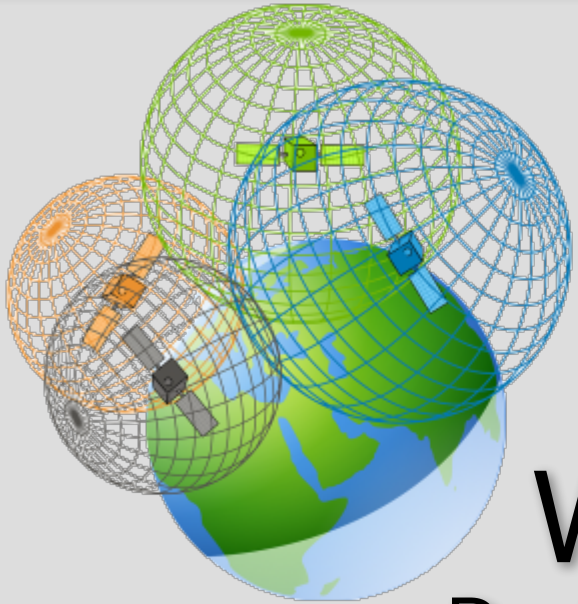
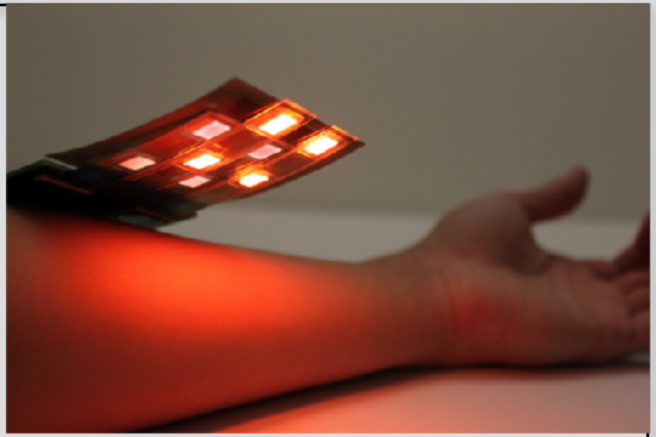


