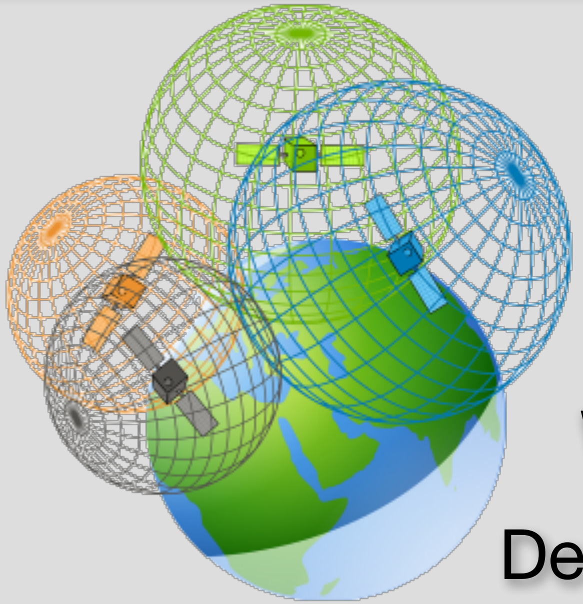
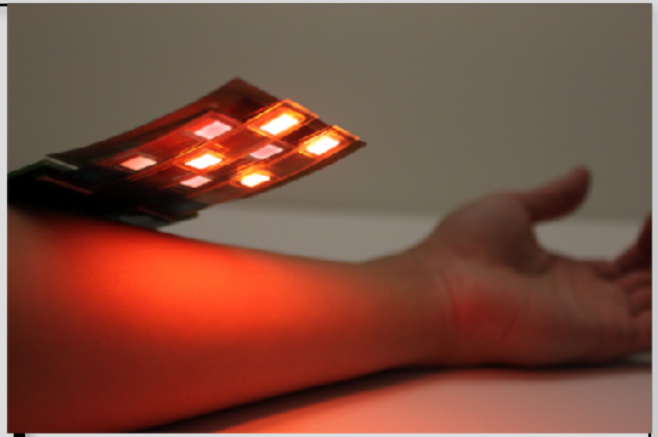




Welcome to EECS 16A!

Designing Information Devices and Systems I

Ana Arias and Miki Lustig

Fall 2022

Lecture 14A
Least Squares Apps

Least Squares

$$\begin{bmatrix} - & \vec{a}_1^T & - \\ - & \vec{a}_2^T & - \\ & \vdots & \\ - & \vec{a}_N^T & - \end{bmatrix} \begin{bmatrix} | \\ \vec{b} - \hat{b} \\ | \end{bmatrix} = 0$$

$$A^T\left(\vec{b} - A\hat{x}\right) = 0$$

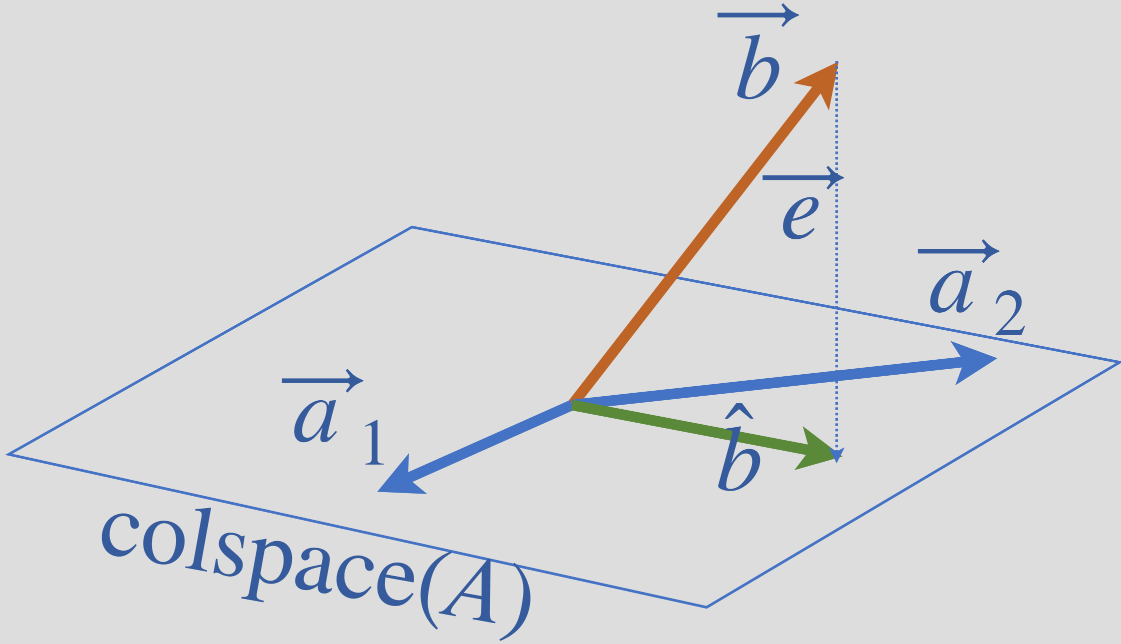
$$A^T\vec{b} - A^T A\hat{x} = 0$$

$$A^T A\hat{x} = A^T\vec{b}$$

If A is full Rank, then $A^T A$ is invertible

$$\hat{x} = (A^T A)^{-1} A^T \vec{b}$$

$$\hat{b} = A(A^T A)^{-1} A^T \vec{b}$$



$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_N \\ | & | & & | \end{bmatrix}$$

$A\vec{x} \in \text{colspace}(A)$

Find $\hat{b} = A\hat{x}$

⌘3+⌘4

Example 4: Regression

Gauss found Ceres by using Kepler's laws:

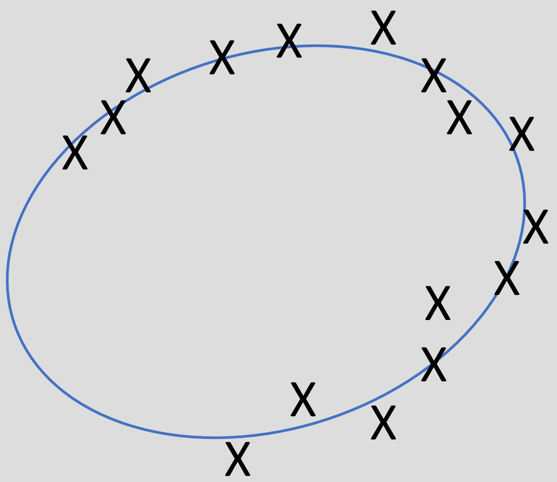
Model: $ax^2+by^2+cxy+dx+ey = 1$

Q: Is this a linear fit?

A: Yes!

Knowns: $(x_1, y_1)\ (x_2, y_2)\ \cdots (x_N, y_N)$

Unknowns: $\overrightarrow{p} = [a\ b\ c\ d\ e]^T$



$$\underbrace{\begin{bmatrix} x_1^2 & y_1^2 & x_1y_1 & x_1 & y_1 \\ x_2^2 & y_2^2 & x_2y_2 & x_2 & y_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_N^2 & y_N^2 & x_Ny_N & x_N & y_N \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}}_{\overrightarrow{p}} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}_{\overrightarrow{y}}$$

$$\hat{p} = (A^TA)^{-1}A^T\overrightarrow{y}$$

Example 5: Exponential Regression

Model: $y = ce^{ax}$

Q: Is this a linear fit?

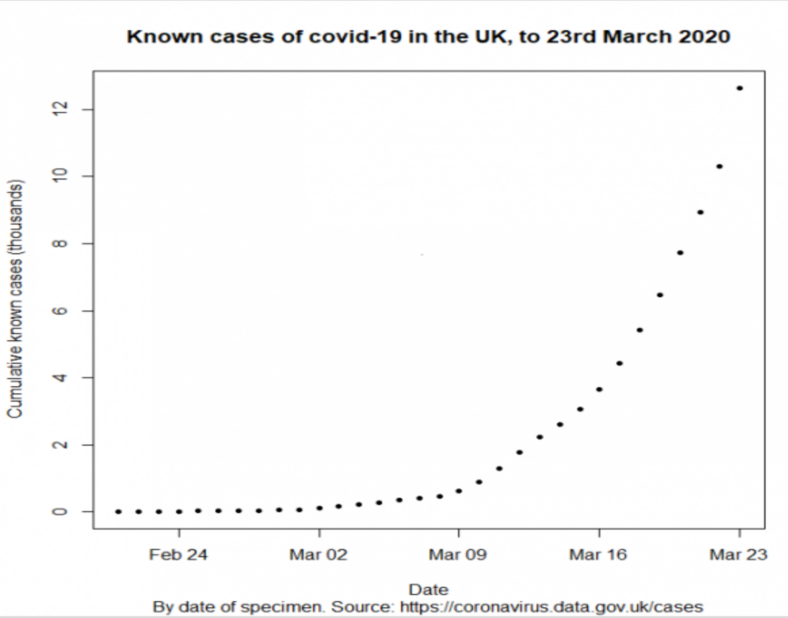
A: No! But, can be made linear…..

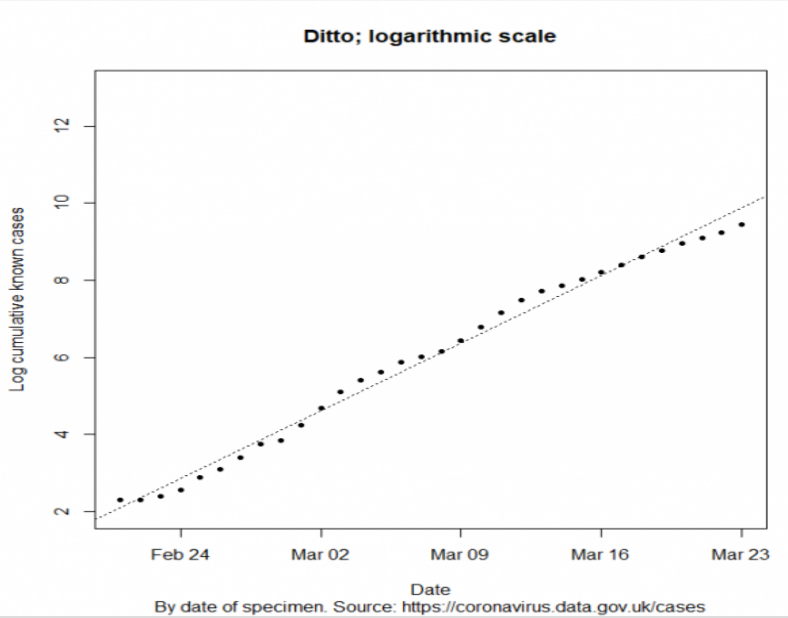
New Model: $\log(y) = \log c + ax = b + ax$

Knowns: $(x_1, \log(y_1)) \; (x_2, \log(y_2)) \cdots (x_N, \log(y_N))$

Unknowns: $\vec{p} = [a \; b]^T$

$A \quad \vec{p} \quad \vec{y}$





https://www.cebm.net/covid-19/exponential-growth-what-it-is-why-it-matters-and-how-to-spot-it/

Example 5: Exponential Regression

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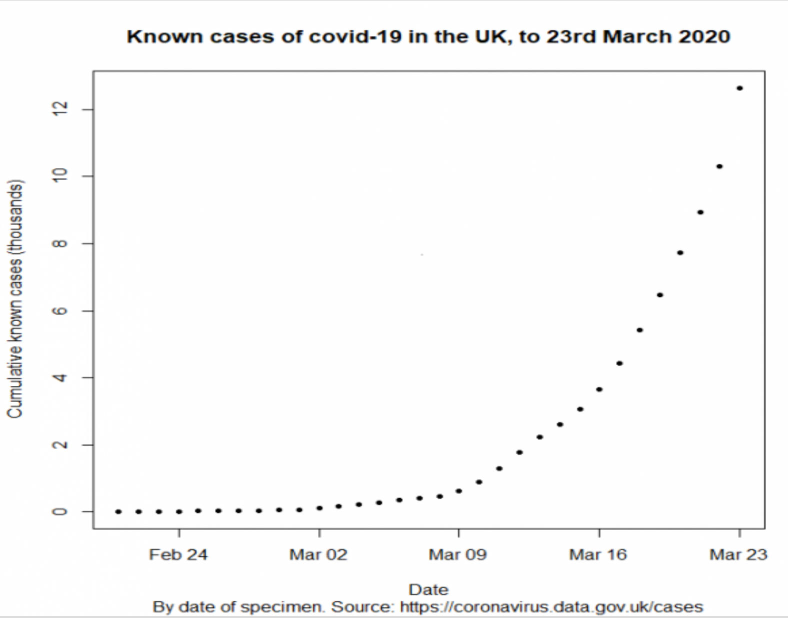
Knowns: $(x_1, \log(y_1))\ (x_2, \log(y_2)) \cdots (x_N, \log(y_N))$

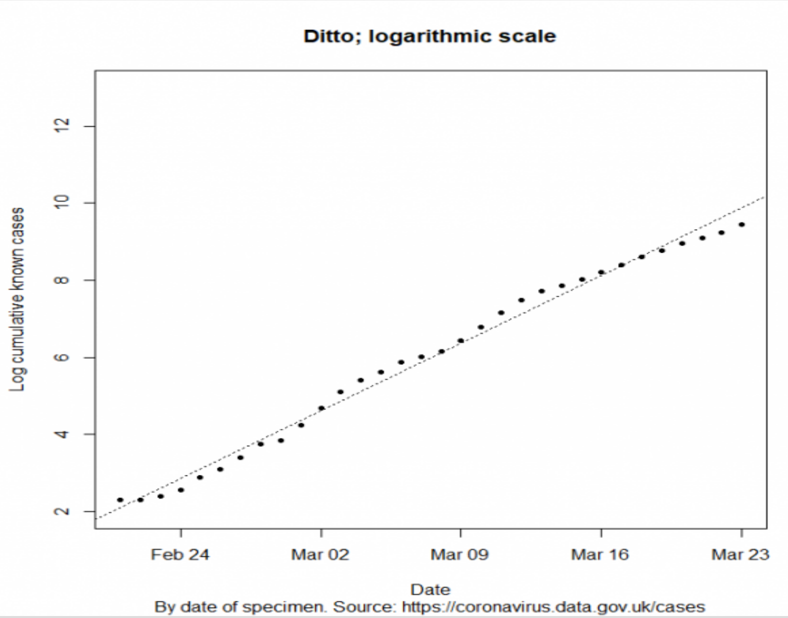
Unknowns: $\vec{p} = [a\ b]^T$

$$\underbrace{\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_N & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{p}} = \underbrace{\begin{bmatrix} \log y_1 \\ \log y_2 \\ \vdots \\ \log y_N \end{bmatrix}}_{\vec{y}}$$

$$\hat{p} = (A^T A)^{-1} A^T \vec{y}$$

$$\hat{c} = e^{\hat{b}}$$





https://www.cebm.net/covid-19/exponential-growth-what-it-is-why-it-matters-and-how-to-spot-it/

