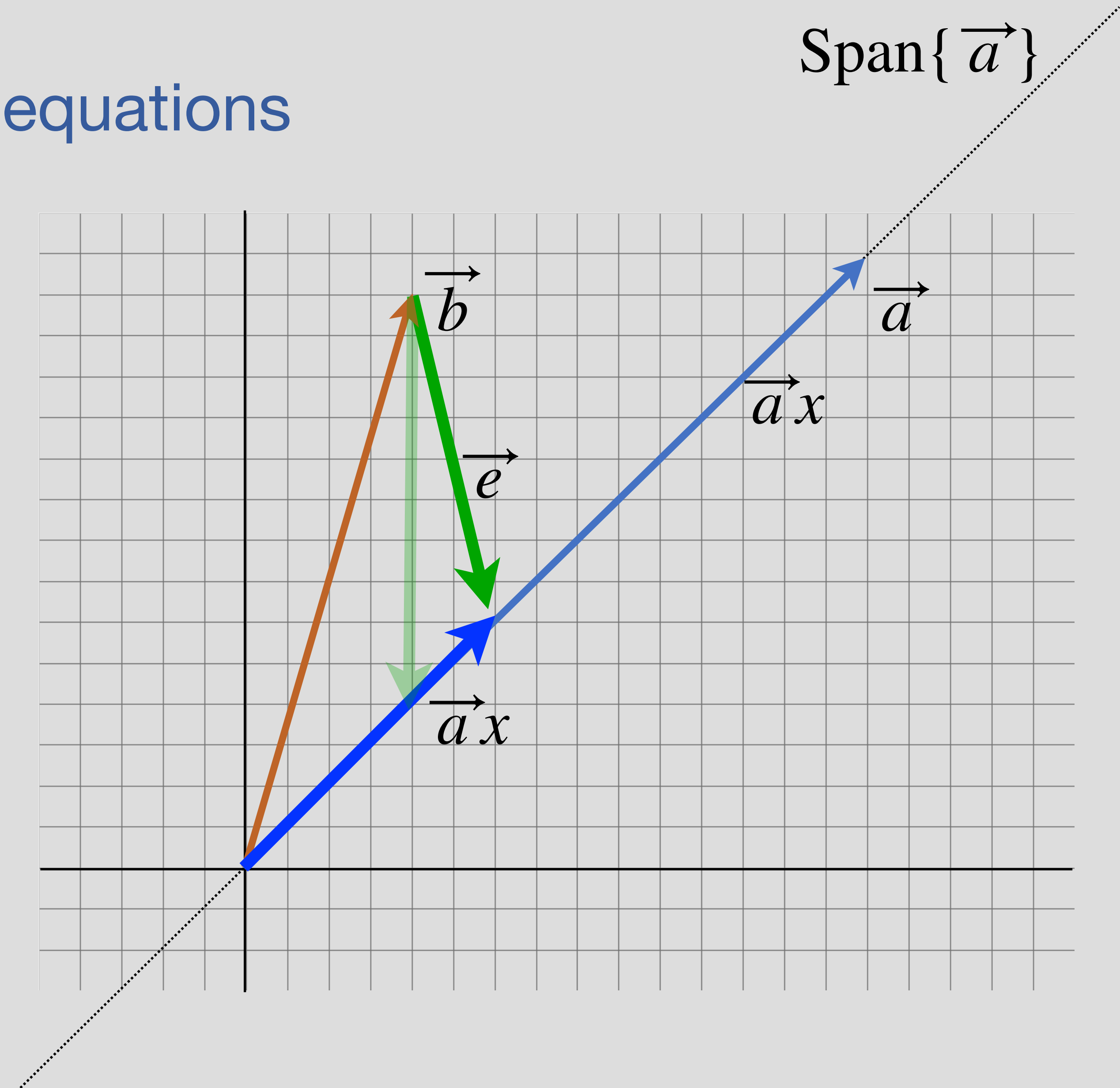