# Welcome to EECS 16A!

## Designing Information Devices and Systems I

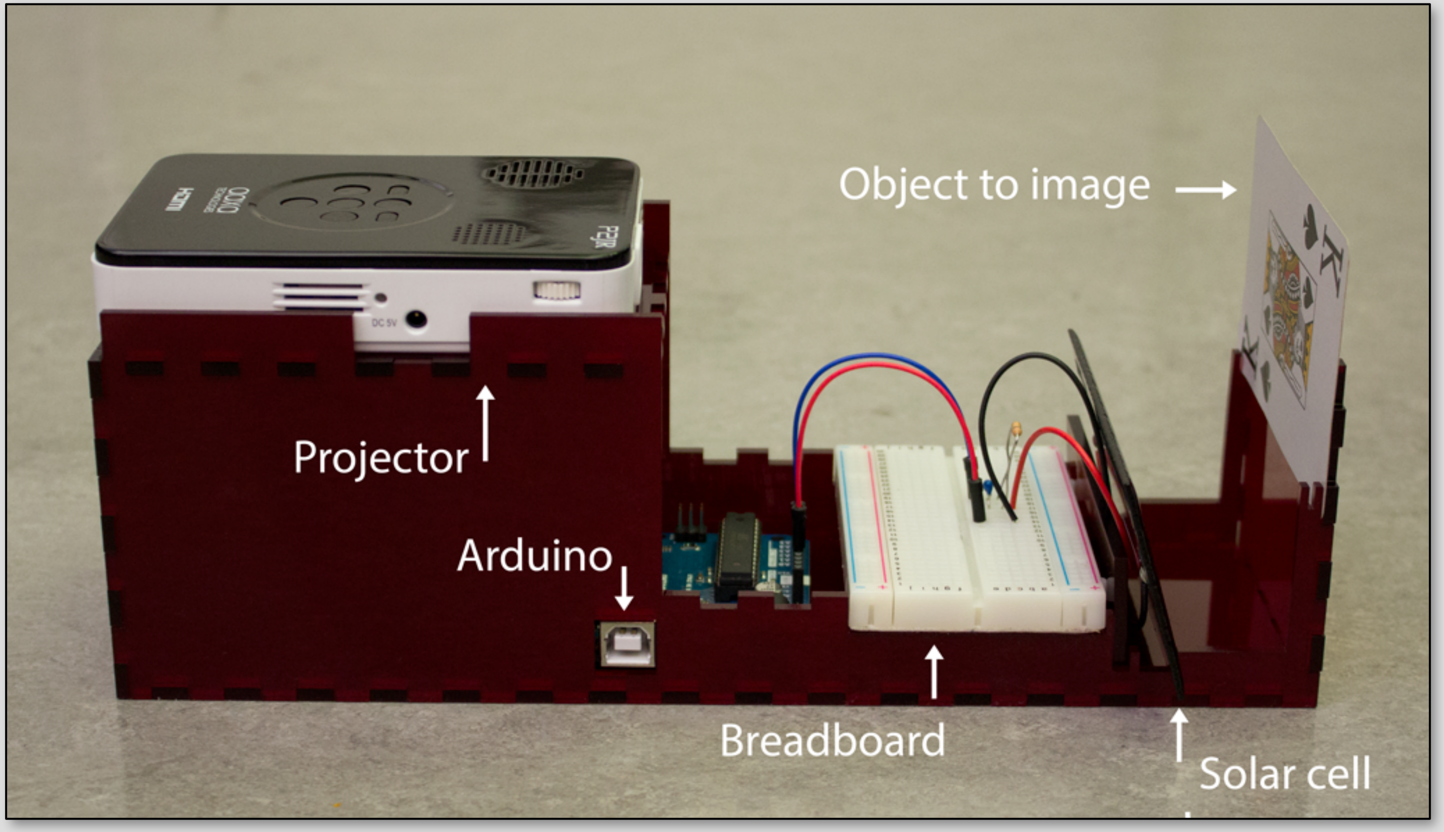
Ana Arias and Miki Lustig

2022

Lecture 1A

Tomography and Linear Equations

# Module 1: Imaging



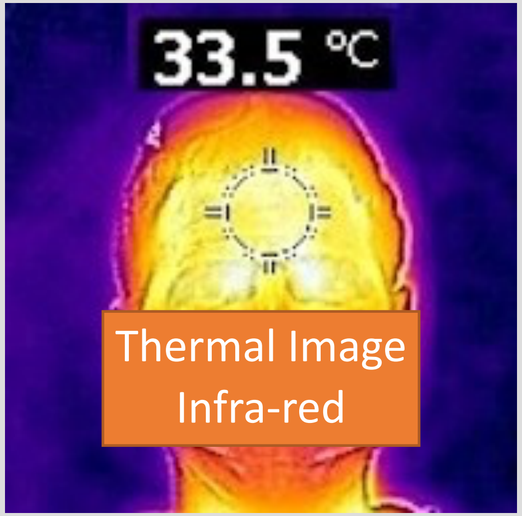
# Image

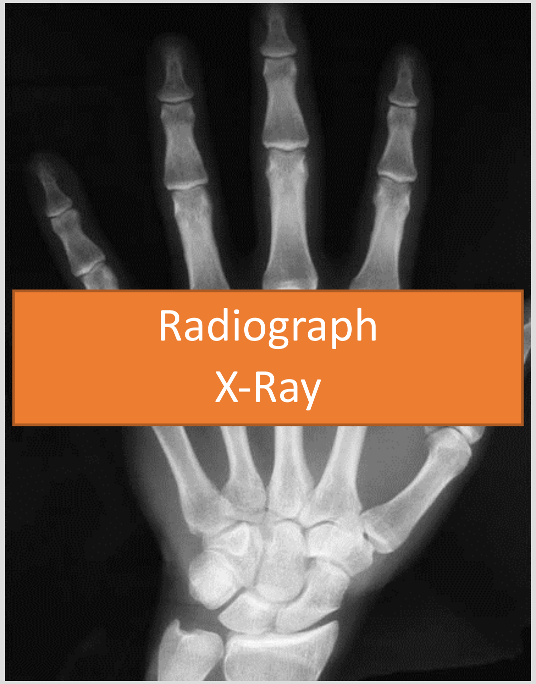
Merriam-Webster: A visual representation of something