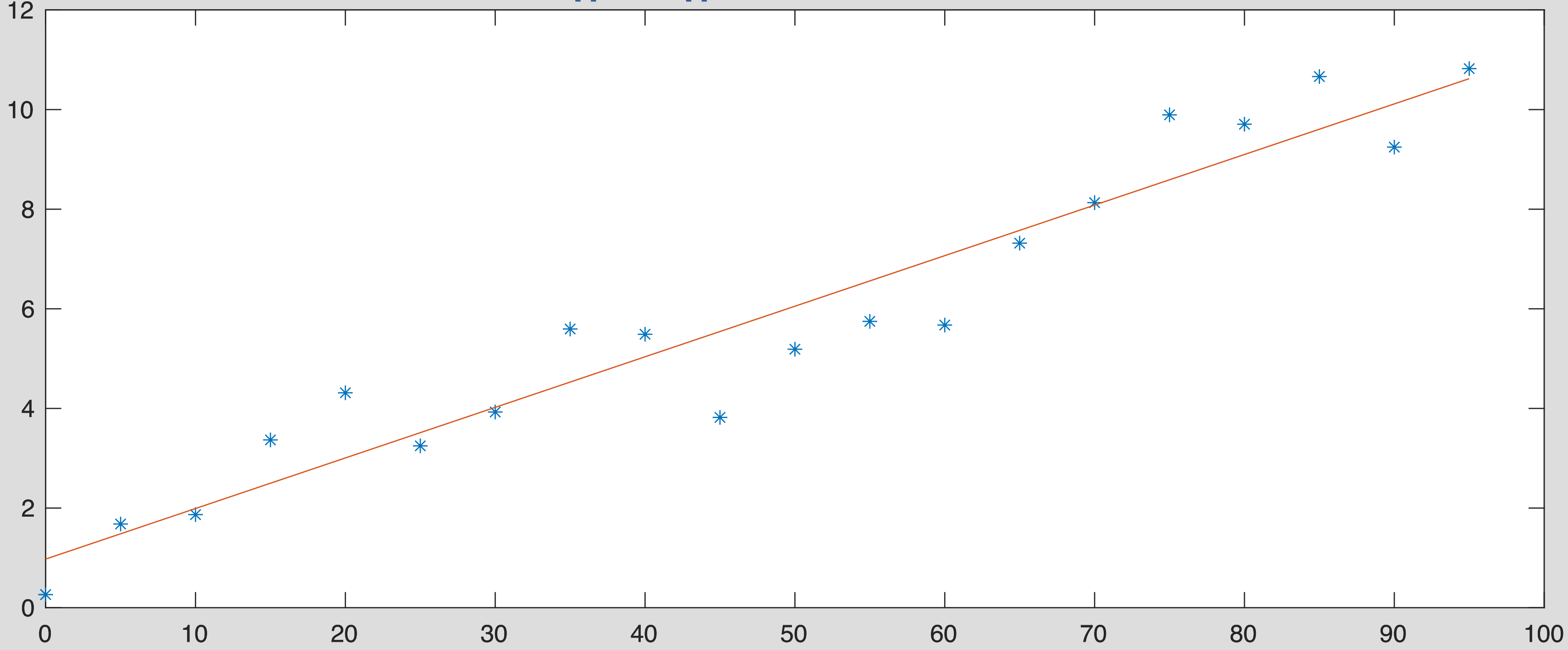
Example 6: Over Fitting

- Consider noisy measurements of $y = 0.1x + 1$:

$$\vec{p} = [0.1015 \quad 0.9757]^T$$

Model: $y = ax + b$

$$\|\vec{e}\| = 3.85$$

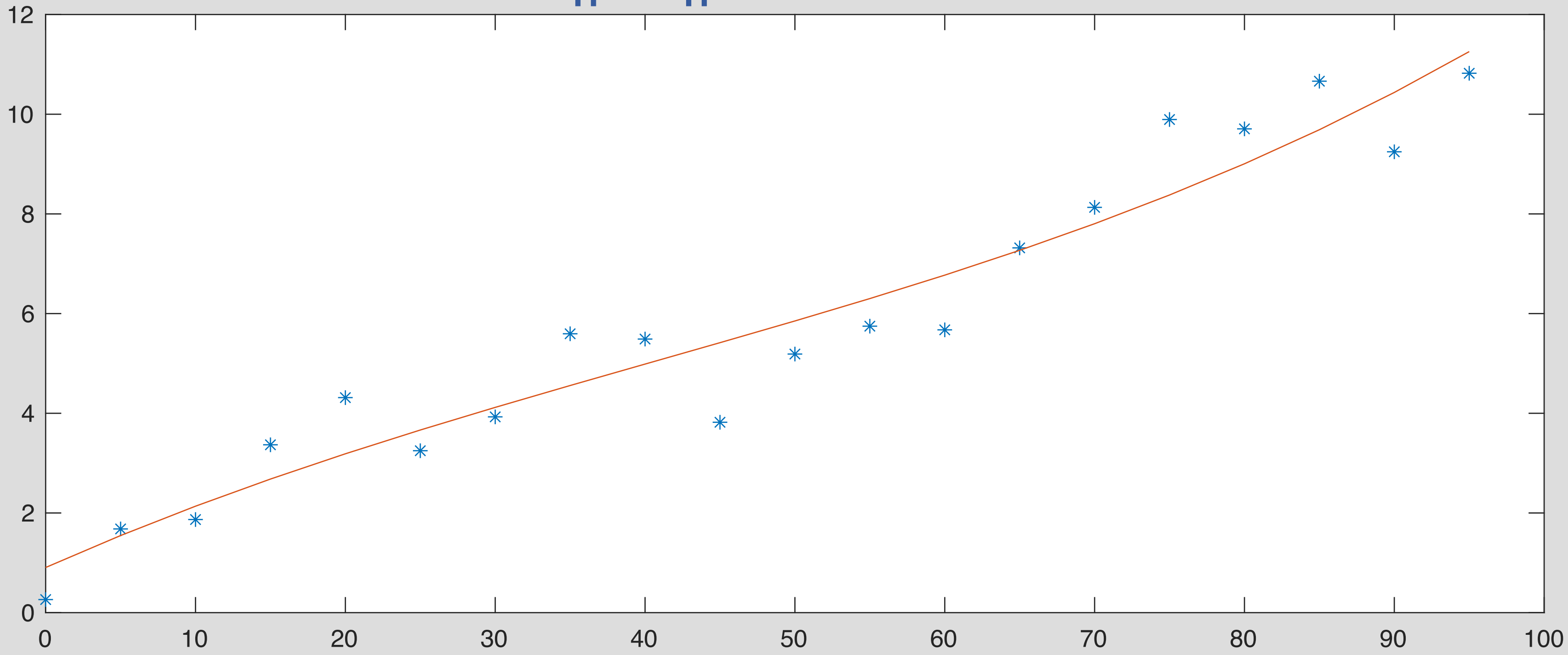


Example 6: Over Fitting

- Consider noisy measurements of $y = 0.1x + 1$:

Model: $y = ax^3 + bx^2 + cx + d$

$$\|\vec{e}\| = 3.71$$

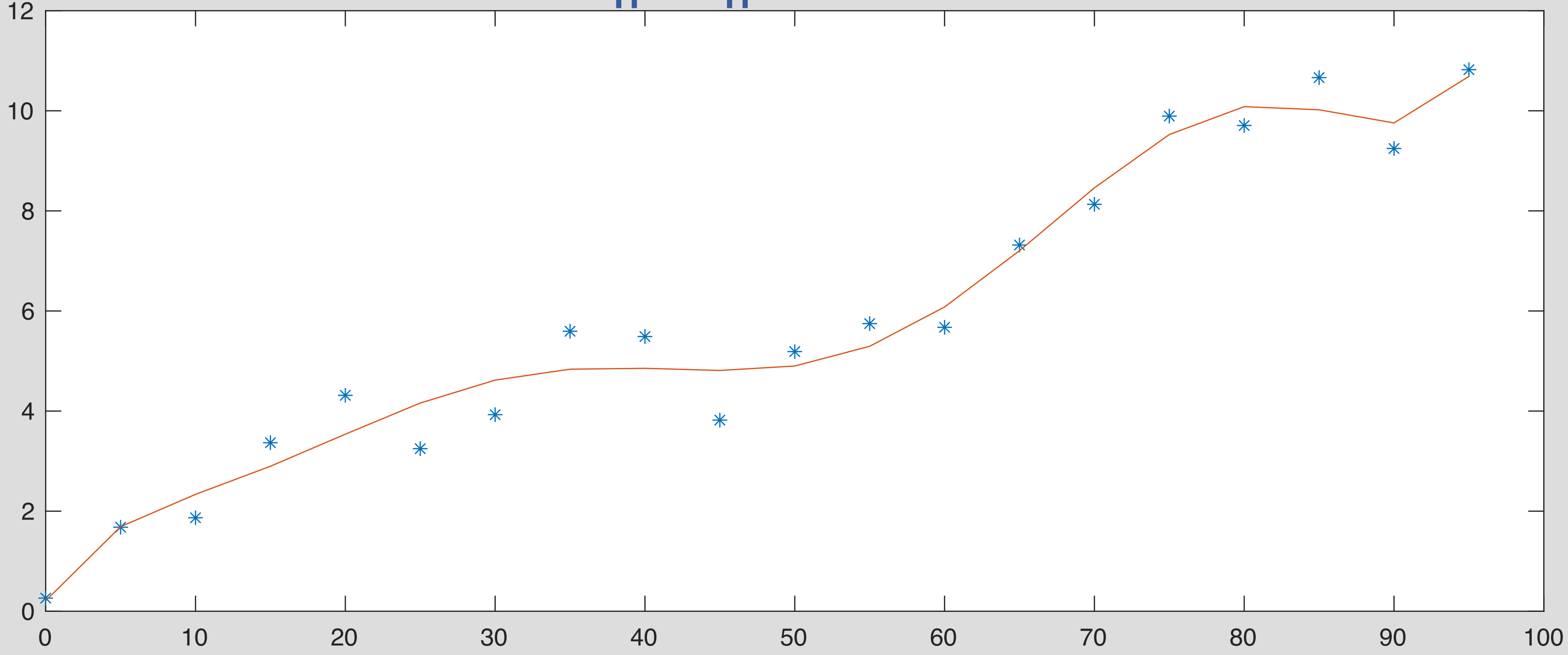


Example 6: Over Fitting

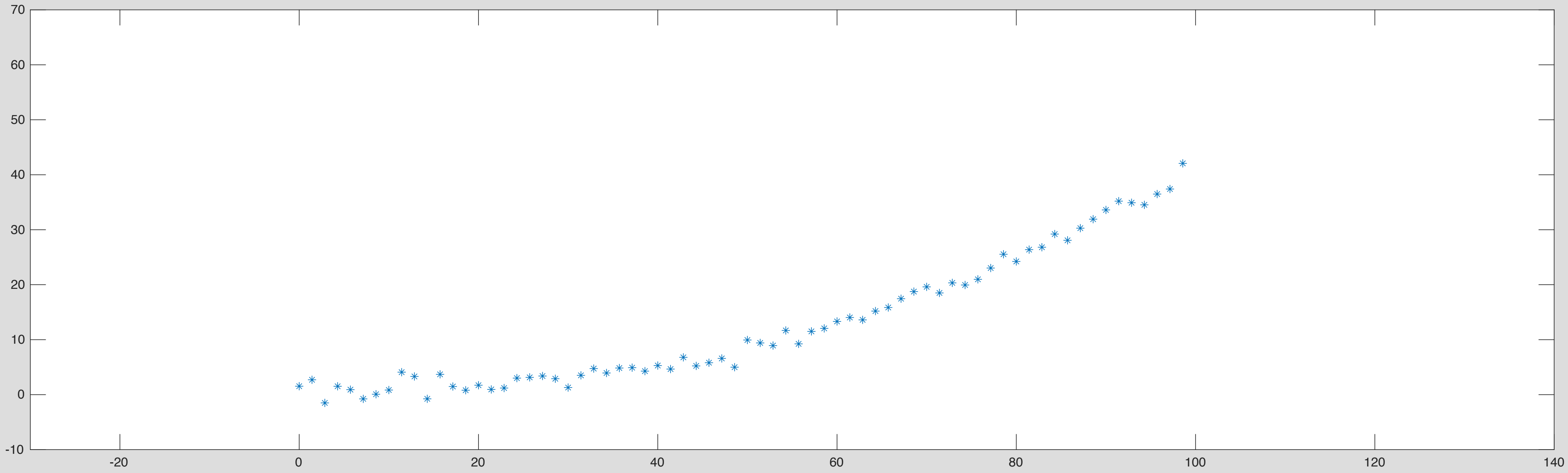
- Consider noisy measurements of $y = 0.1x + 1$:

Model: $y = ax^7 + bx^6 + cx^5 + dx^4 + ex^3 + fx^2 + gx + h$

$$\|\overrightarrow{e}\| = 2.42$$

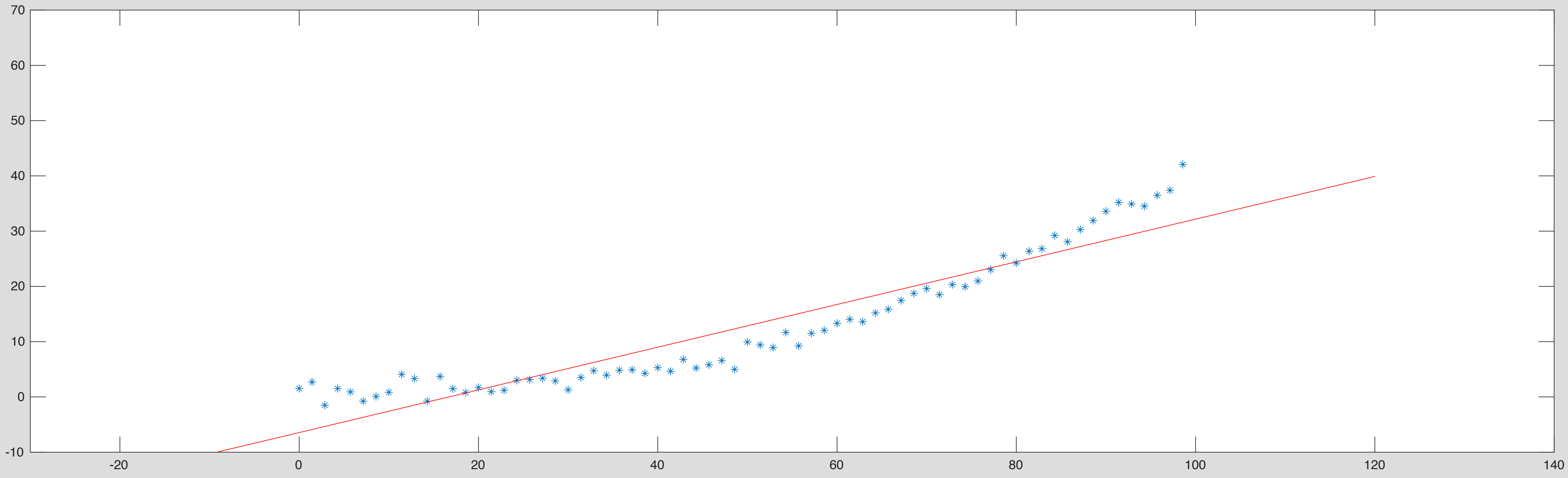


Example 6: Model Order Selection



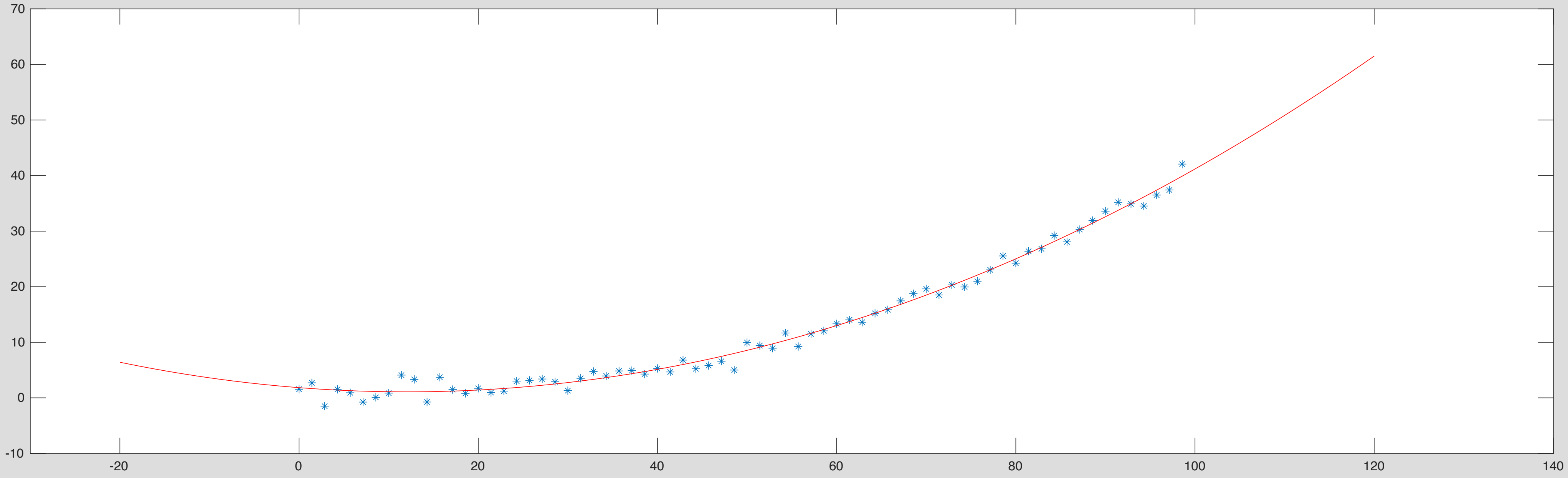
Example 6: Model Order Selection

Model: $y = ax + b$



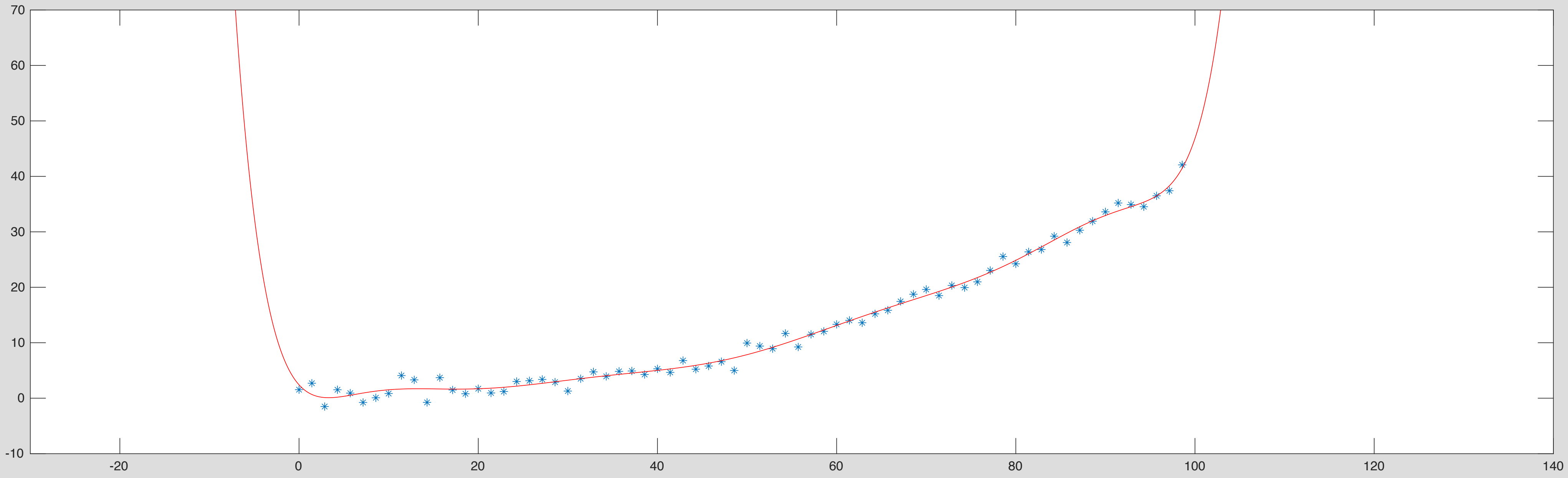
Example 6: Model Order Selection

Model: $y = ax^2 + bx + c$

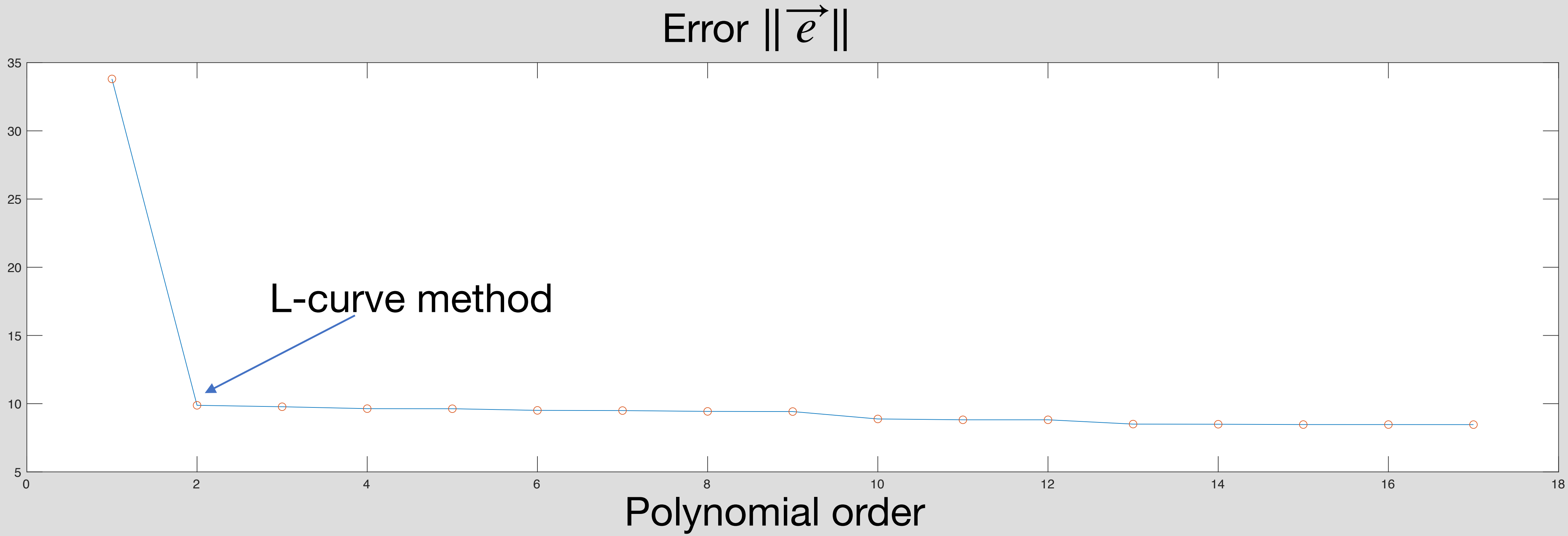


Example 6: Model Order Selection

Model: $y = ax^{10} + bx^9 + cx^8 + dx^7 + ex^6 + fx^5 + gx^4 + hx^3 + ix^2 + jx + k$



Example 6: Model Order Selection

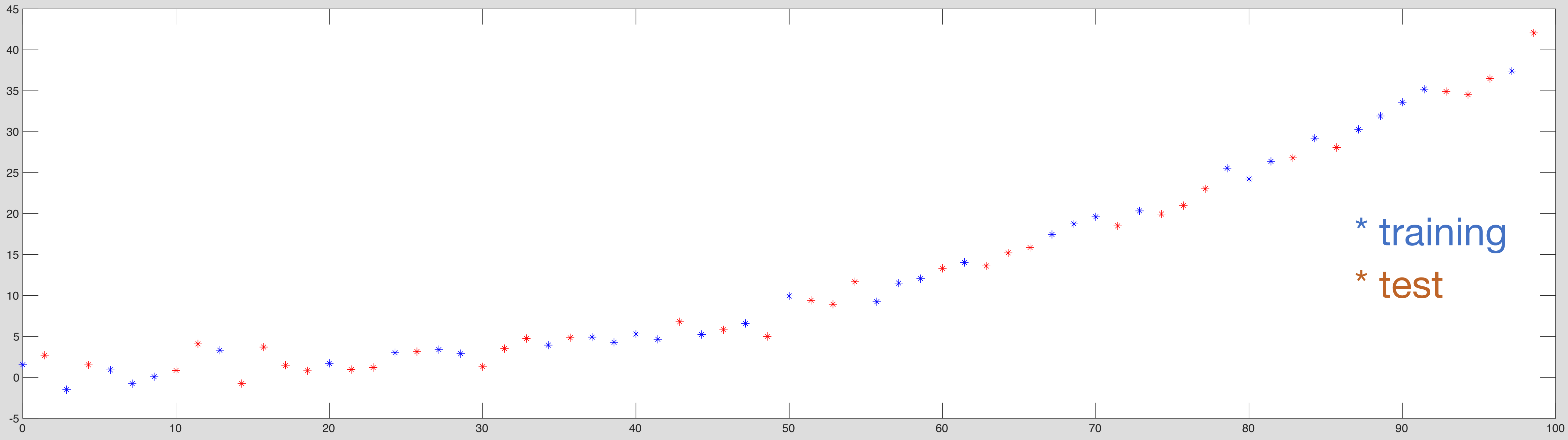


Example 6: Model Order Selection

Split data into training / test sets

Fit model on training set

Evaluate error on test set



Example 6: Model Order Selection

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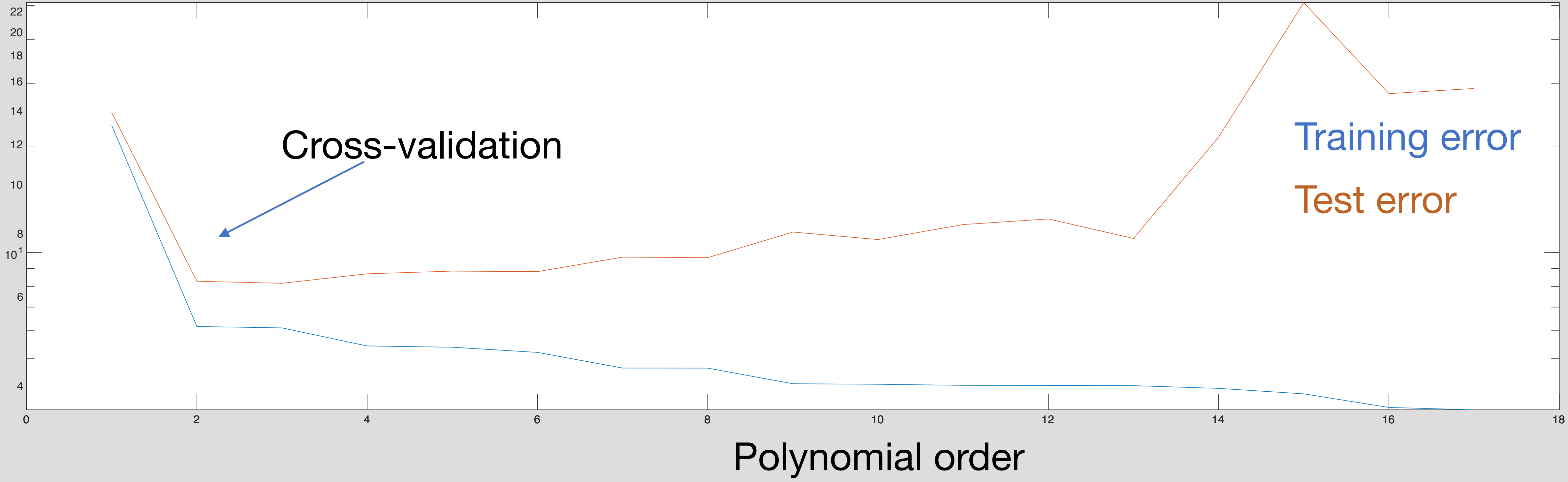
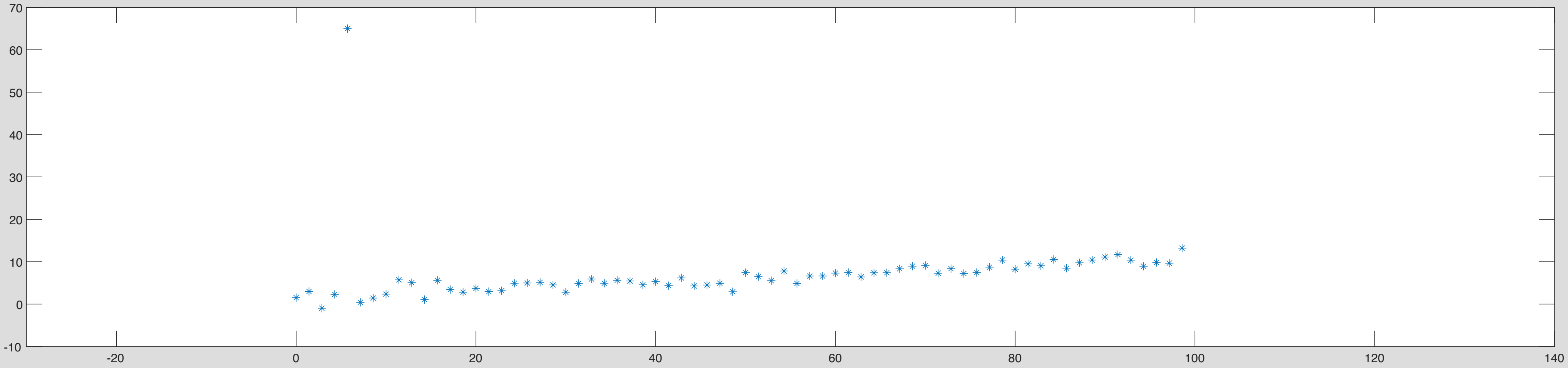