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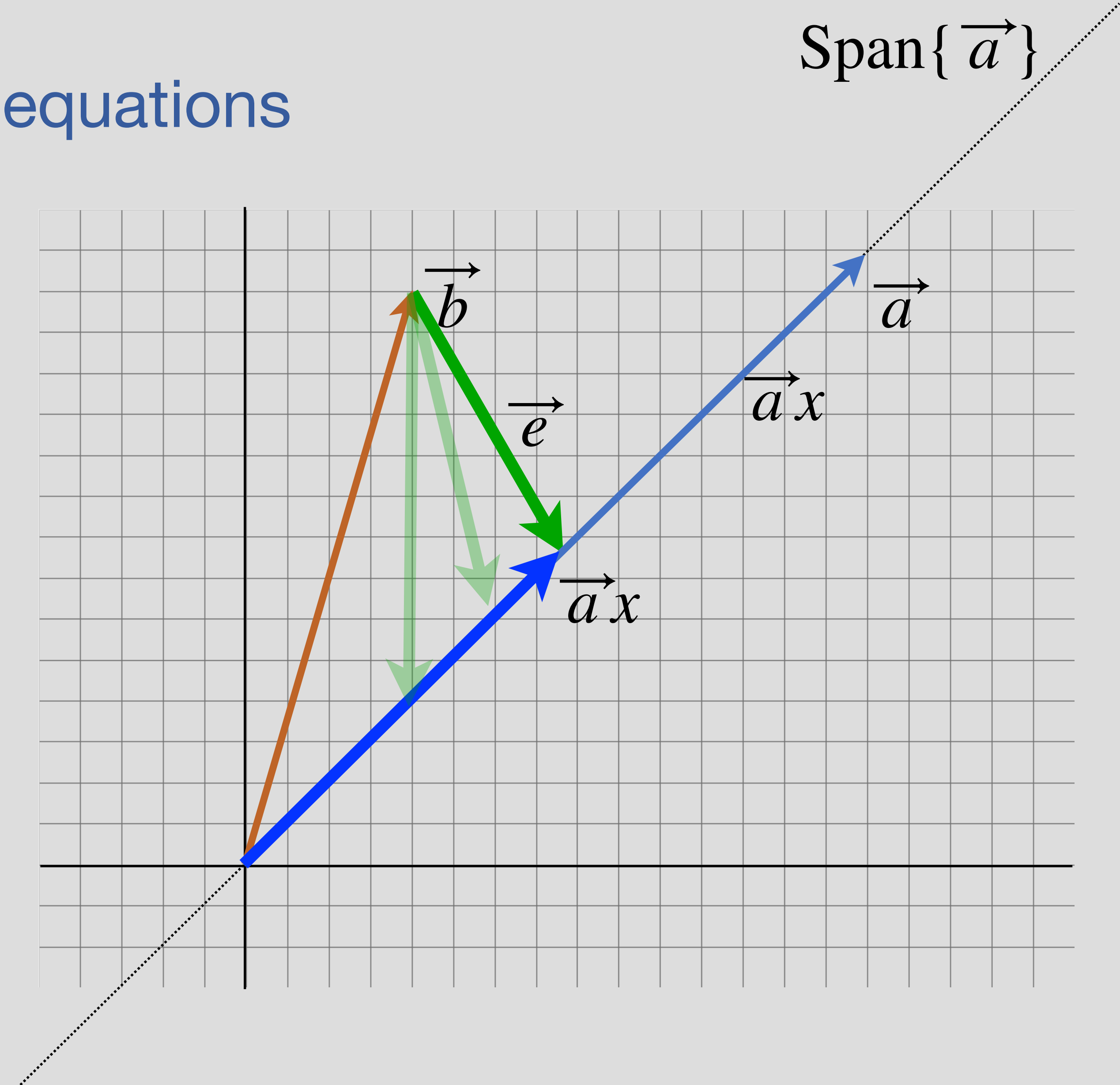