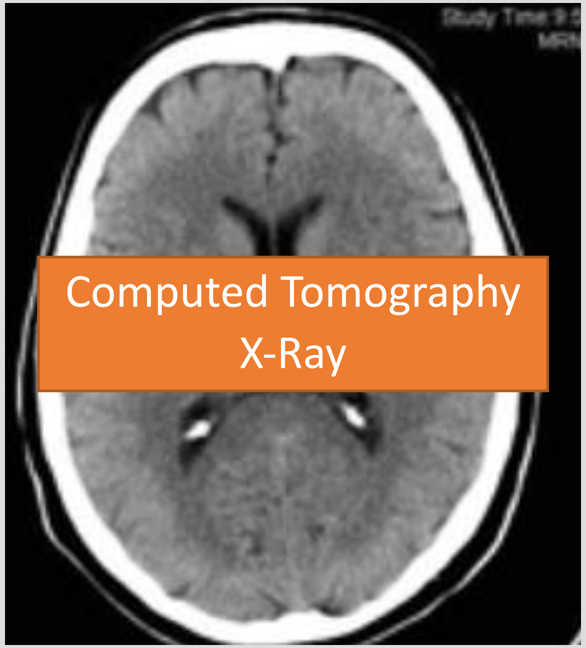
# Imaging

Merriam-Webster: the action or the process of producing an image

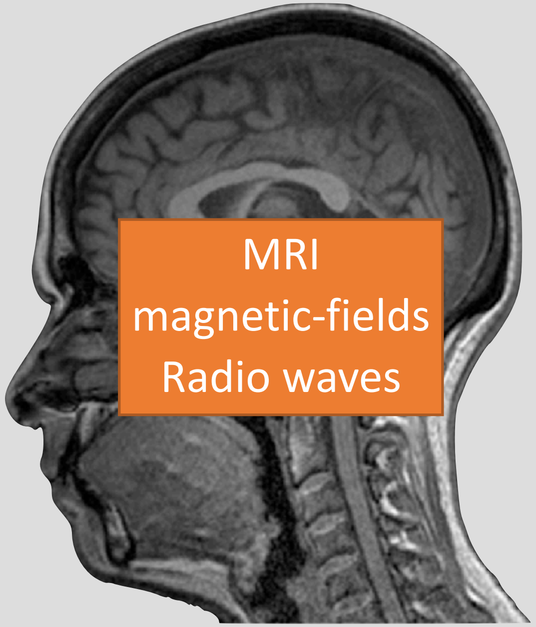
# Different Images

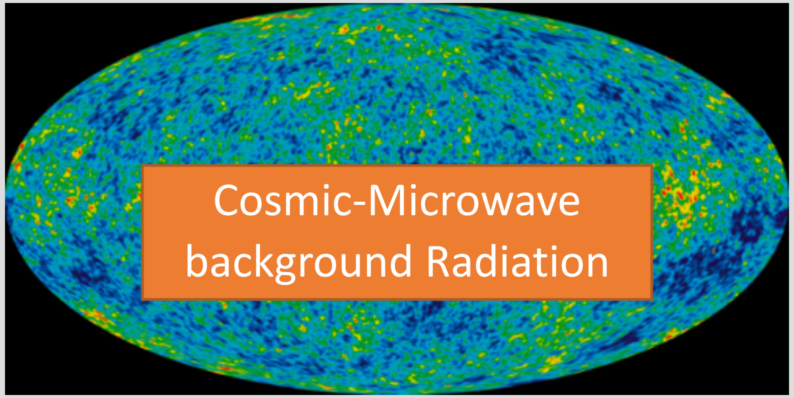


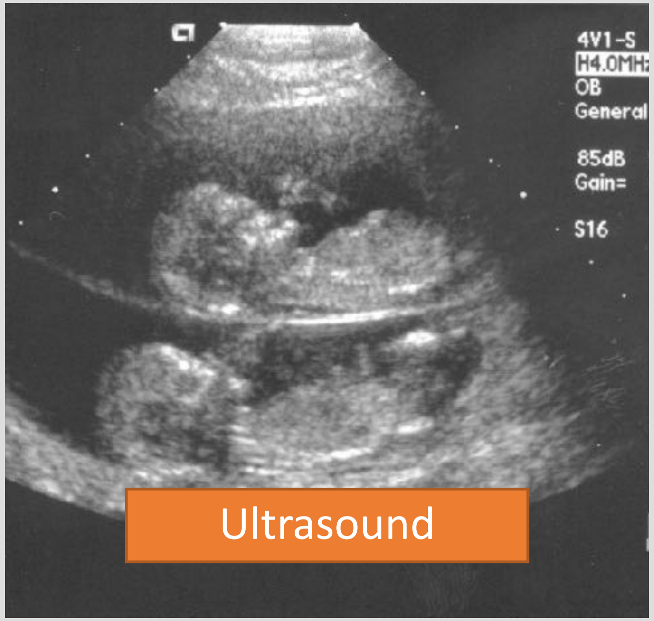



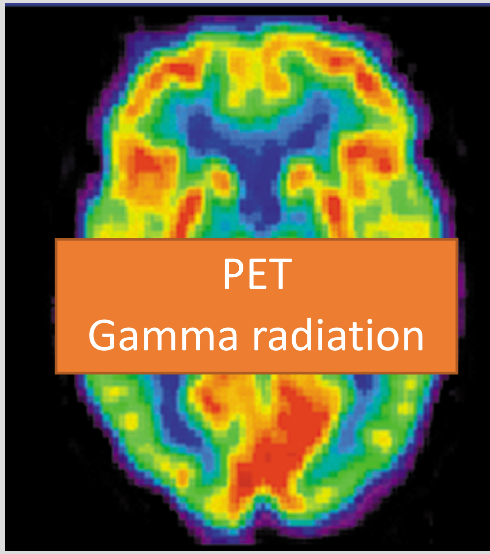





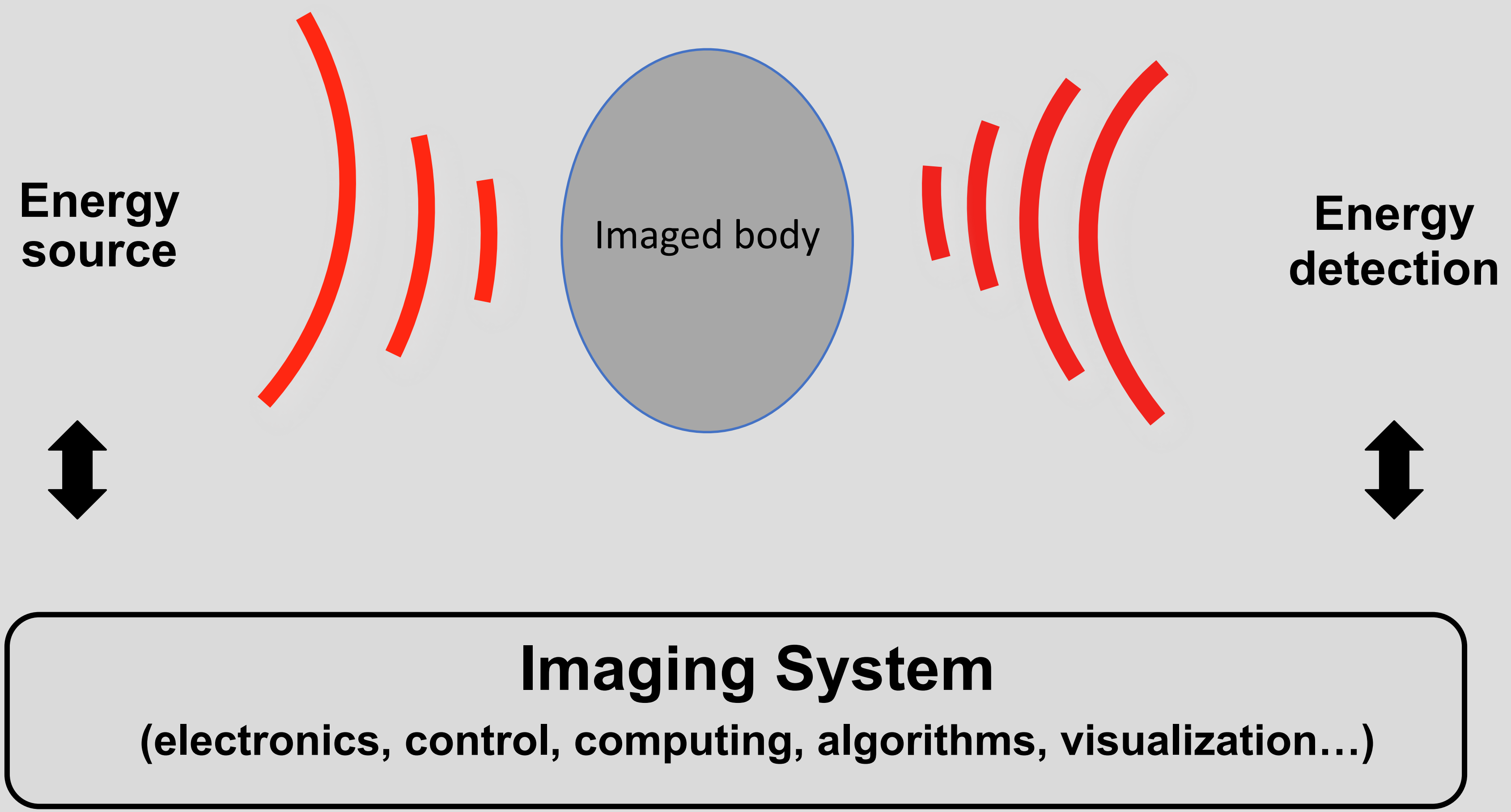


# Imaging Systems in General

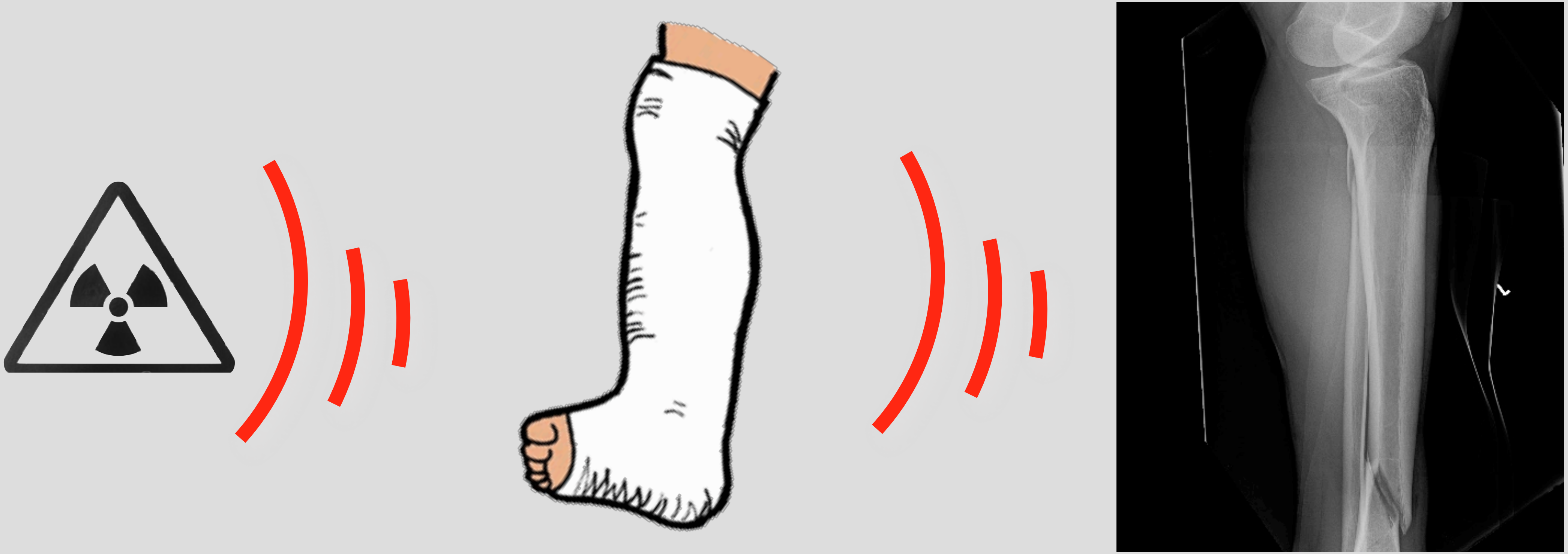