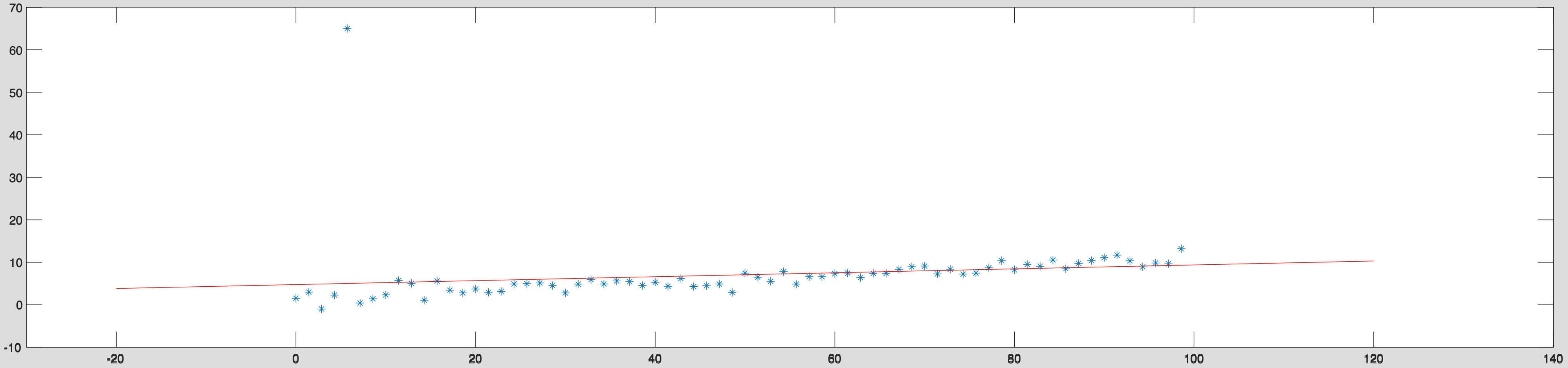
Example 7: Outlier

Model: $y = ax + b$



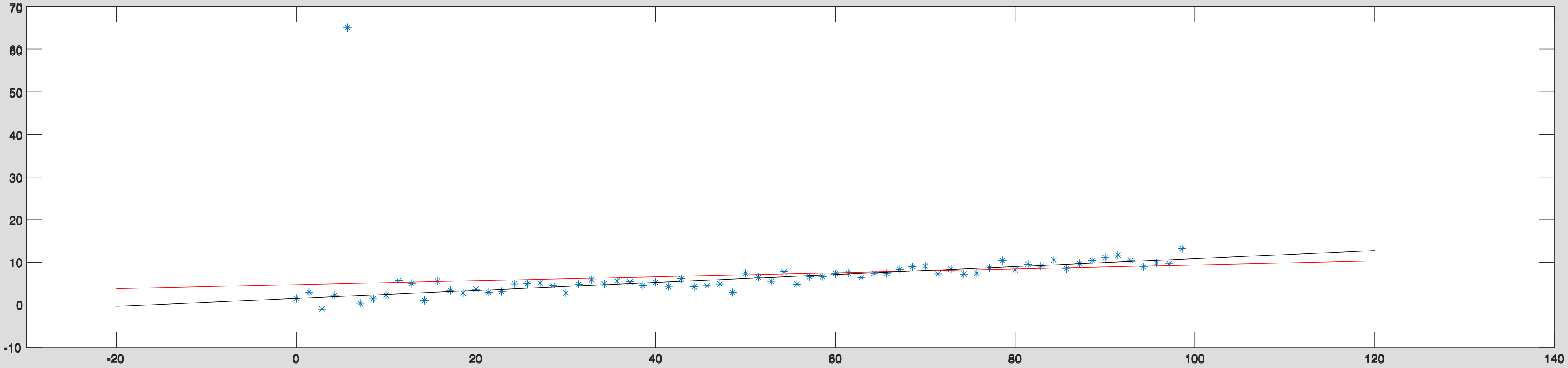
Example 7: Outlier

Model: $y = ax + b$



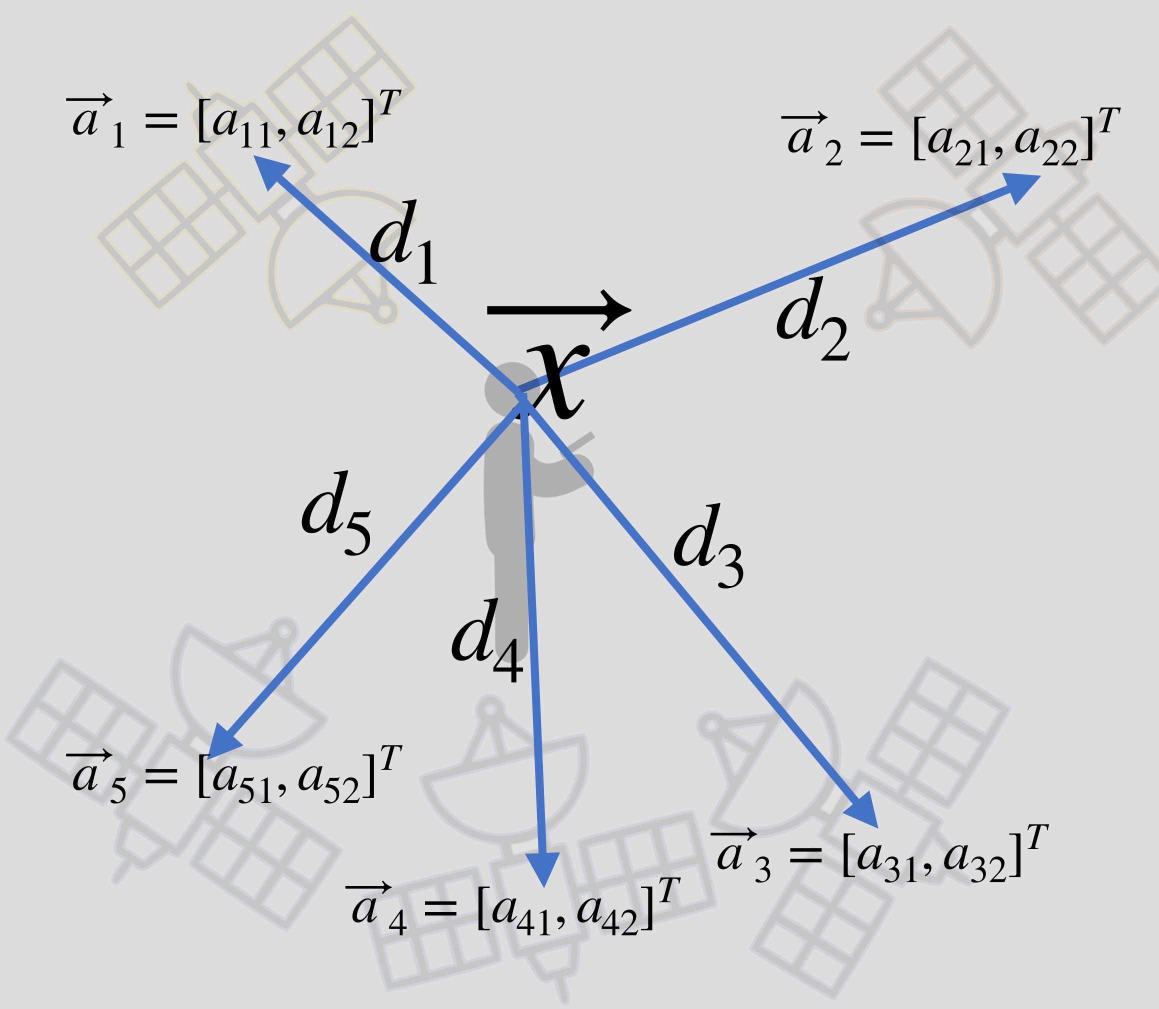
Example 7: Outlier

Model: $y = ax + b$

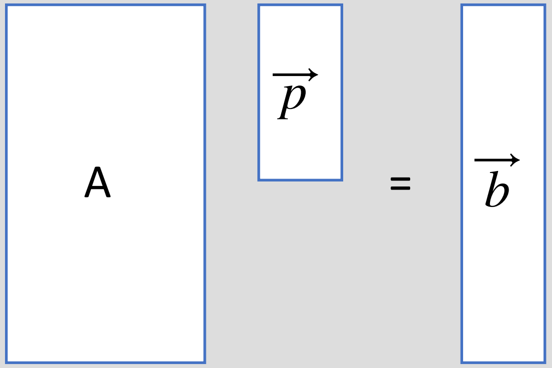


Multi-Lateration

$$2(\vec{a}_1 - \vec{a}_2)^T\vec{x} - 2C^2\Delta\tau_2\tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 + C^2(\Delta\tau_2)^2$$
$$2(\vec{a}_1 - \vec{a}_3)^T\vec{x} - 2C^2\Delta\tau_3\tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 + C^2(\Delta\tau_3)^2$$
$$2(\vec{a}_1 - \vec{a}_4)^T\vec{x} - 2C^2\Delta\tau_4\tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_4\|^2 + C^2(\Delta\tau_4)^2$$
$$2(\vec{a}_1 - \vec{a}_5)^T\vec{x} - 2C^2\Delta\tau_5\tau_1 = \|\vec{a}_1\|^2 - \|\vec{a}_5\|^2 + C^2(\Delta\tau_5)^2$$



More equations than unknowns



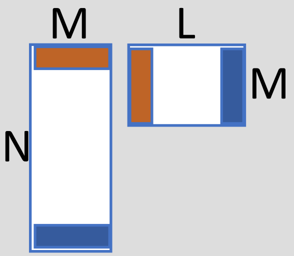
Over-determined — Solve via Least-Squares

Q: How do we know if A^TA is invertible?

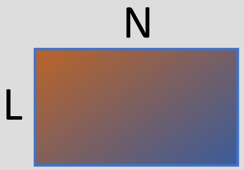
A: if A is full rank!?!?

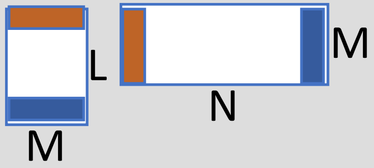
$$\hat{p} = (A^TA)^{-1}A^T\vec{y}$$

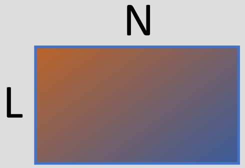
Matrix Transposes

$(AB)^T$ $\Big($ 

$\Big)^T =$ $\Big($ 

$\Big)^T \in \mathbb{R}^{L \times N}$ 

$B^T A^T$ 

$\in \mathbb{R}^{L \times N}$ 

$$(AB)^T = B^T A^T$$

Invertibility of A^TA

- Invertible ⇒ Trivial null space ⇒ Linear independent cols/rows….

The matrix A^TA is invertible <u>iff</u> $\text{Null}\left(A^TA\right) = \vec{0}$

Theorem: $\text{Null}\left(A^TA\right) = \text{Null}\left(A\right)$

Proof: (1) show that if $\vec{w} \in \text{Null}(A)$, then $\vec{w} \in \text{Null}(A^TA)$

(2) show that if $\vec{v} \in \text{Null}(A^TA)$, then $\vec{v} \in \text{Null}(A)$

(1). $\vec{w} \in \text{Null}(A)$

$$A\vec{w} = \vec{0}$$

$$A^TA\vec{w} = A^T\vec{0}$$

$$A^TA\vec{w} = \vec{0}$$ ✅

(2). $\vec{v} \in \text{Null}(A^TA)$

$$A^TA\vec{v} = \vec{0}$$

Need to show $A\vec{v} = \vec{0}$

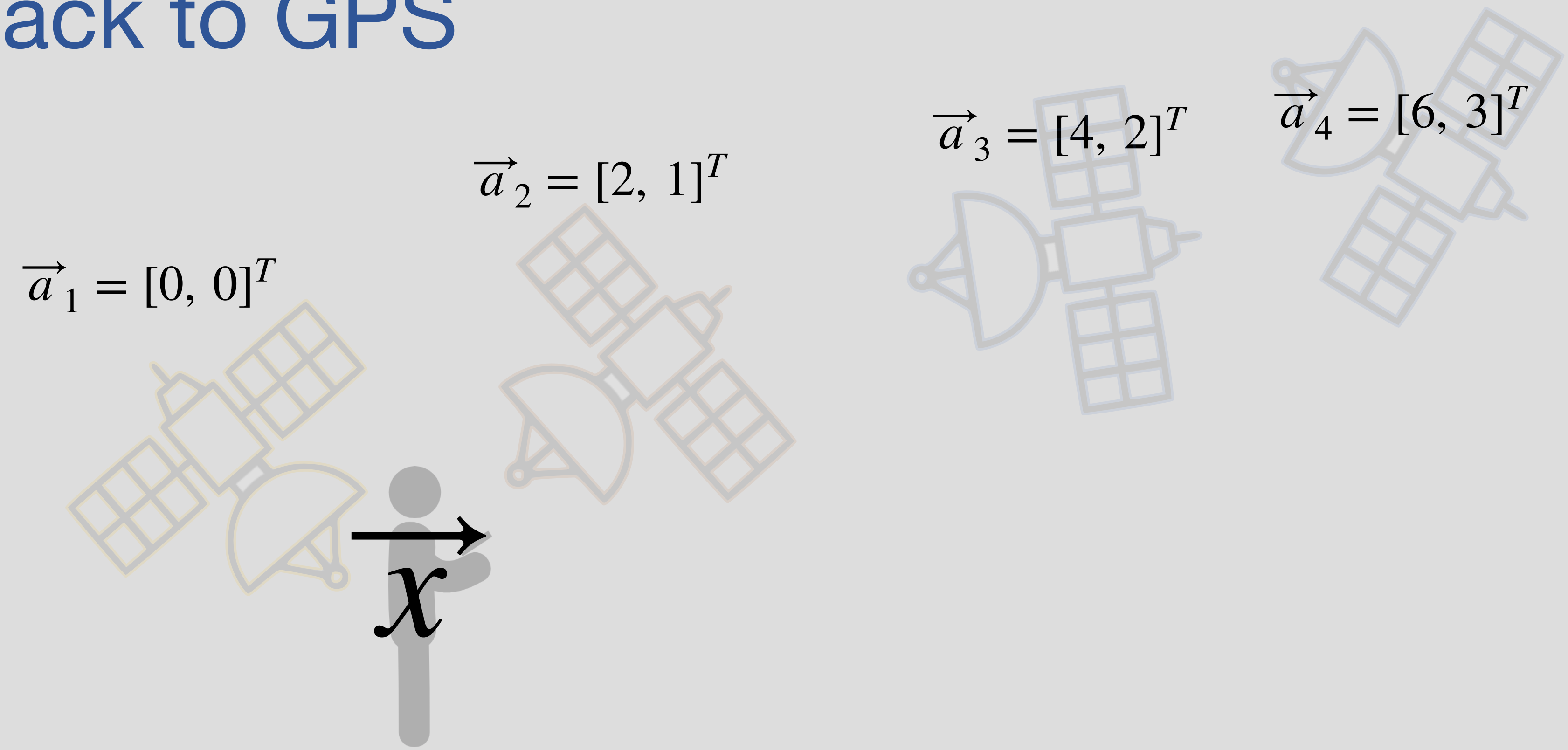
Or… $\|A\vec{v}\| = 0$

$$\|A\vec{v}\|^2 = (A\vec{v})^T(A\vec{v})$$

$$= \vec{v}^TA^T(A\vec{v})$$

$$= \vec{v}^T(A^TA\vec{v}) = 0$$ ✅

Back to GPS



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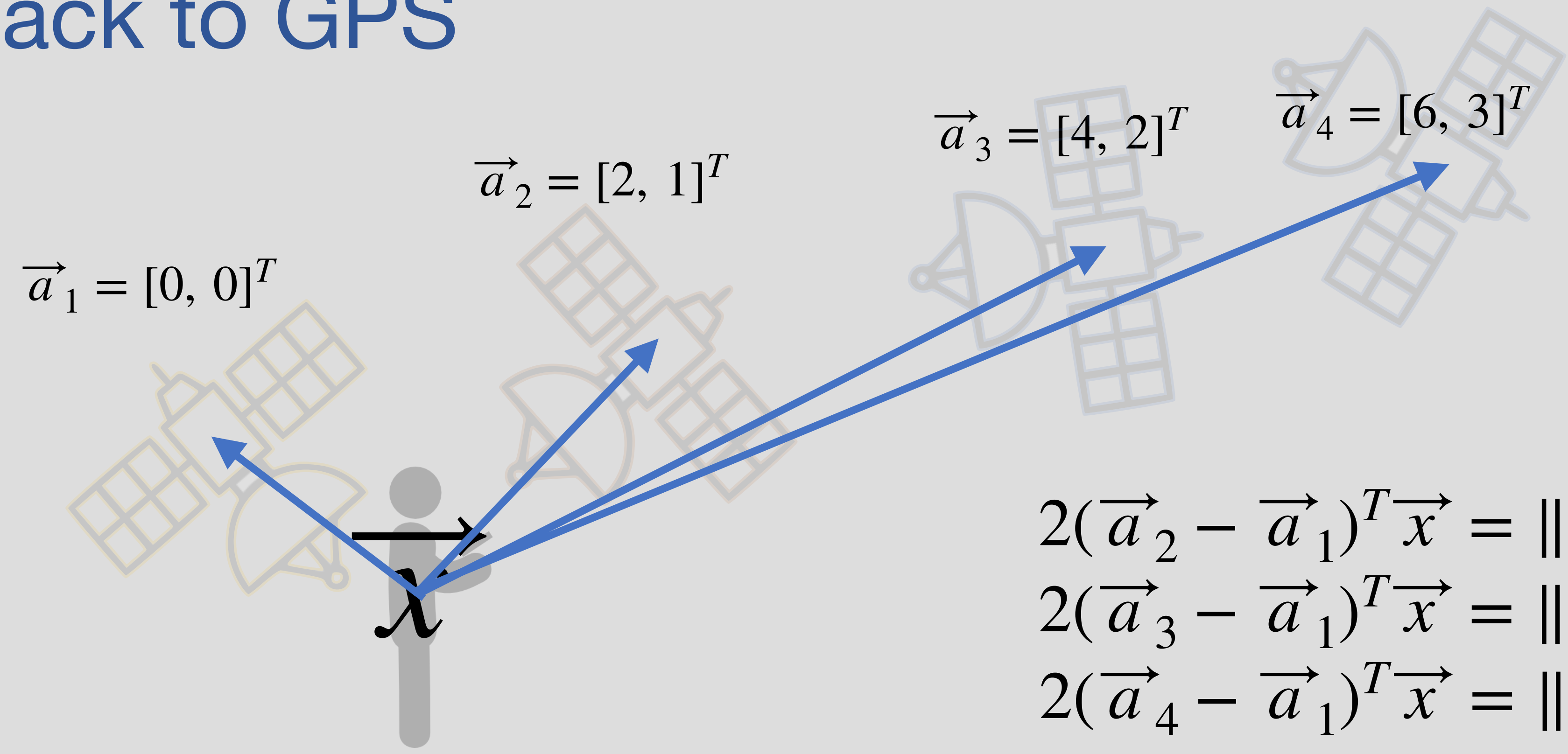
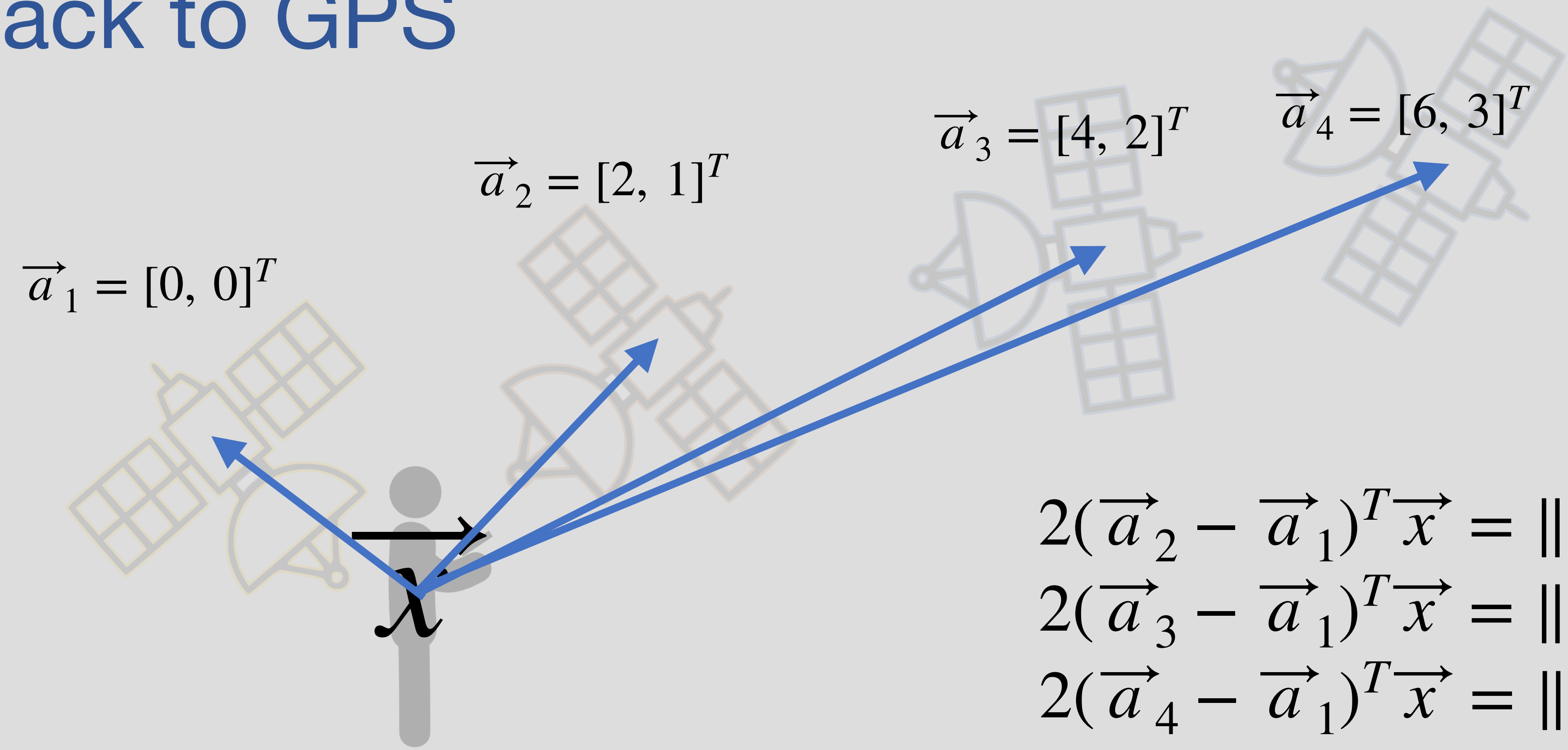
Back to GPS



$$2(\vec{a}_2 - \vec{a}_1)^T \vec{x} = \|\vec{a}_2\|^2 - \|\vec{a}_1\|^2 + d_1^2 - d_2^2$$
$$2(\vec{a}_3 - \vec{a}_1)^T \vec{x} = \|\vec{a}_3\|^2 - \|\vec{a}_1\|^2 + d_1^2 - d_3^2$$
$$2(\vec{a}_4 - \vec{a}_1)^T \vec{x} = \|\vec{a}_4\|^2 - \|\vec{a}_1\|^2 + d_1^2 - d_4^2$$