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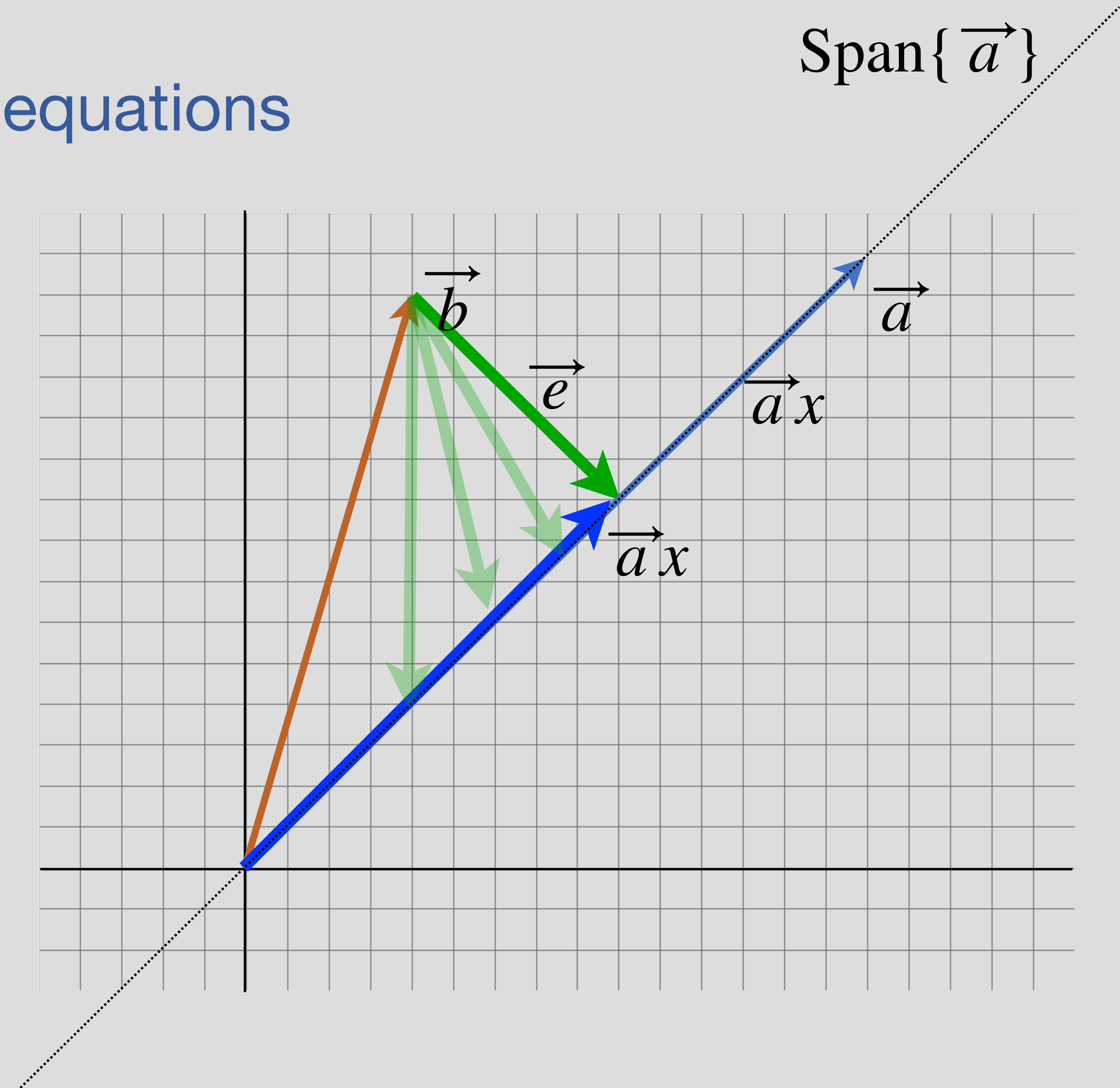
$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$, one unknown, two equations

Solution:

find $\hat{x}$ that has the smallest error

$$\| \vec{e} \| = \| \vec{a}\hat{x} - \vec{b} \| \leq \| \vec{a}x - \vec{b} \|$$



Example: a scalar problem

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$, one unknown, two equations