“Medical imaging” circa 1632

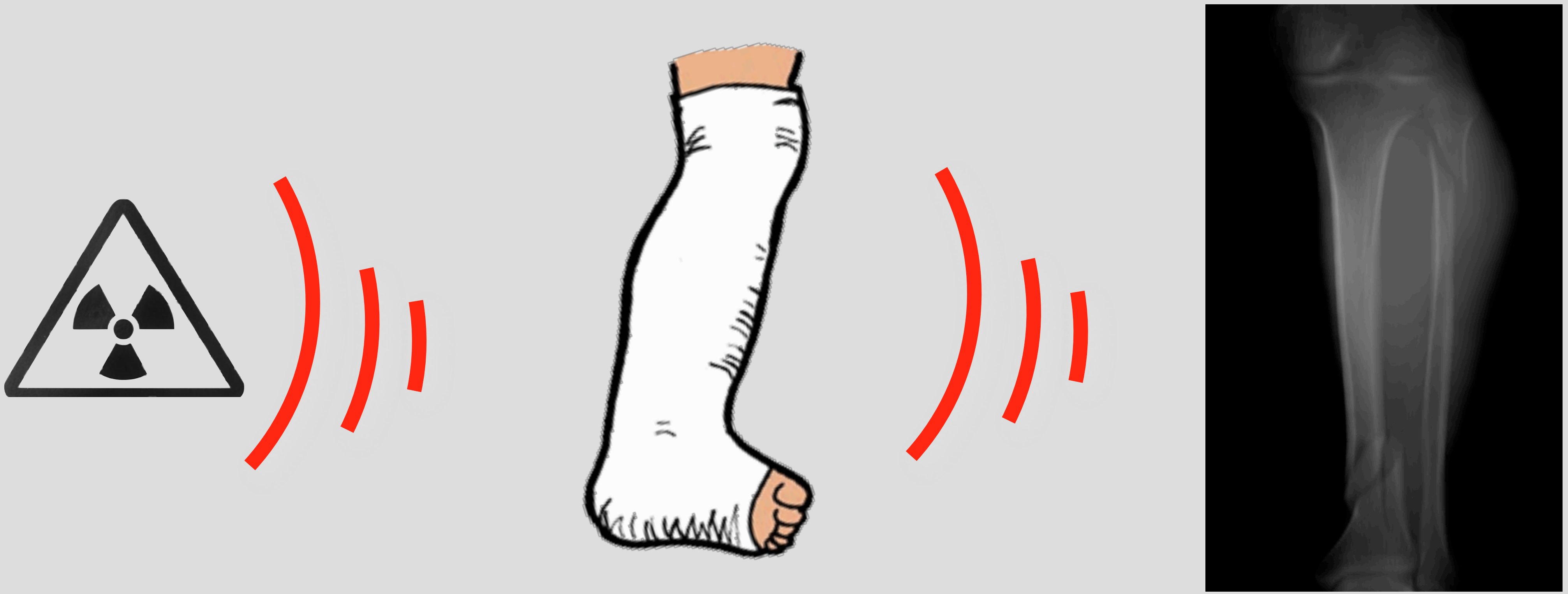
“The Anatomy Lesson of Dr. Nicolaes Tulp”, Rembrandt

Mauritshuis, The Hague

# Projection Xray



# Projection Xray



# Tomography

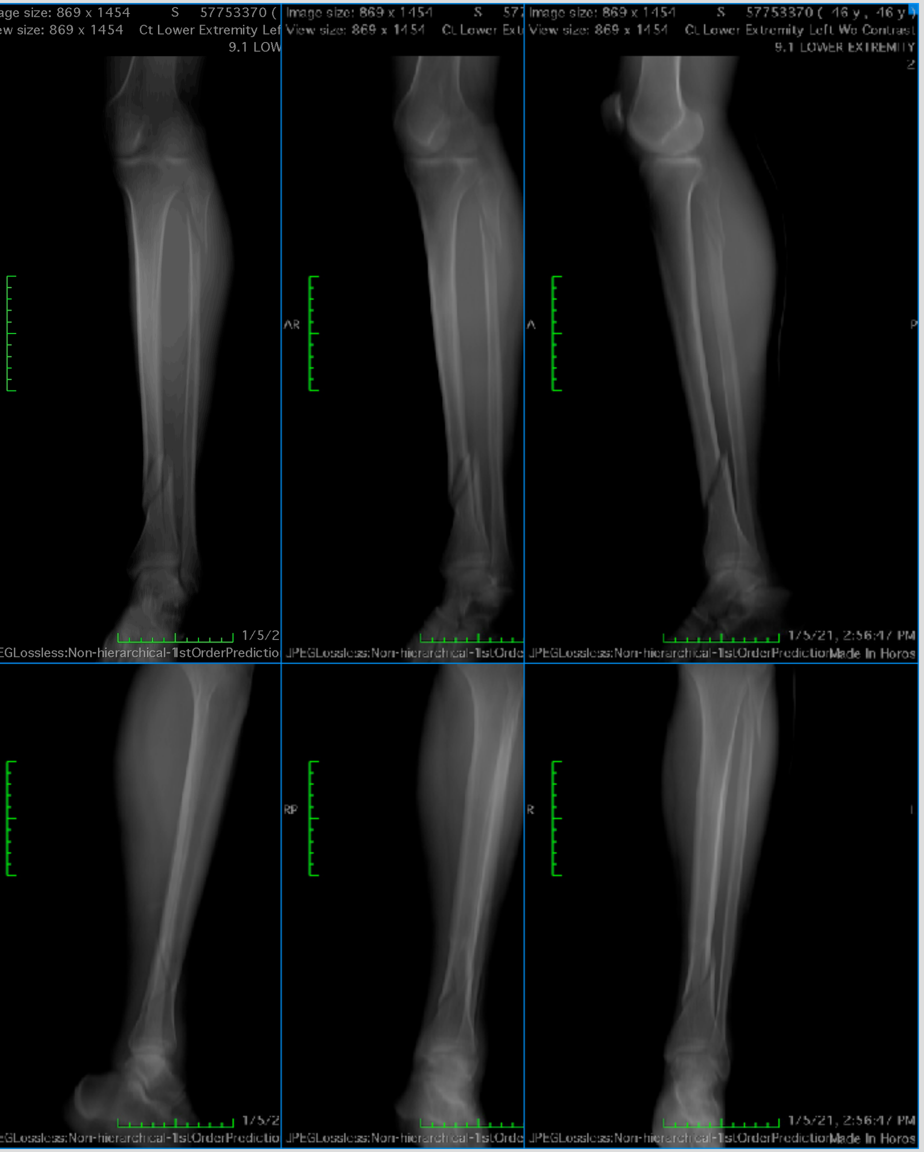
'tomo' – slice

'graphy' – to write

Assume it is not desirable to slice open leg. How does tomography visualizes cross-sectional slices?