=

Back to GPS



$$2(\vec{a}_2 - \vec{a}_1)^T \vec{x} = \|\vec{a}_2\|^2 - \|\vec{a}_1\|^2 + d_1^2 - d_2^2$$
$$2(\vec{a}_3 - \vec{a}_1)^T \vec{x} = \|\vec{a}_3\|^2 - \|\vec{a}_1\|^2 + d_1^2 - d_3^2$$
$$2(\vec{a}_4 - \vec{a}_1)^T \vec{x} = \|\vec{a}_4\|^2 - \|\vec{a}_1\|^2 + d_1^2 - d_4^2$$

$$\begin{bmatrix} 4 & 2 \\ 8 & 4 \\ 12 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \|\vec{a}_2\|^2 + d_1^2 - d_2^2 \\ \|\vec{a}_3\|^2 + d_1^2 - d_3^2 \\ \|\vec{a}_4\|^2 + d_1^2 - d_4^2 \end{bmatrix}$$

Back to GPS

$$\begin{bmatrix} 4 & 2 \\ 8 & 4 \\ 12 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \|\vec{a}_2\|^2 + d_1^2 - d_2^2 \\ \|\vec{a}_3\|^2 + d_1^2 - d_3^2 \\ \|\vec{a}_4\|^2 + d_1^2 - d_4^2 \end{bmatrix}$$

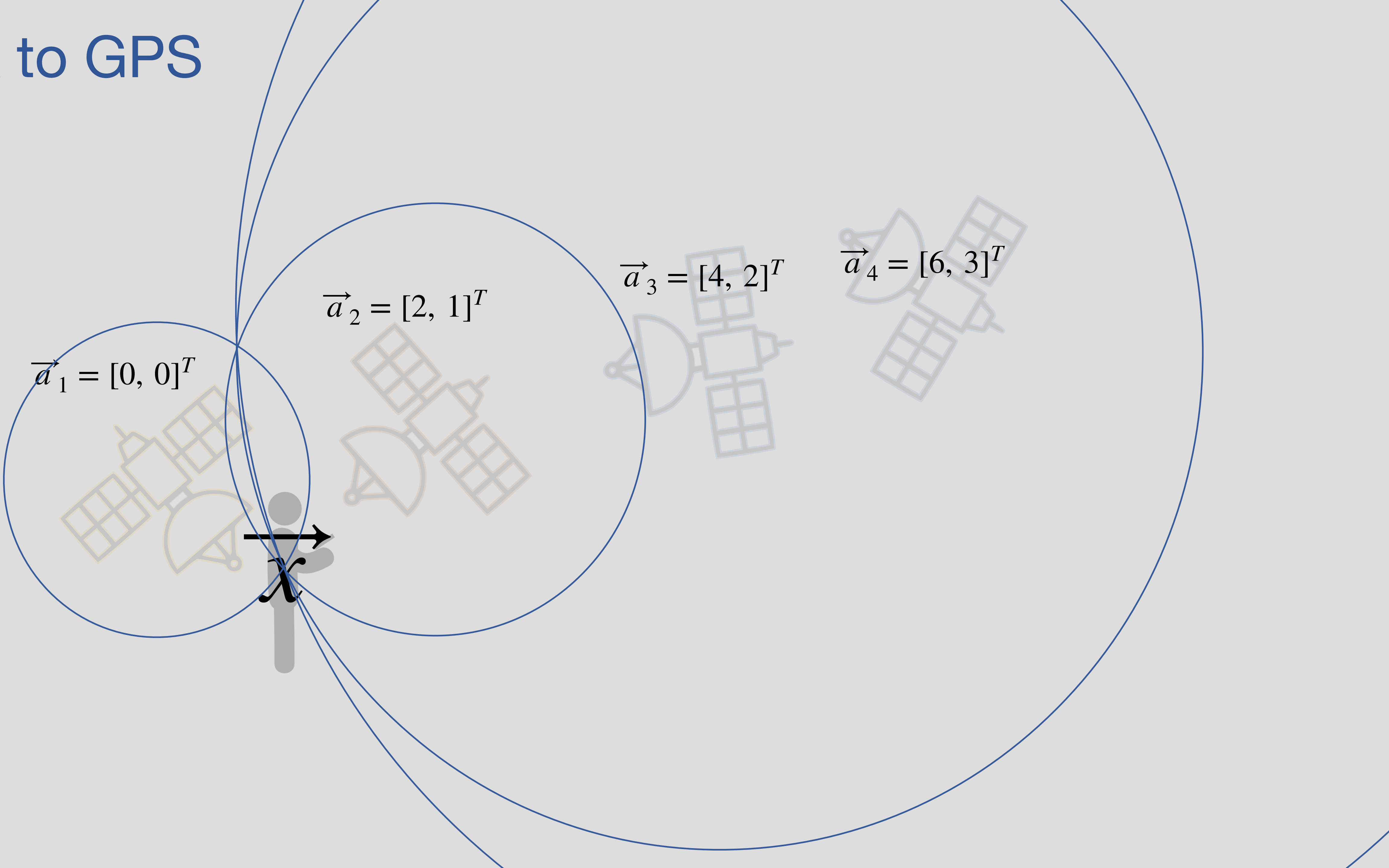
$$\begin{bmatrix} 4 & 8 & 12 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 8 & 4 \\ 12 & 6 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

Back to GPS

$$\begin{bmatrix} 4 & 2 \\ 8 & 4 \\ 12 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \|\vec{a}_2\|^2 + d_1^2 - d_2^2 \\ \|\vec{a}_3\|^2 + d_1^2 - d_3^2 \\ \|\vec{a}_4\|^2 + d_1^2 - d_4^2 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 8 & 12 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 8 & 4 \\ 12 & 6 \end{bmatrix} = \begin{bmatrix} 224 & 112 \\ 112 & 56 \end{bmatrix}$$

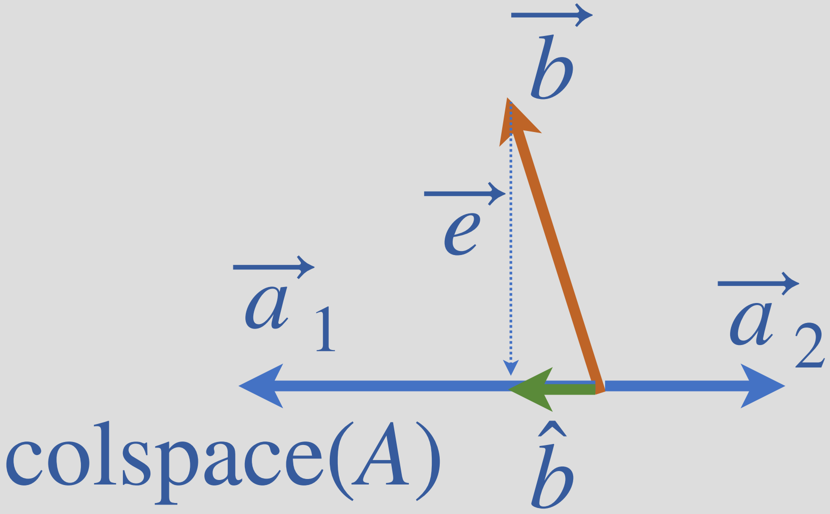
Back to GPS



Invertibility of $A^T A$

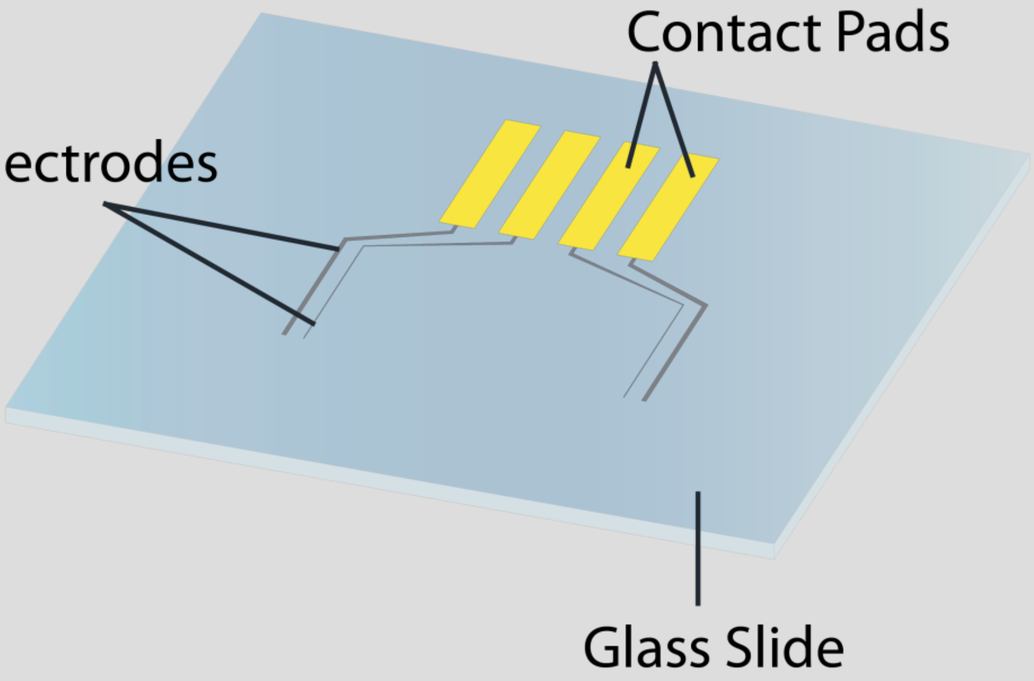
- What if $A^T A$ is not invertible

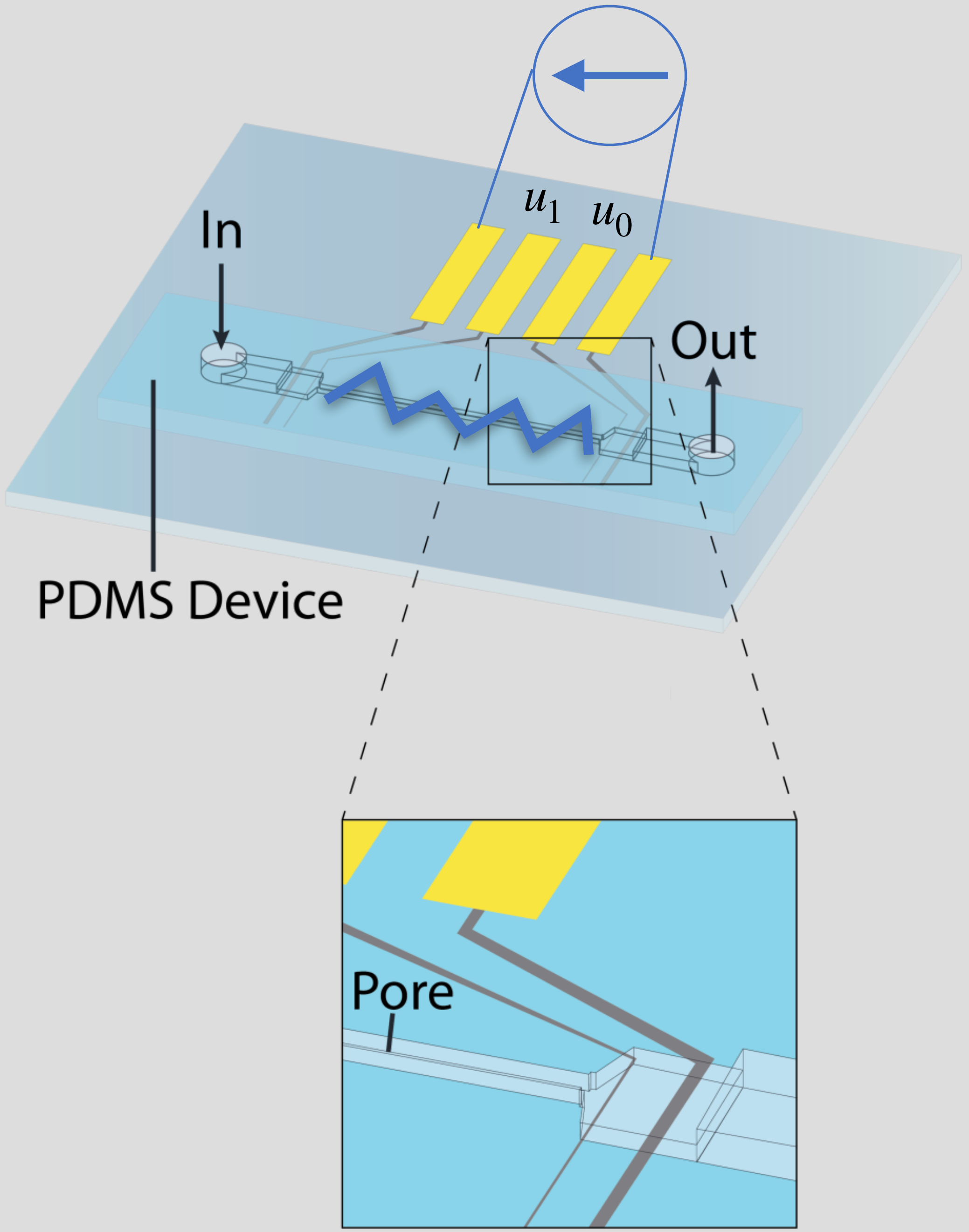
$$A^T A \hat{x} = A^T \vec{b}$$



A: $\hat{x}$ will have infinite solutions with the same $\vec{e} = A\hat{x} - \vec{b}$