Solution:

find $\hat{x}$ that has the smallest error

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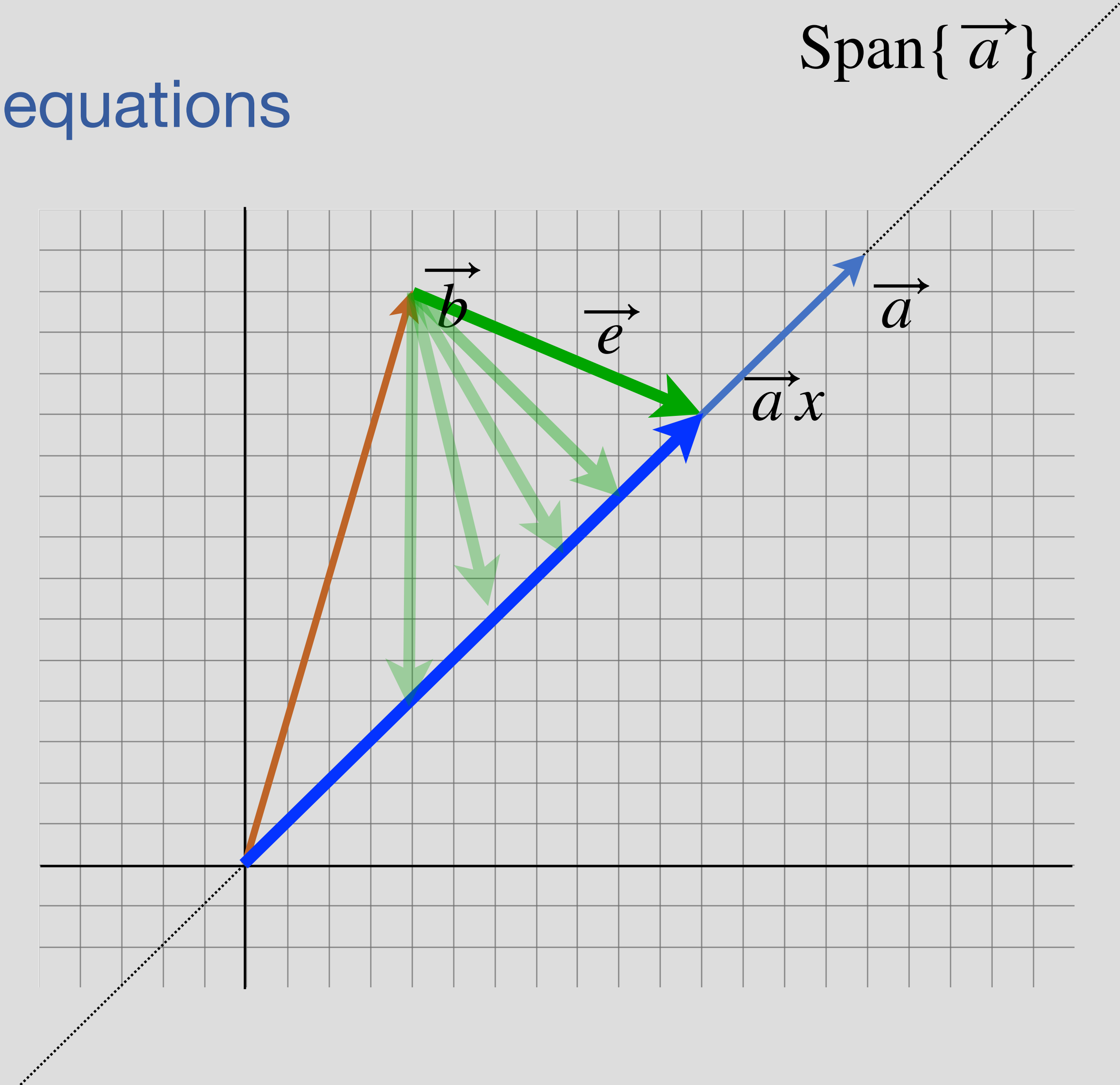