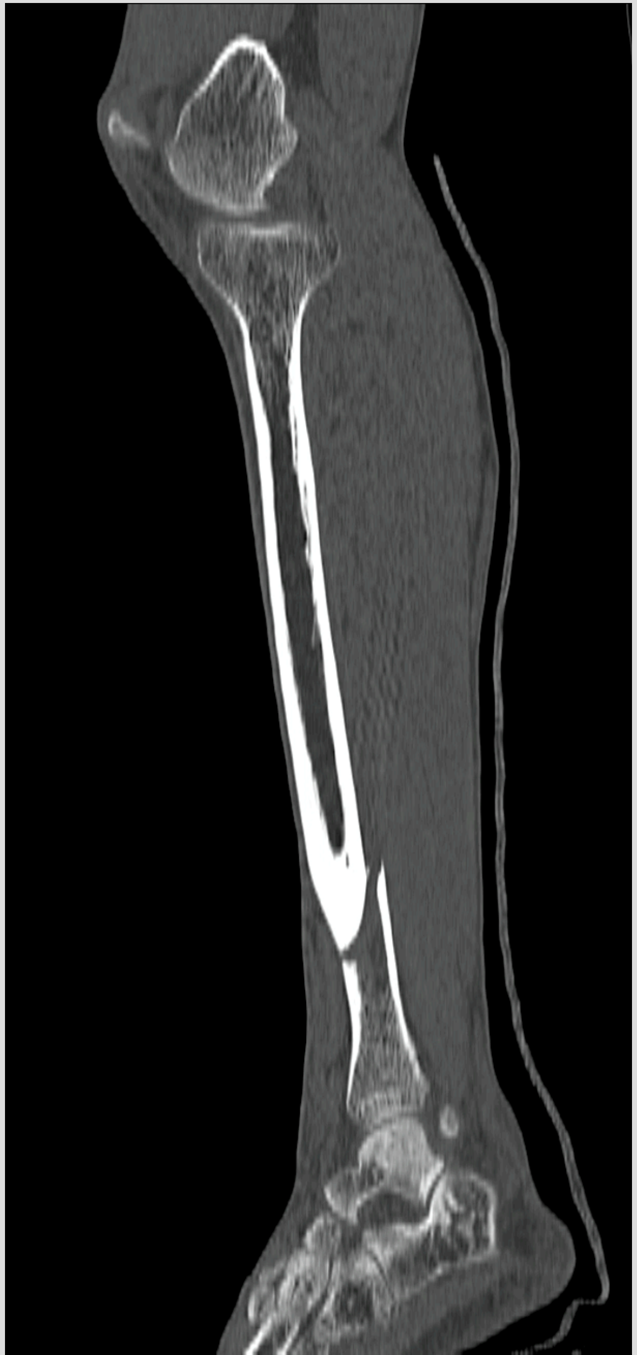
# From Projections

## Projections

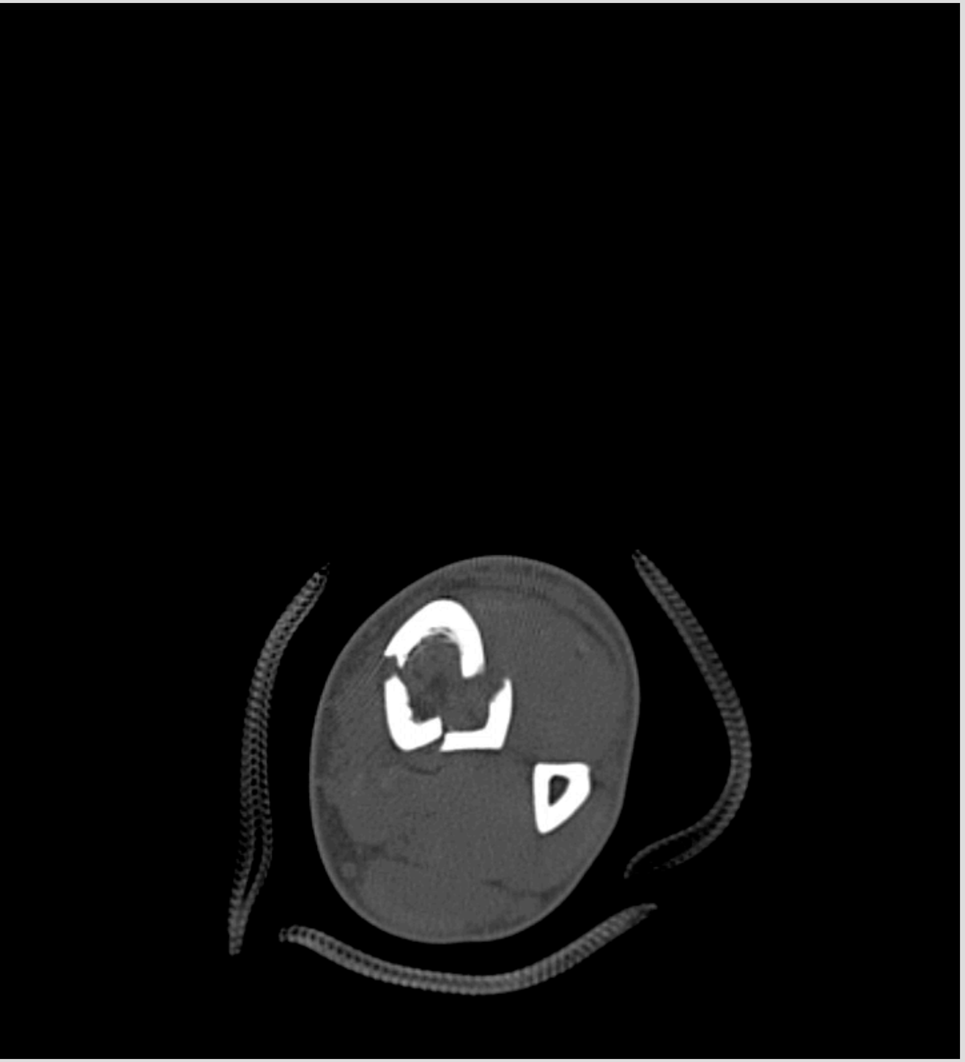


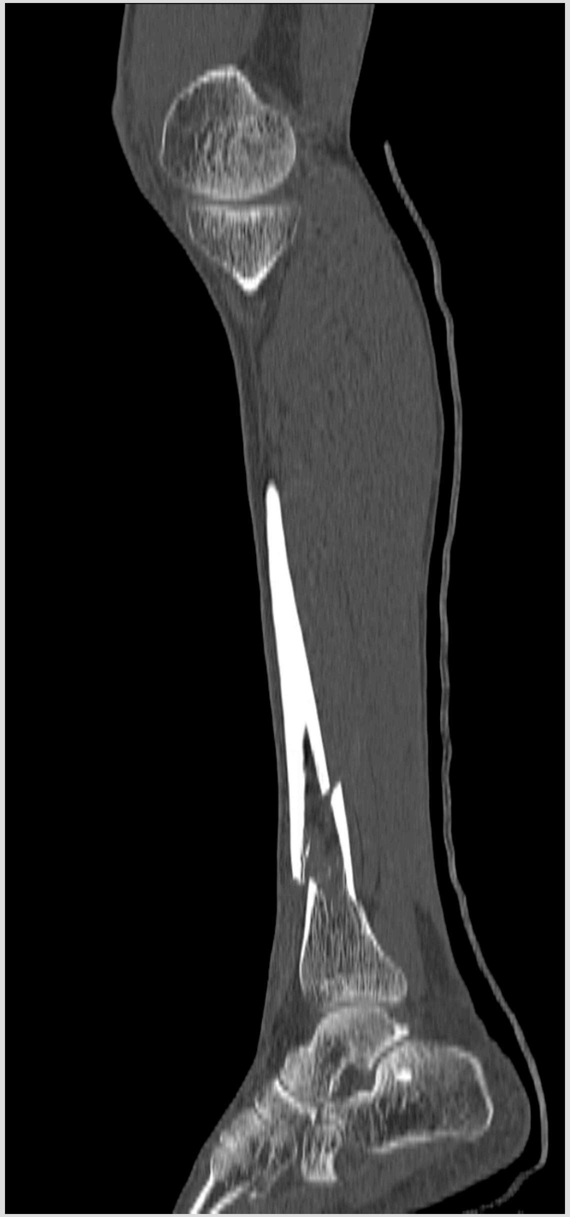

## Axial Slices

## Sagittal Slices

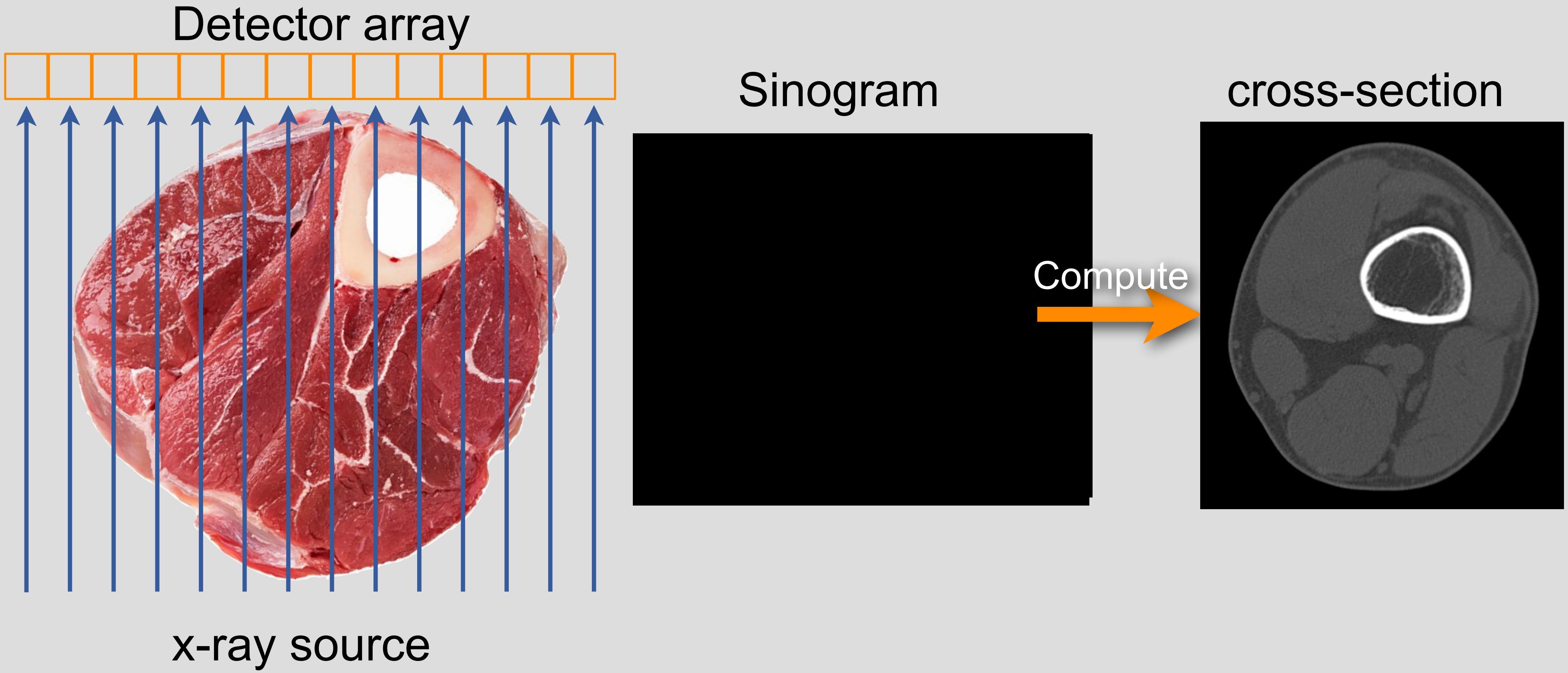


# 3D Rendering from Slices





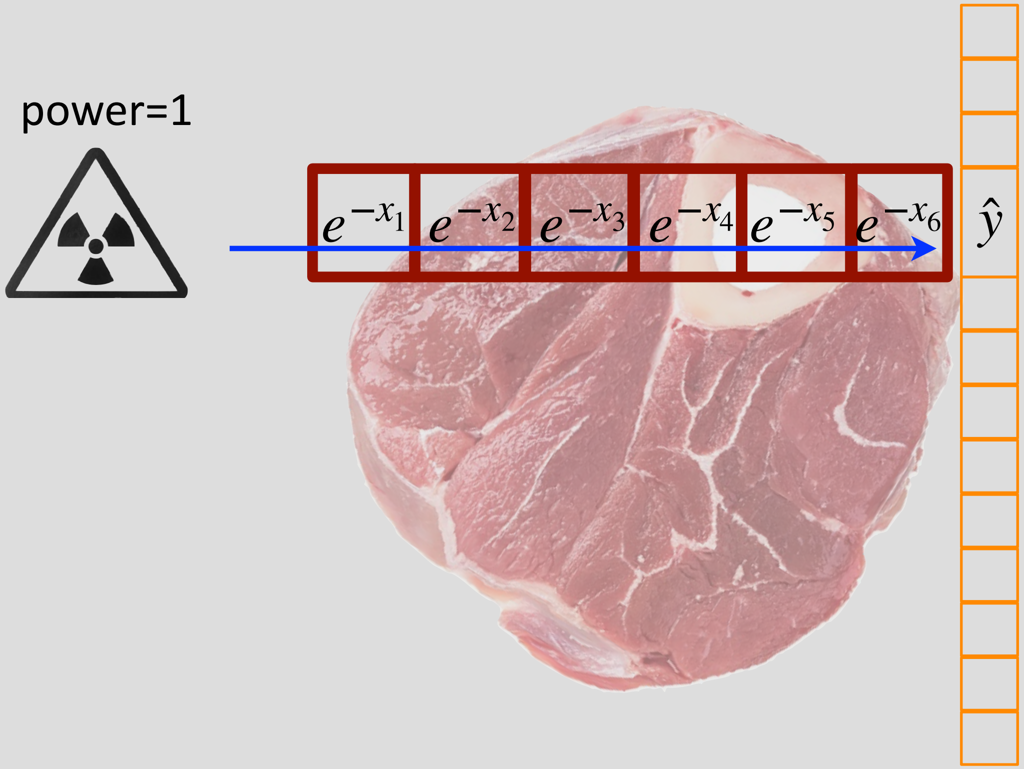