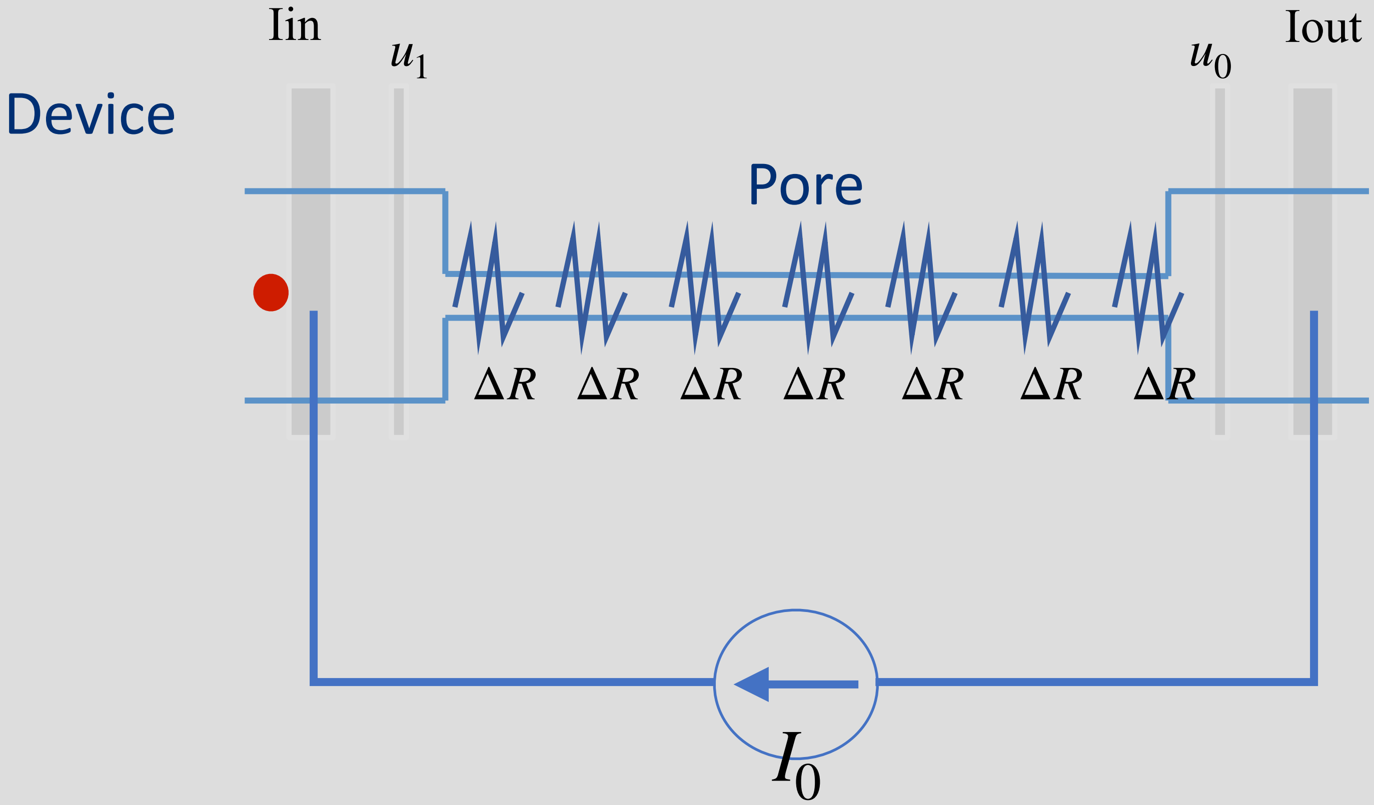
Resistive Pulse Sensing





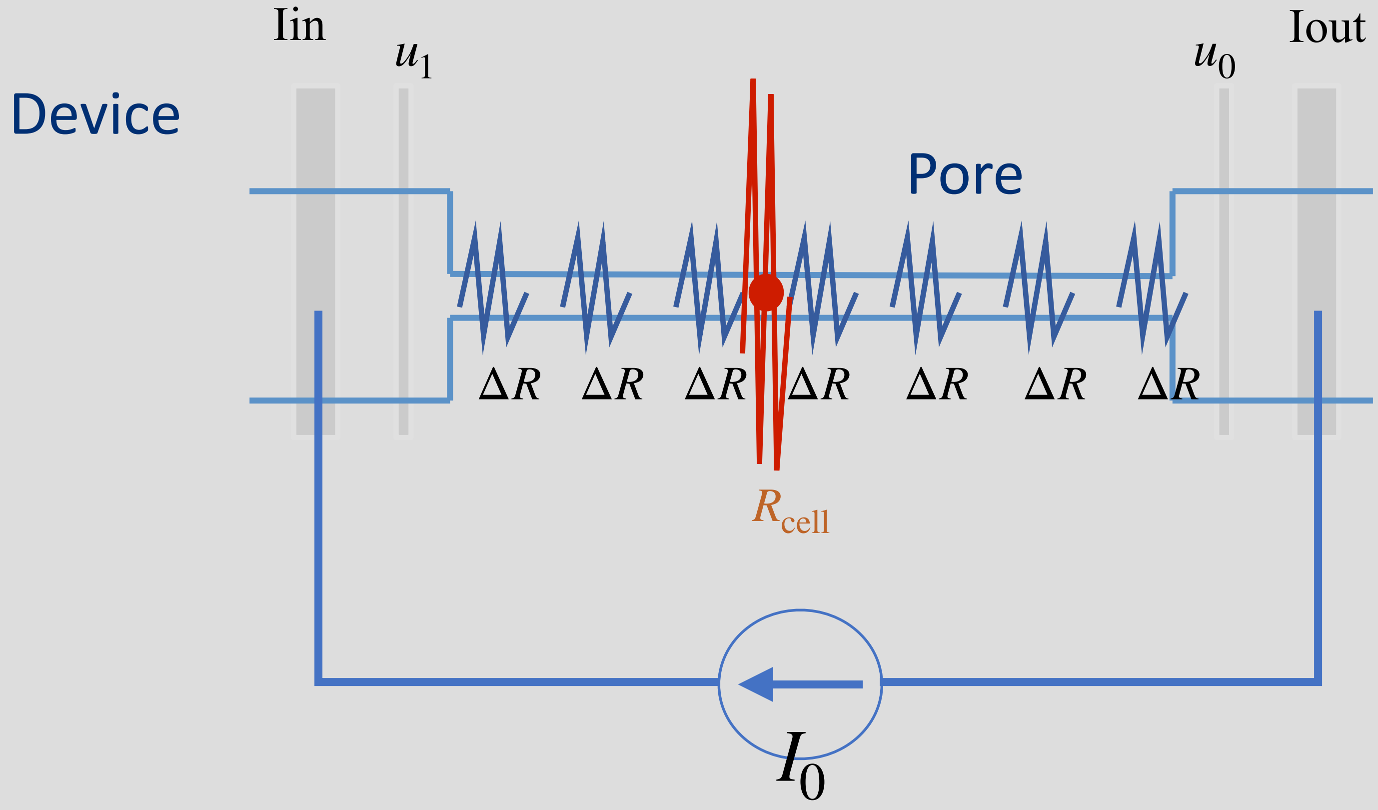
Prof. Lydia Sohn M.E.