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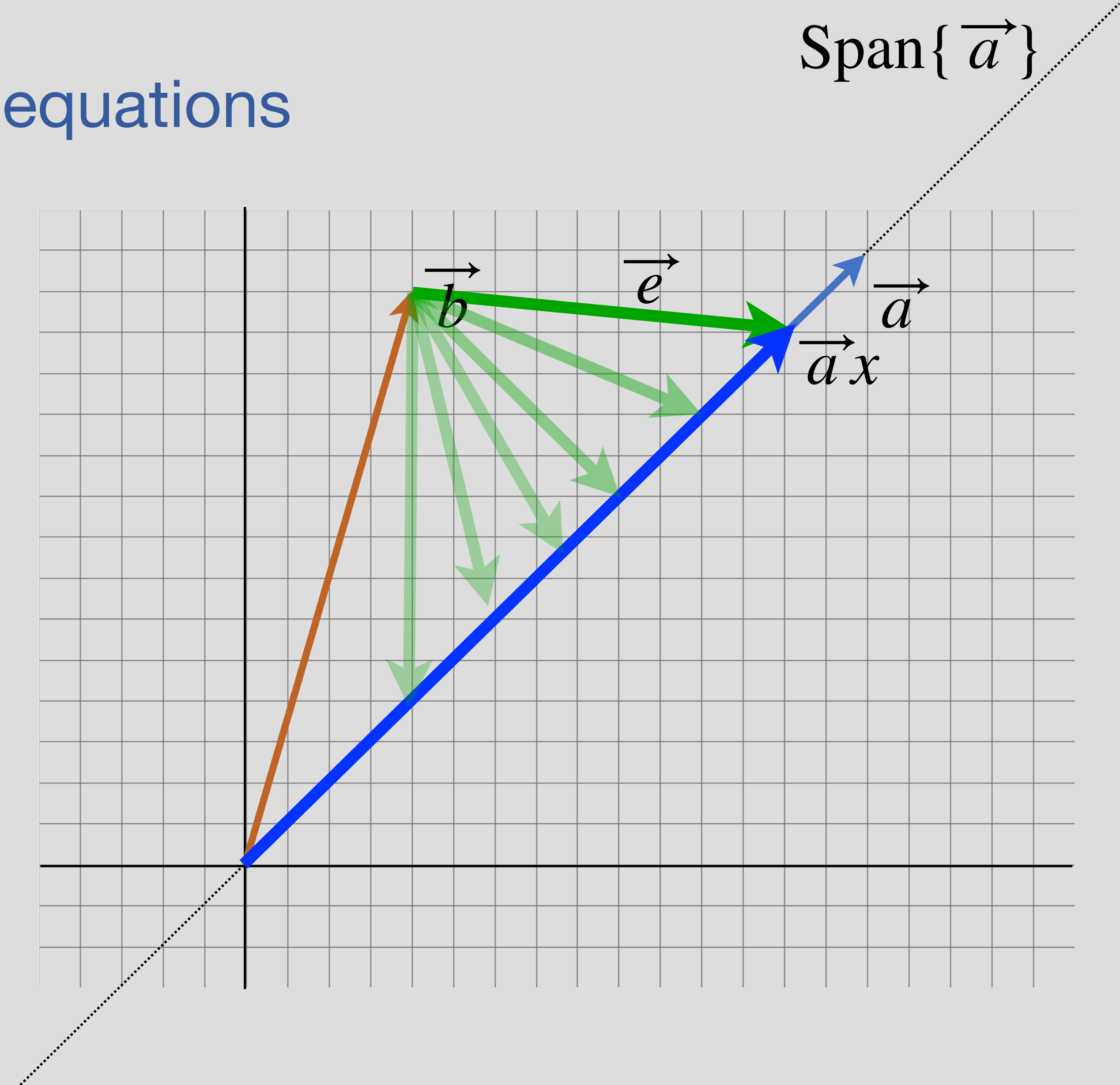