# Computed Tomography



# Computed Tomography

http://www.youtube.com/watch?v=4gklQHM19aY&feature=related

# Modeling Tomography




$$1 \cdot e^{-(x_1+x_2+x_3+x_4+x_5+x_6)} = \hat{y}$$

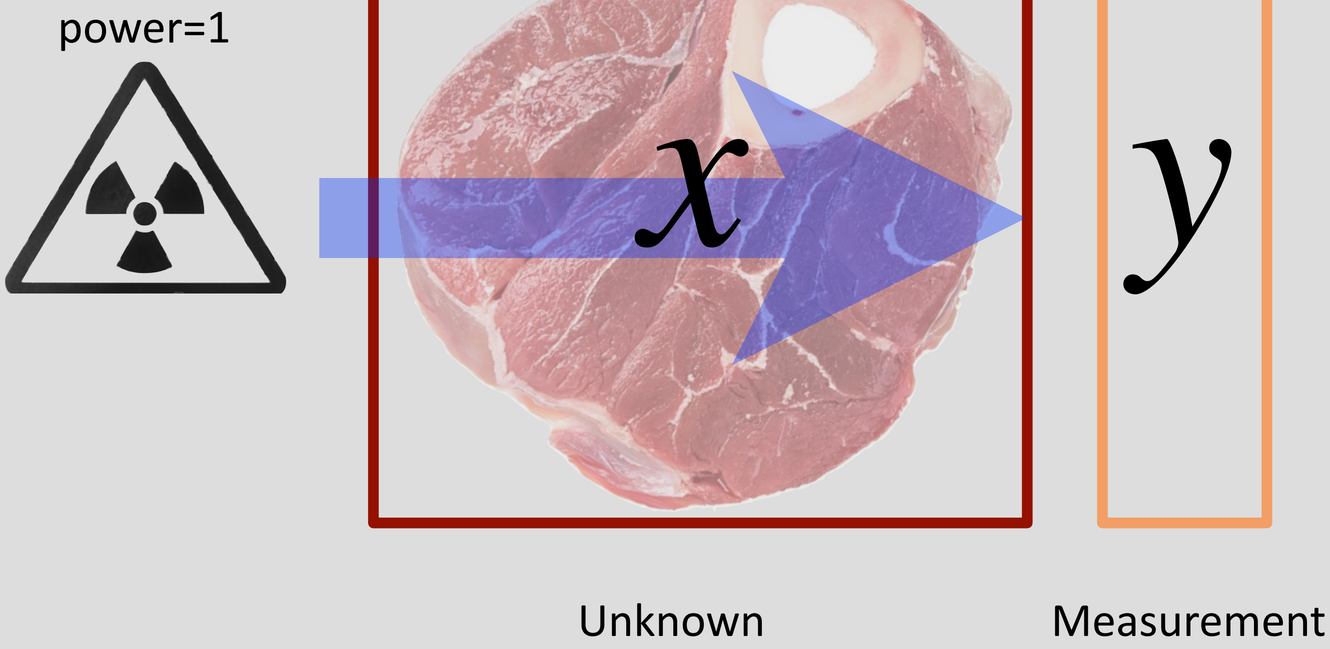
$$\log\{e^{-(x_1+x_2+x_3+x_4+x_5+x_6)}\} = \log\{\hat{y}\}$$

$$y = -\,log\{\hat{y}\}$$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = y$$