Resistive Pulse Sensing



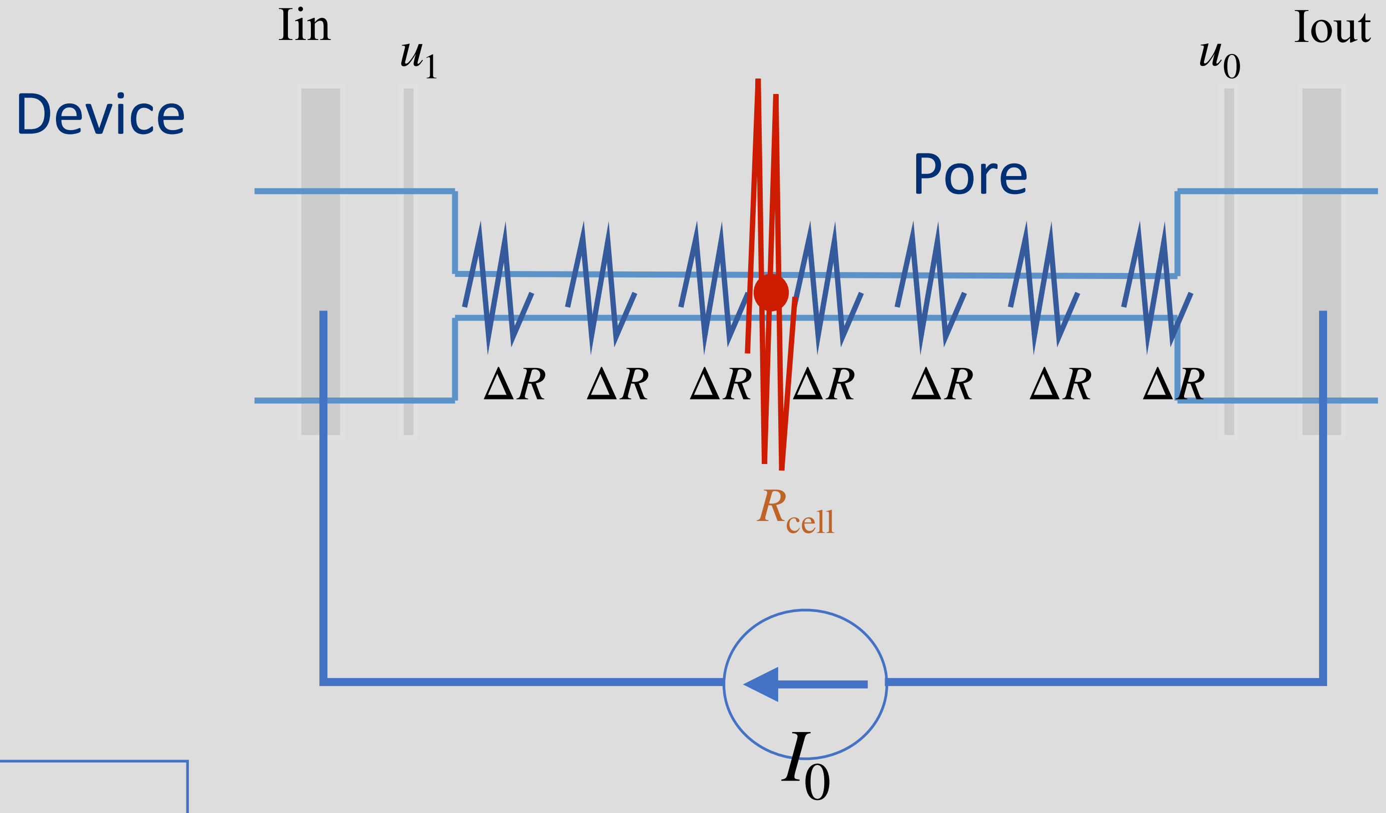
$$u_1 - u_0 = I_0 R$$

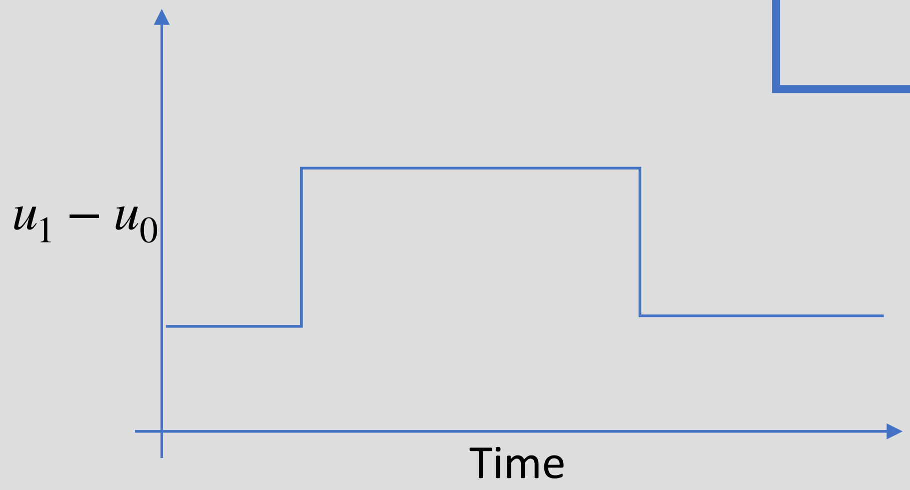
Resistive Pulse Sensing



$$u_1 - u_0 = I_0 R$$

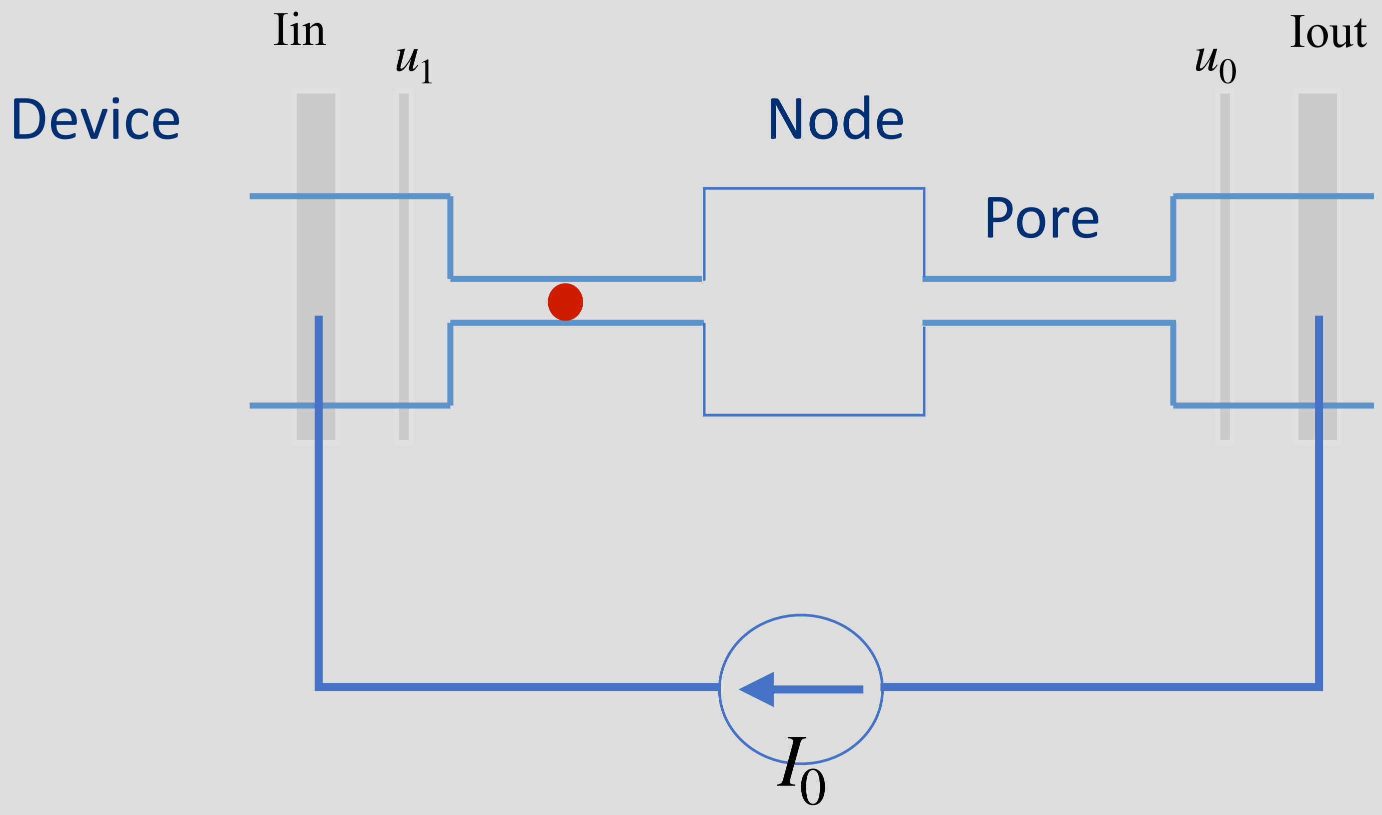
Resistive Pulse Sensing

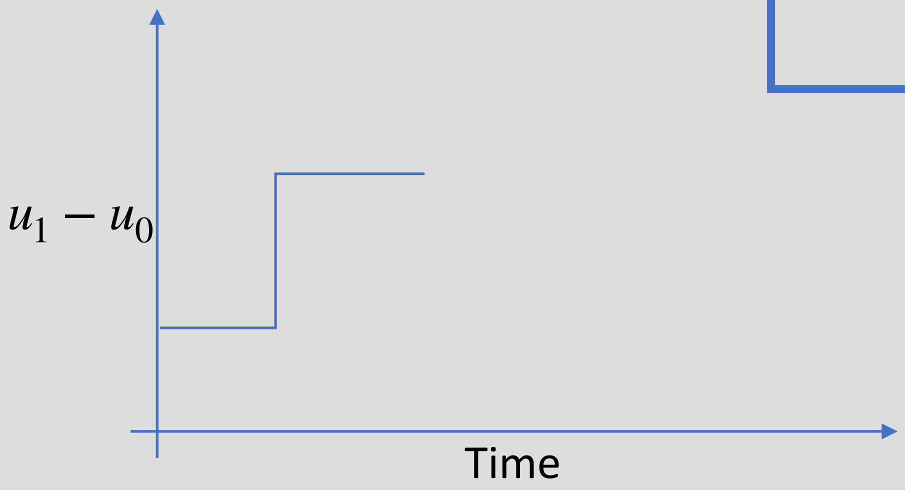




$$u_1 - u_0 = I_0(R + R_{\text{cell}})$$

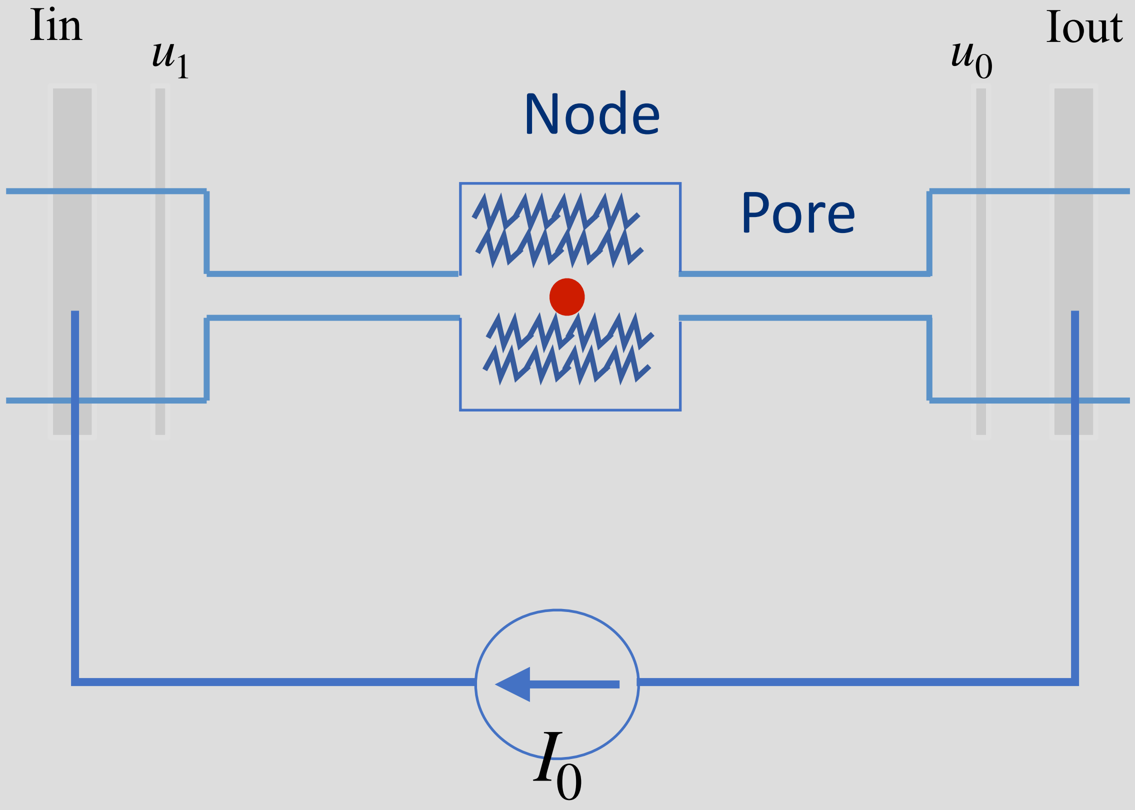
Node-Pore Sensing

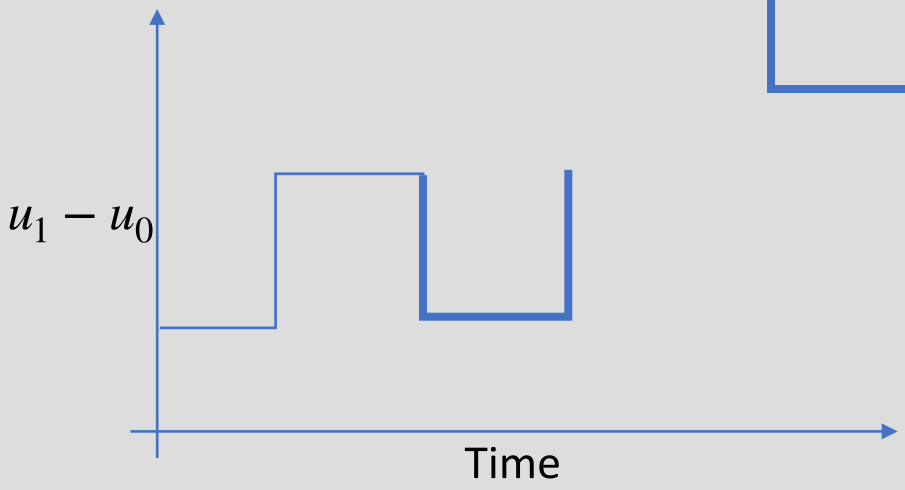




$$u_1 - u_0 = I_0(R + R_{cell} + R_{node})$$

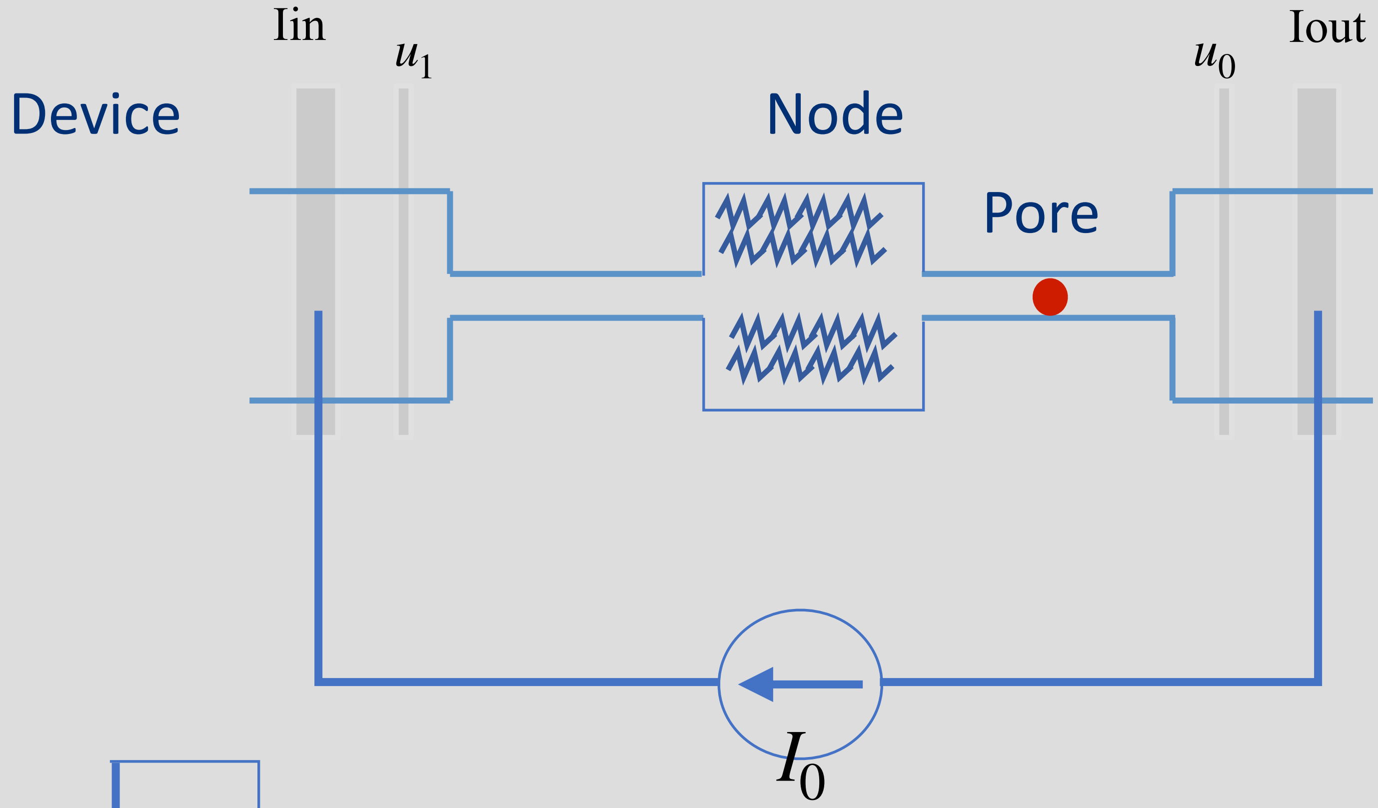
Node-Pore Sensing

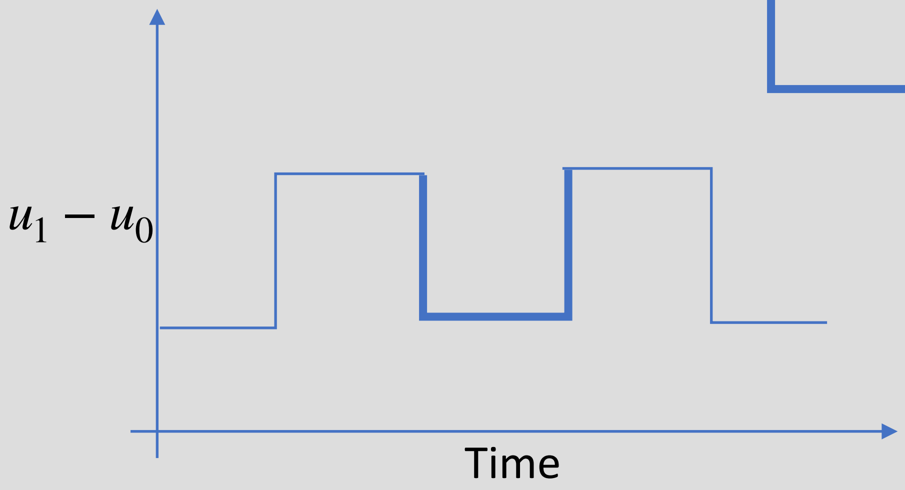




$$u_1 - u_0 = I_0(R + R_{cell} \| R_{node})$$

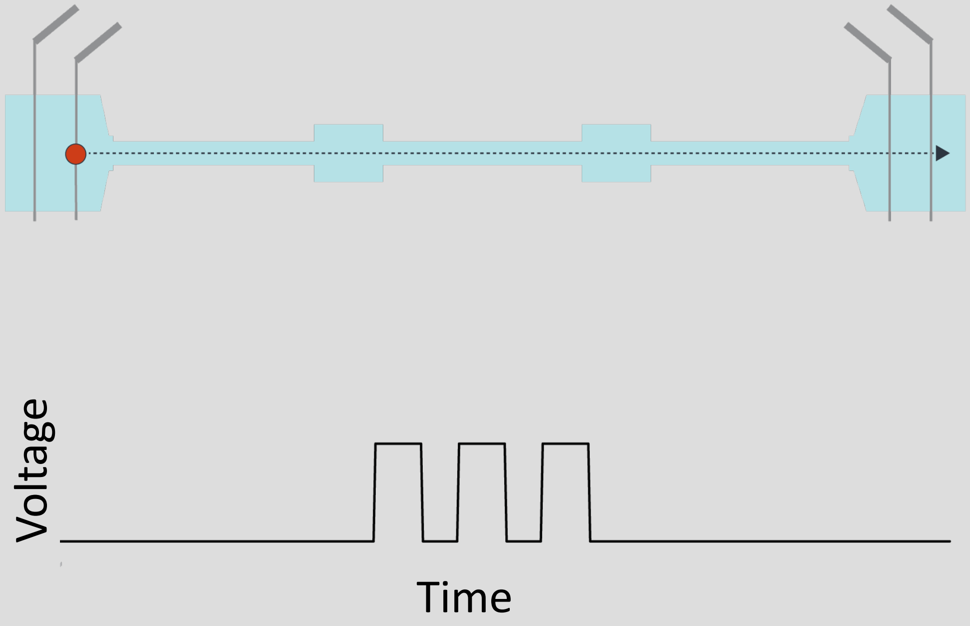
Node-Pore Sensing

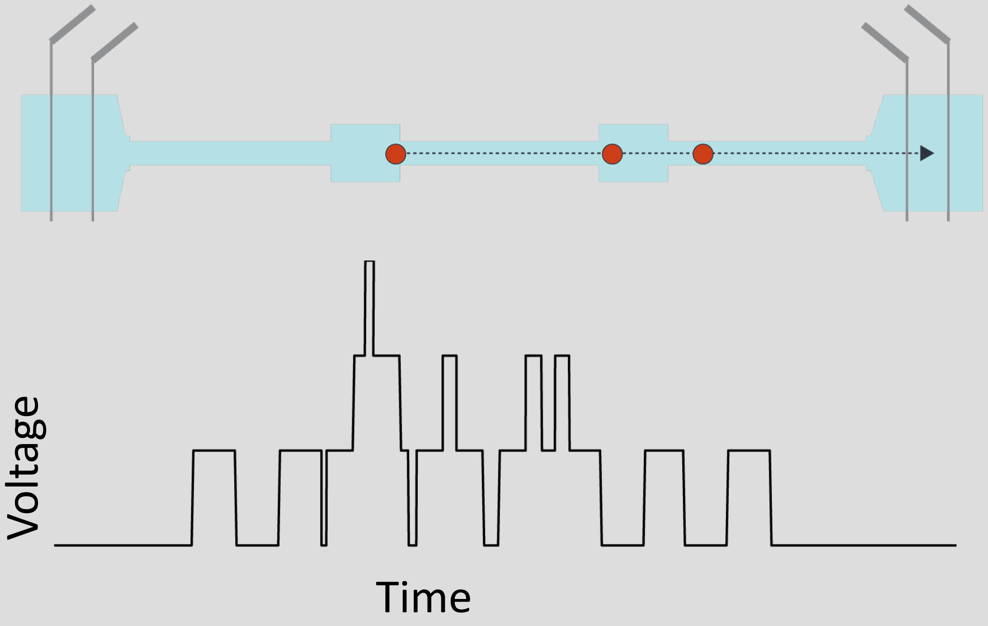




$$u_1 - u_0 = I_0(R + R_{cell} + R_{node})$$

Sensing Complexities

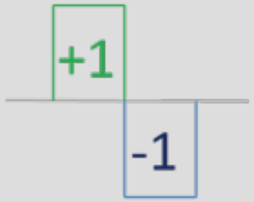




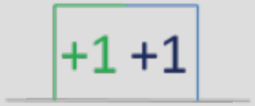
Barker Codes

- 9 uniques sequences

Barker 2 : +1 −1 or +1 +1 ->



or



Barker 3 : +1 +1 −1

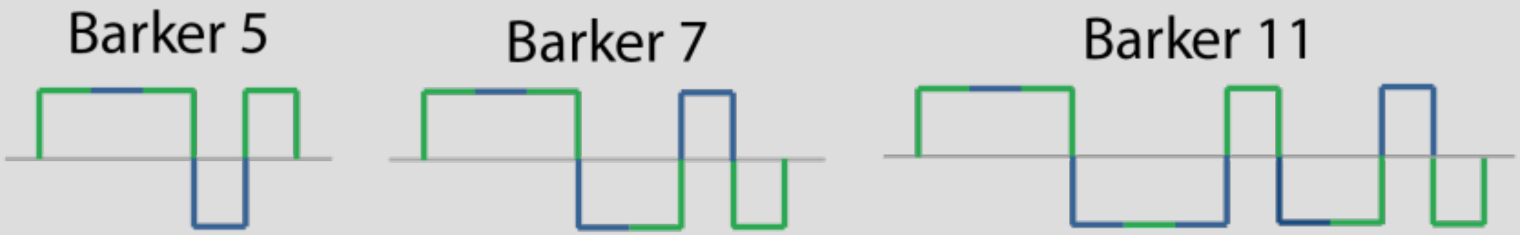
Barker 4 : +1 +1 −1 +1 or +1 +1 +1 +1 −1