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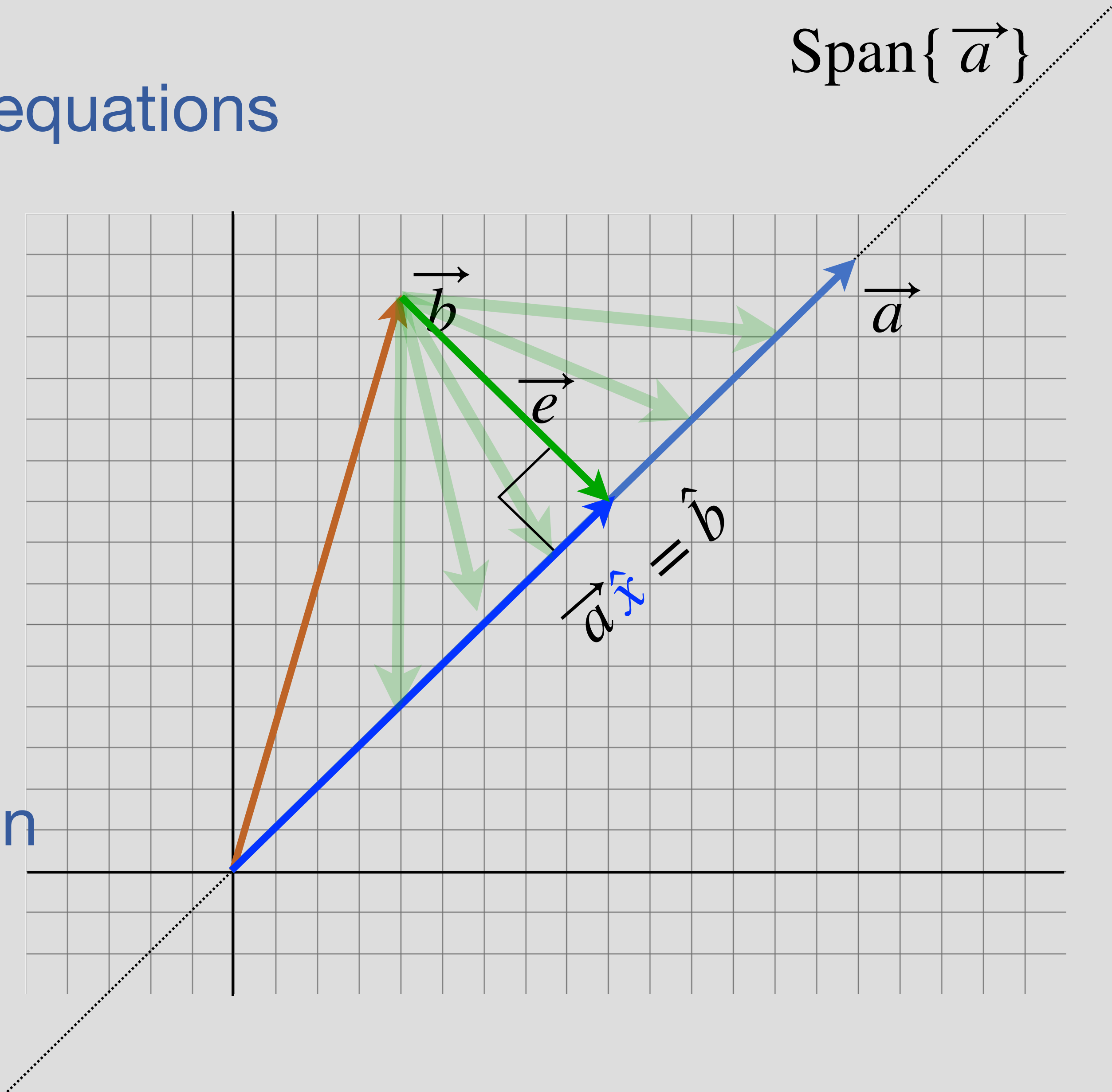
Theorem:

shortest distance between a point and a line is the orthogonal projection



Projections

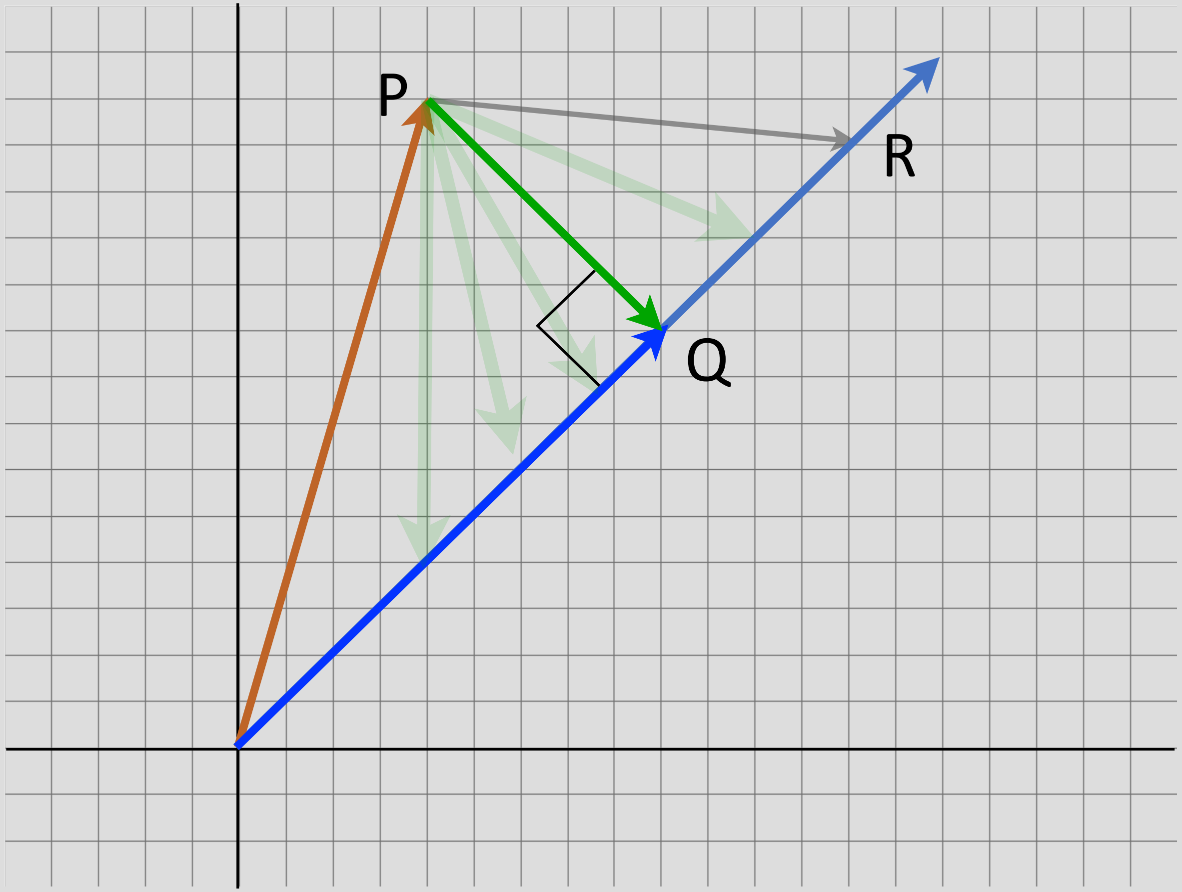
Theorem:
shortest distance between a point and line is the orthogonal projection

Proof:

Pythagoras: $(PR)^2 = (PQ)^2 + (QR)^2$ > 0

$$(PR)^2 > (PQ)^2$$

$$(PR) > (PQ)$$



Projections

find $\hat{x}$ that has the smallest error

$$\|\vec{e}\| = \|\vec{a}\hat{x} - \vec{b}\| \leq \|\vec{a}x - \vec{b}\|$$

Need to find the orthogonal projection!