…. or y is the sum of x-ray attenuation coefficients along a line

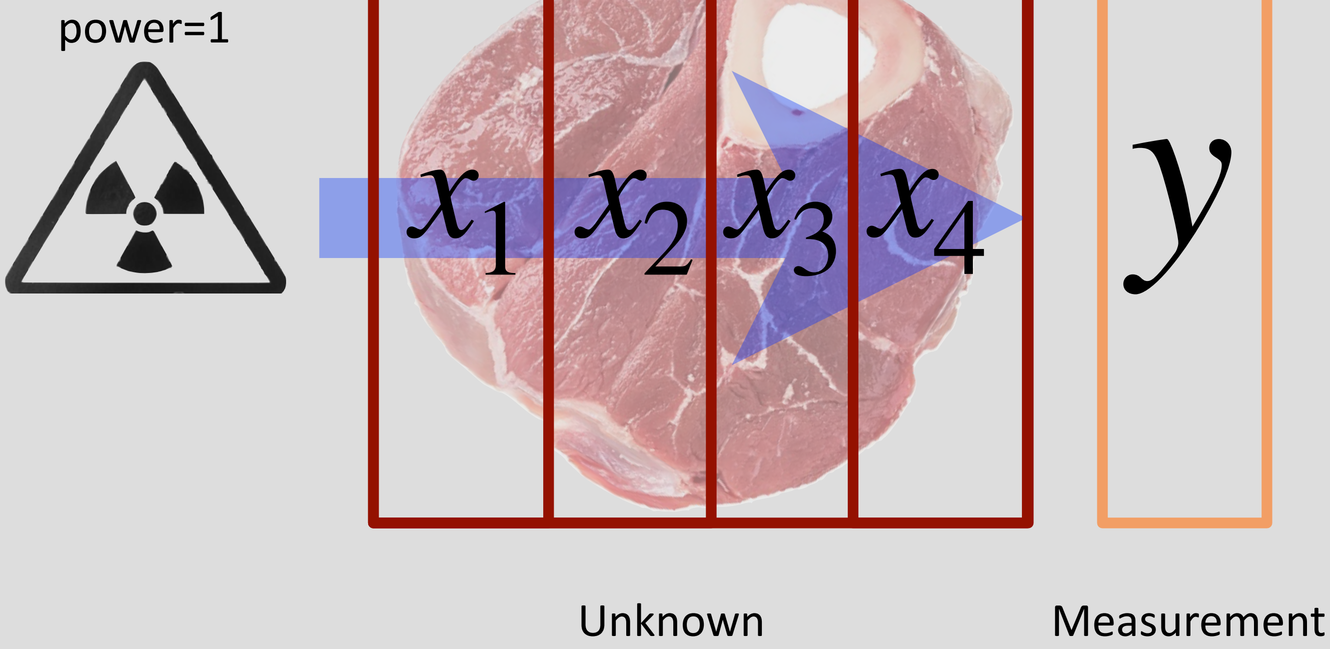
# Modeling Tomography




$$y = 1 \cdot x$$

$$x = y$$

# Modeling Tomography



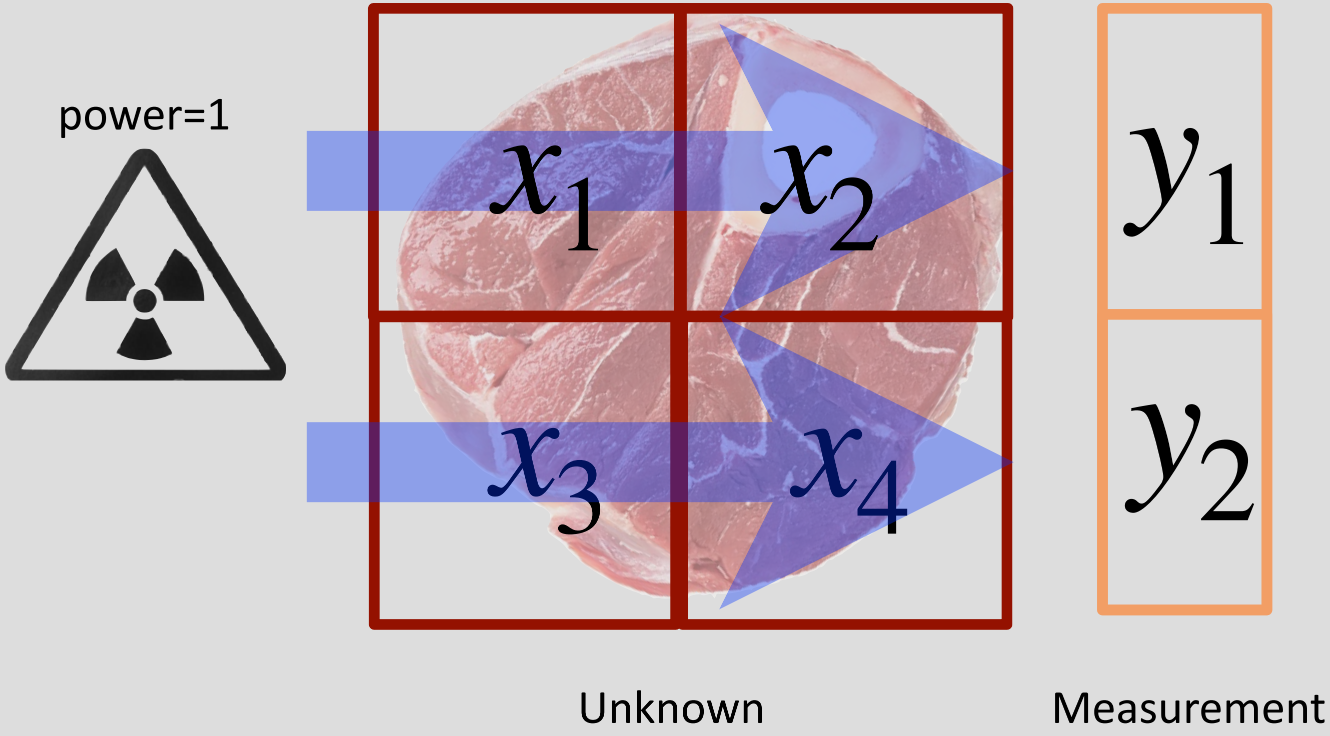

$$y = x_1 + x_2 + x_3 + x_4$$

↓

1 equation 4 unknowns!

😭