Barker 5 : +1 +1 +1 −1+1

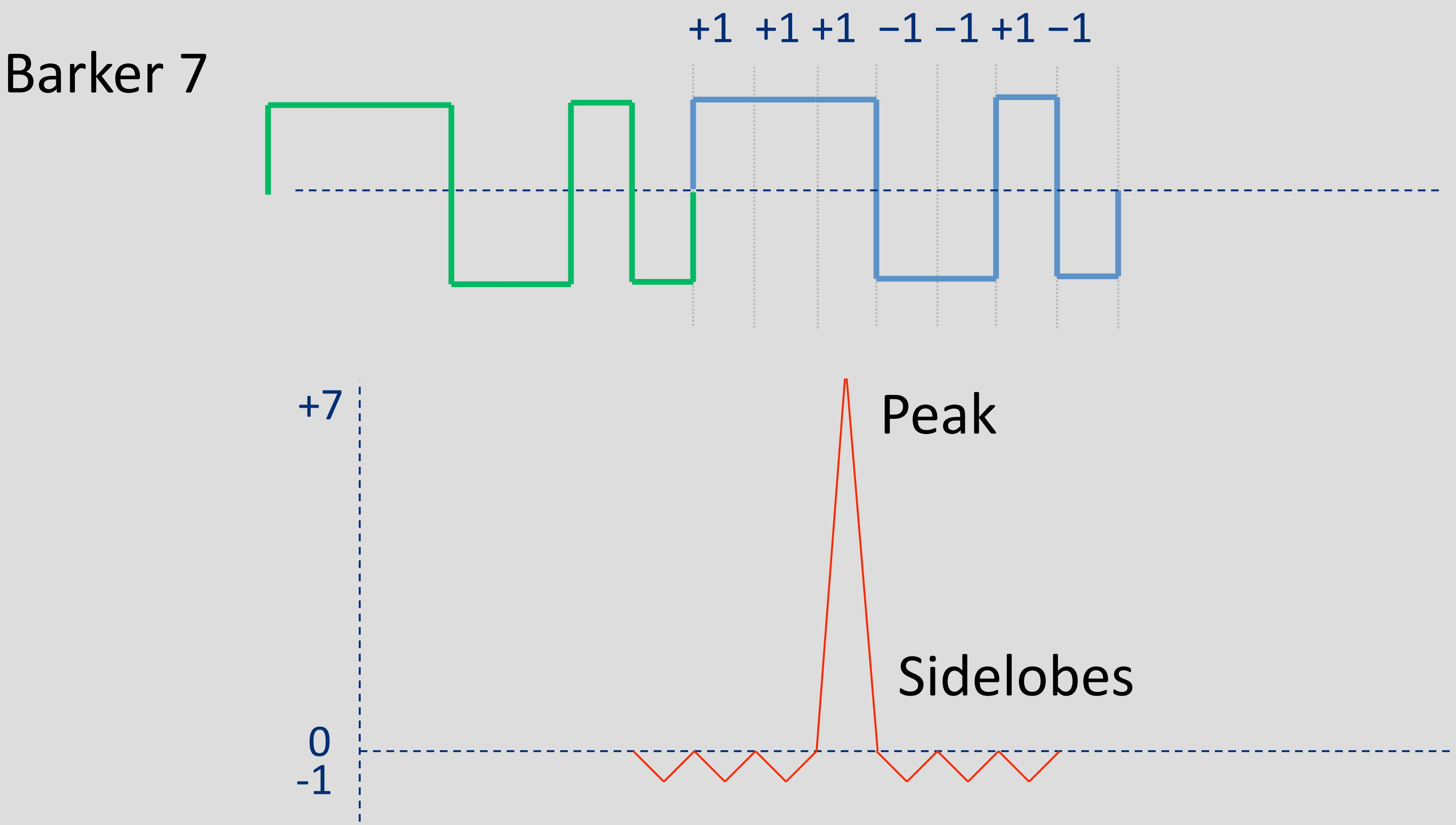
Barker 7 : +1 +1 +1 −1 −1 +1 −1

Barker 11 : +1 +1 +1 −1 −1 −1 +1 −1−1 +1 −1

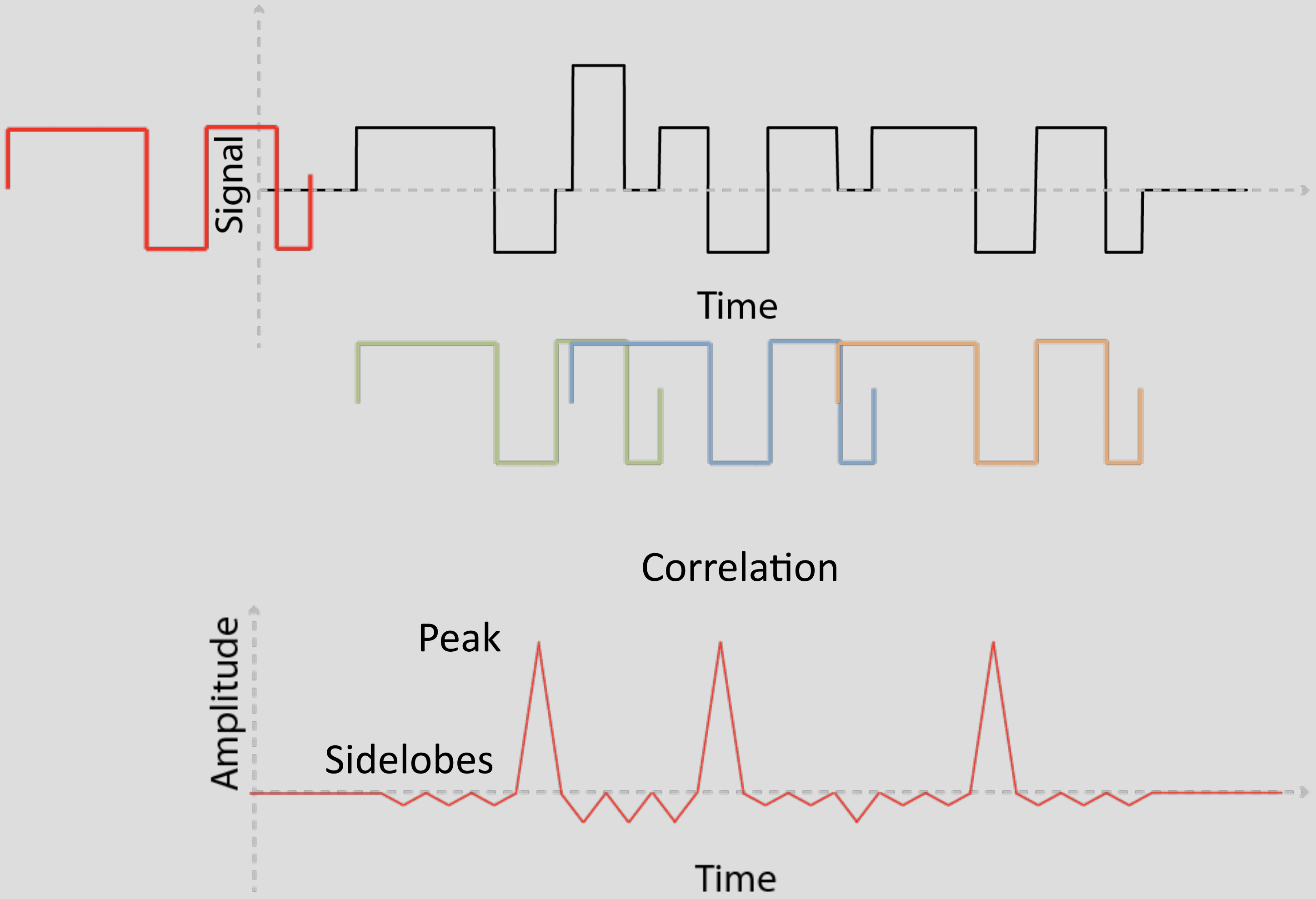
Barker 13 : +1 +1 +1 +1 +1 −1 −1 +1 +1 −1 +1 −1 +1



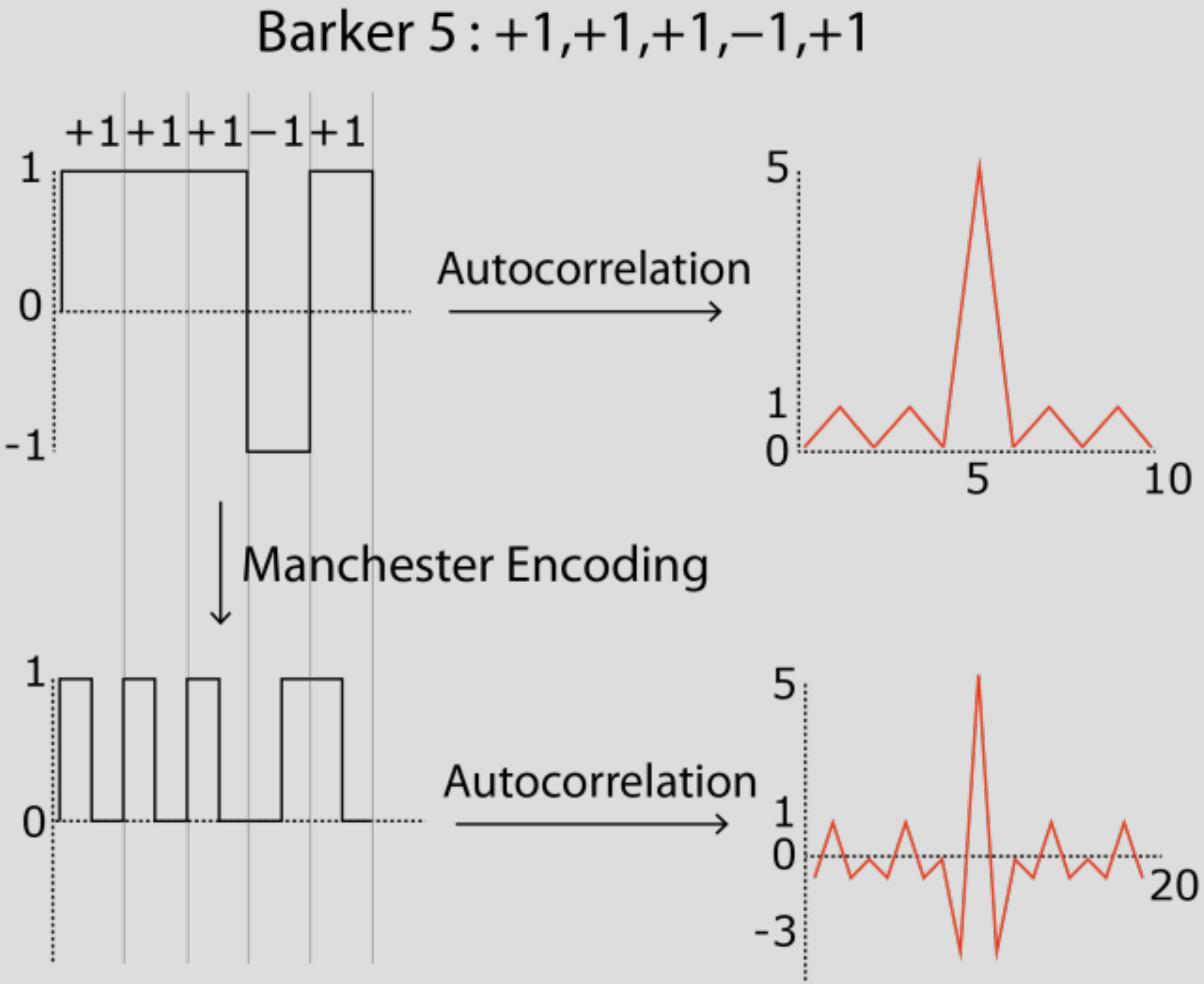
Auto-Correlation of Barker Codes



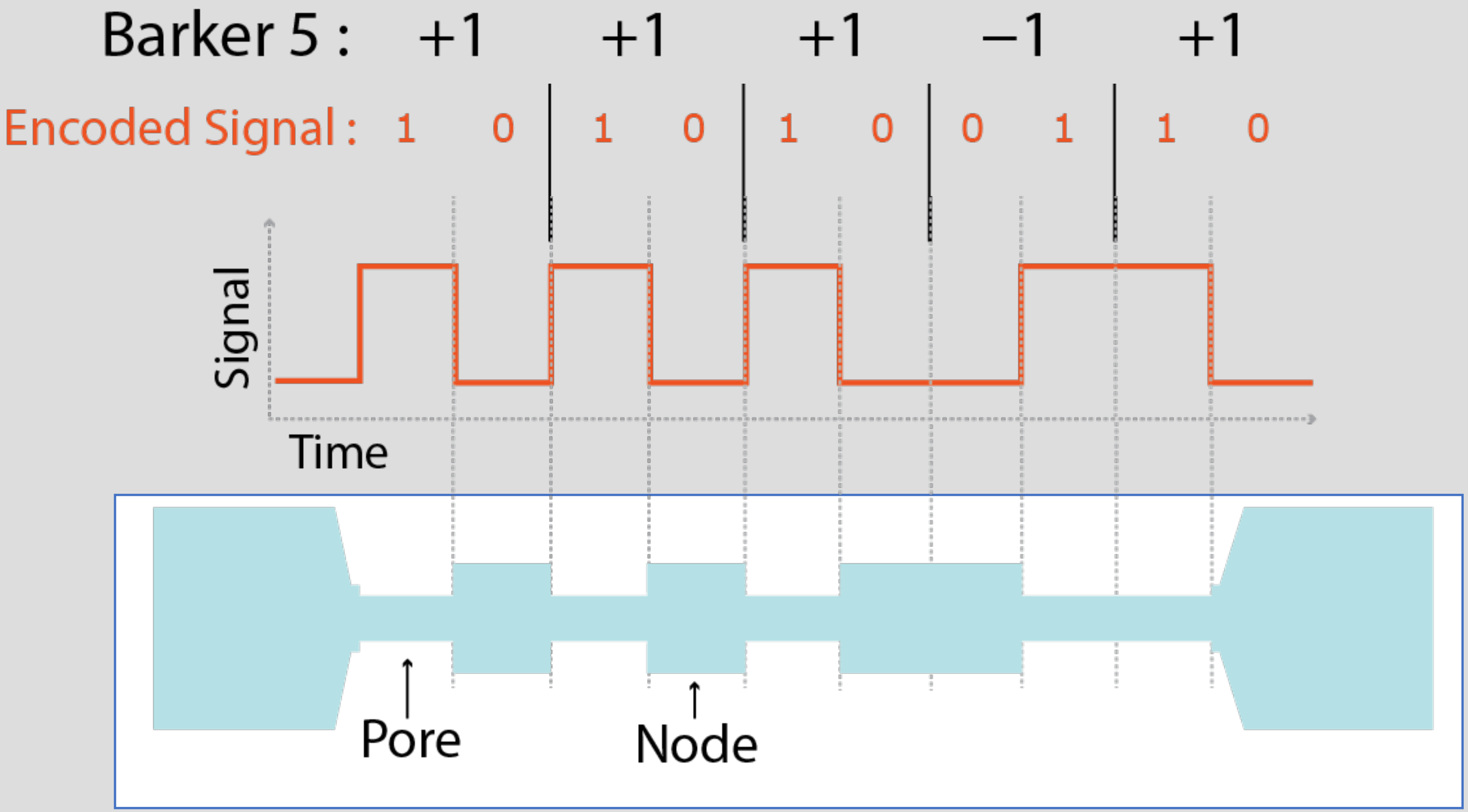
Cross Correlation with Barker Codes



Implementing Barker Codes in NPS



Encoding a Channel

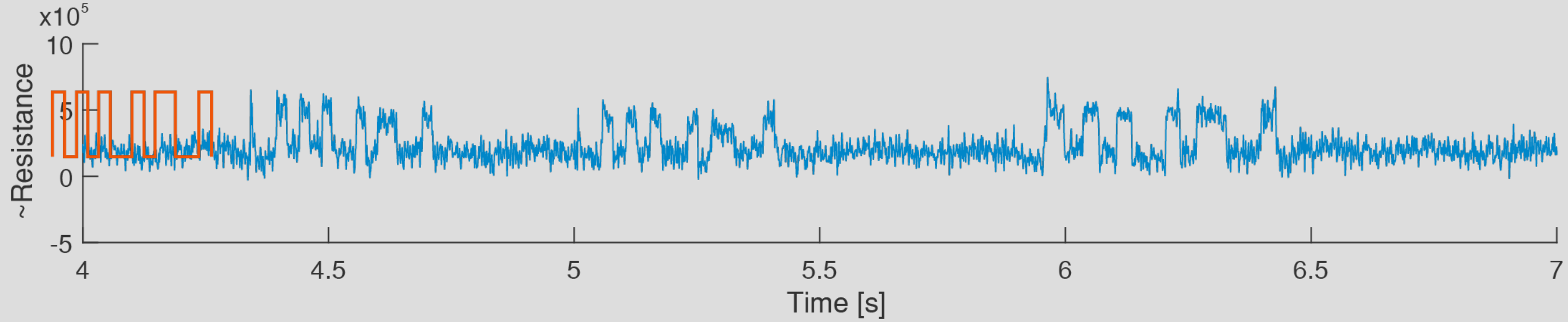


(Kellman et. al IEEE Sens. J 18(8):3068-79)

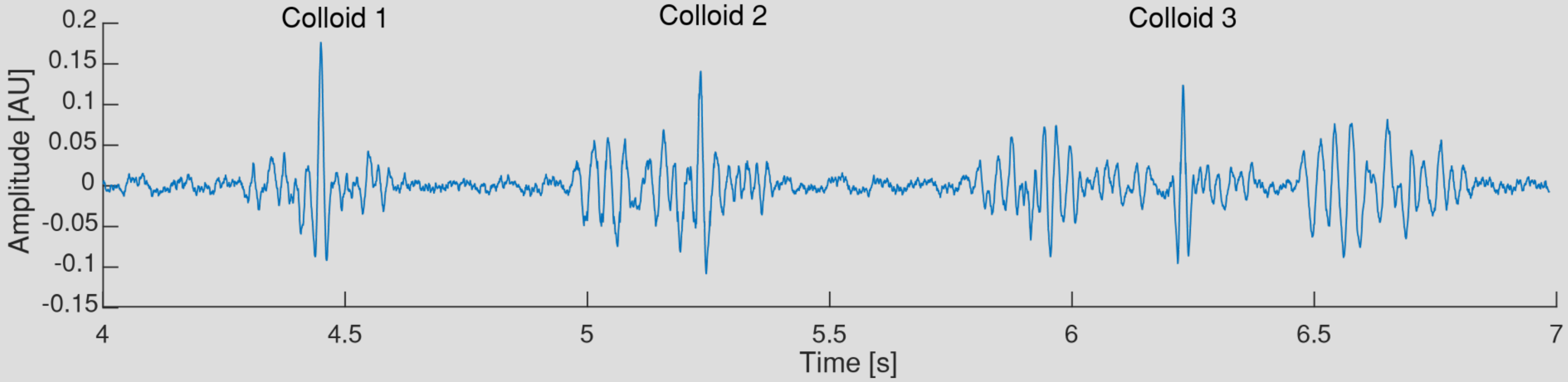
https://www.ncbi.nlm.nih.gov/pmc/articles/PMC6034687/

Real Data

26 mm/s



Correlation plot



Speed and Time