We know: $\vec{e} \perp \hat{b}$, $\vec{e} \perp \vec{a}$

$$\langle \vec{e}, \vec{a} \rangle = 0$$

$$\langle \vec{b} - \hat{b}, \vec{a} \rangle = 0$$

$$\langle \vec{b}, \vec{a} \rangle - \langle \hat{b}, \vec{a} \rangle = 0$$

$$\langle \vec{b}, \vec{a} \rangle = \langle \hat{b}, \vec{a} \rangle$$

$$\langle \vec{b}, \vec{a} \rangle = \langle \vec{a}\hat{x}, \vec{a} \rangle$$

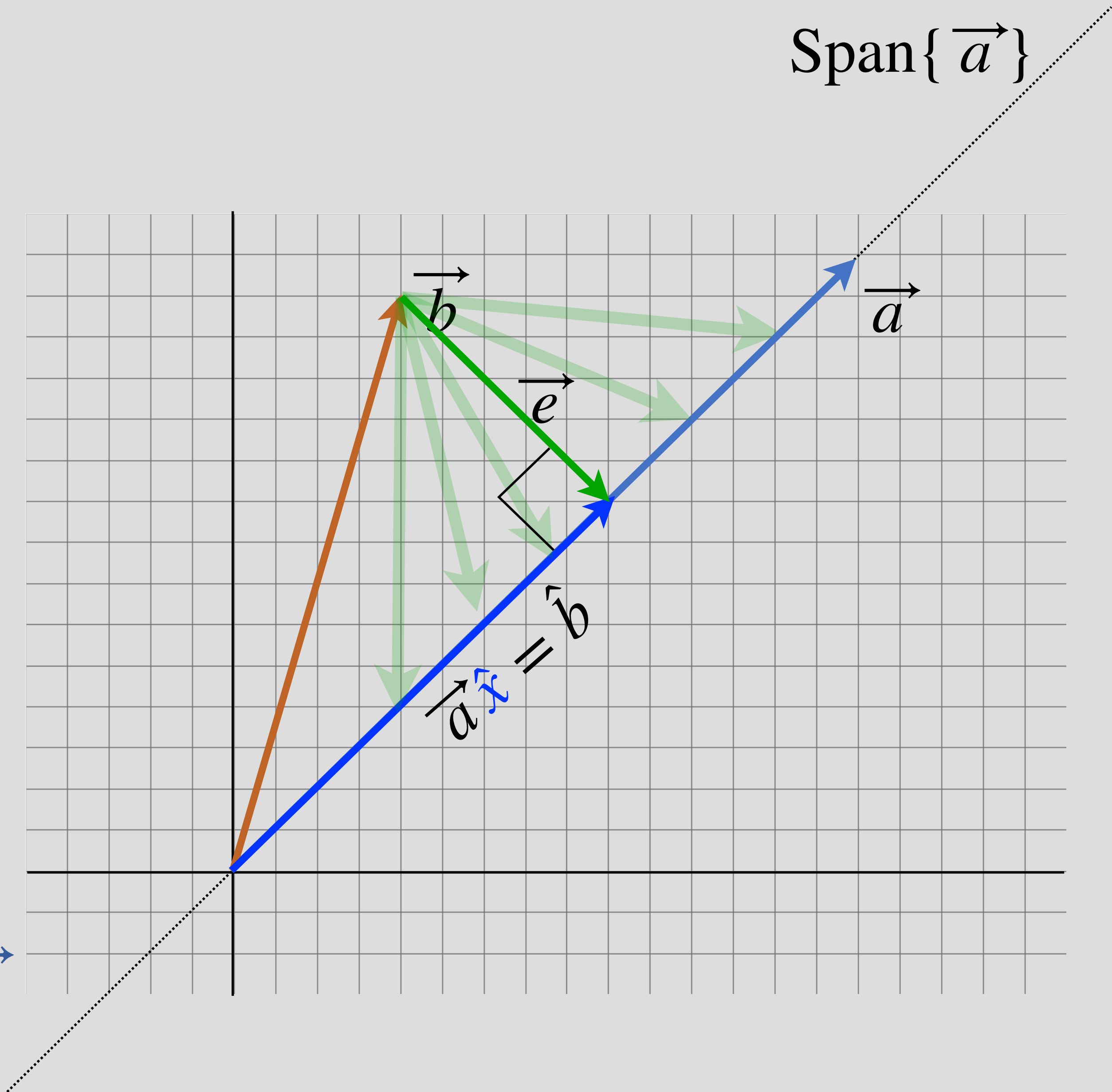
$$\langle \vec{b}, \vec{a} \rangle = \hat{x} \langle \vec{a}, \vec{a} \rangle$$

$$\langle \vec{b}, \vec{a} \rangle = \hat{x}\|\vec{a}\|^2$$

$$\hat{x} = \frac{\langle \vec{b}, \vec{a} \rangle}{\|\vec{a}\|^2}$$

$$\hat{b} = \frac{\langle \vec{b}, \vec{a} \rangle}{\|\vec{a}\|^2}\vec{a}$$

$$\hat{b} = \frac{\vec{b}^T\vec{a}}{\vec{a}^T\vec{a}}\vec{a}$$



Orthogonal Projections

Given vectors $\overrightarrow{a}$, $\overrightarrow{b}$, we say that the orthogonal projection of $\overrightarrow{b}$ onto $\overrightarrow{a}$ is:

$$\text{Proj}_{\overrightarrow{a}}(\overrightarrow{b}) = \frac{\overrightarrow{a}^T \overrightarrow{b}}{\|\overrightarrow{a}\|^2} \overrightarrow{a}$$