# Modeling Tomography

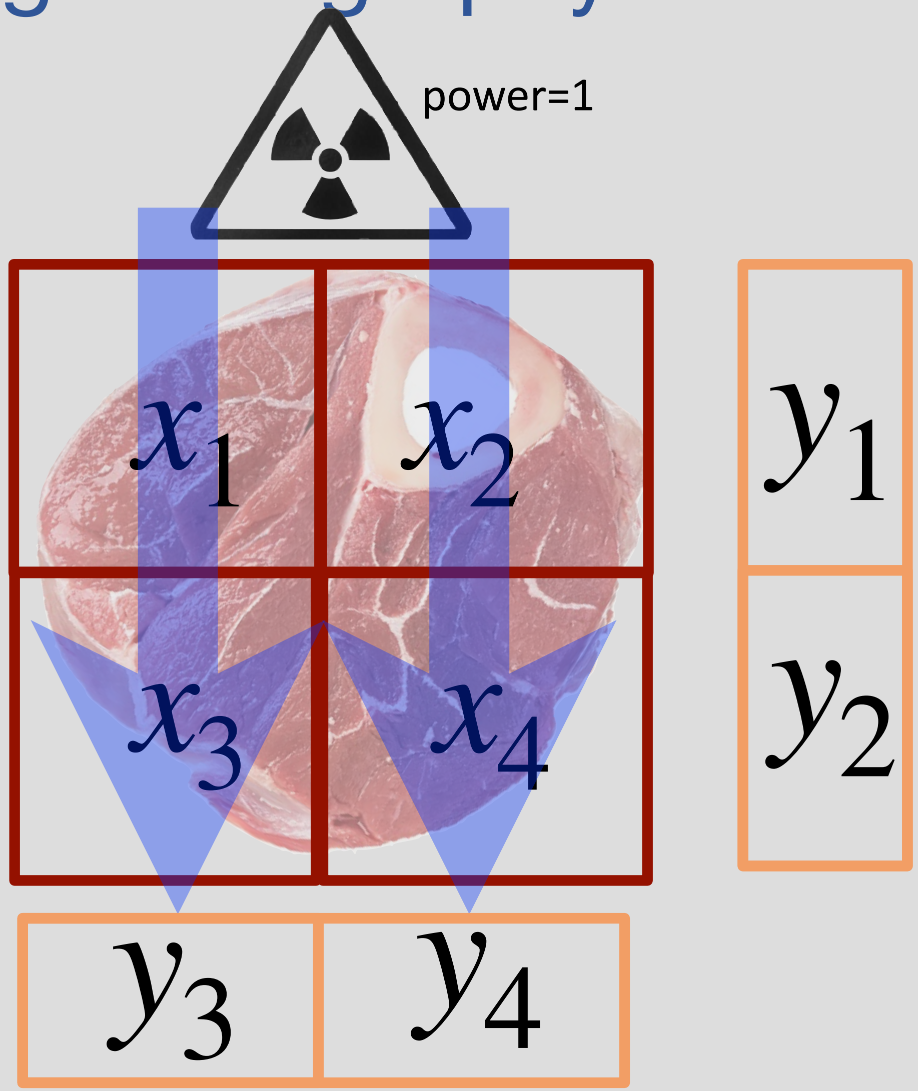



$$y_1 = x_1 + x_2$$
$$y_2 = x_3 + x_4$$

2 equation 4 unknowns!

😭

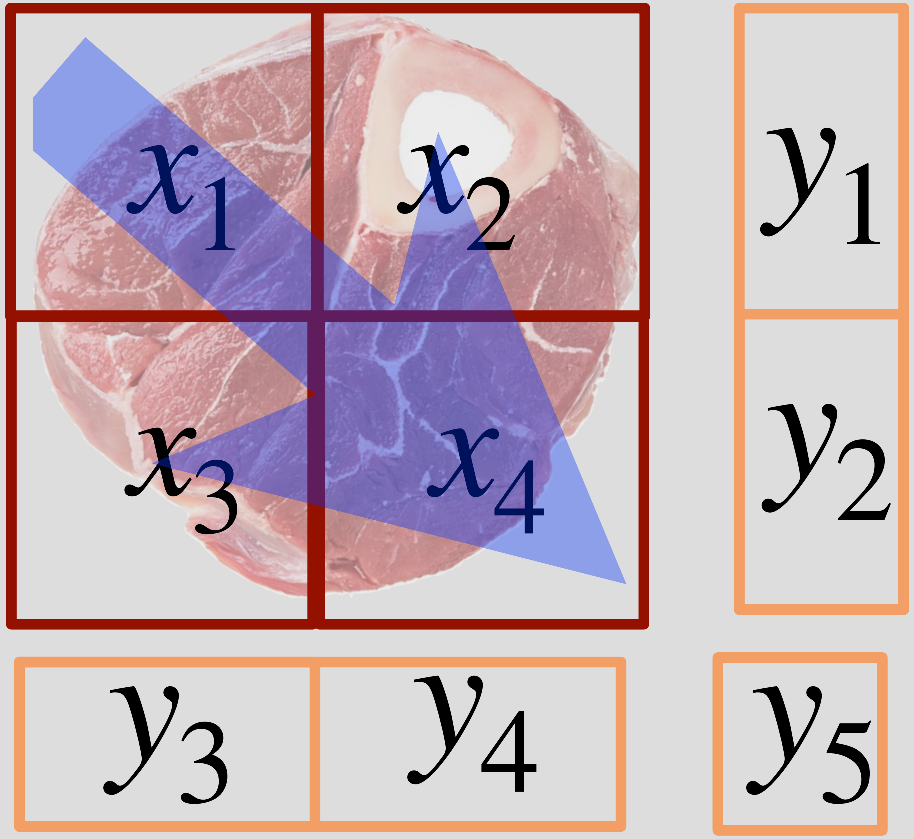
# Modeling Tomography




$$
\begin{aligned}
y_1 &= x_1 + x_2 \\
y_2 &= x_3 + x_4 \\
y_3 &= x_1 + x_3 \\
y_4 &= x_2 + x_4
\end{aligned}
$$

Can we solve this?

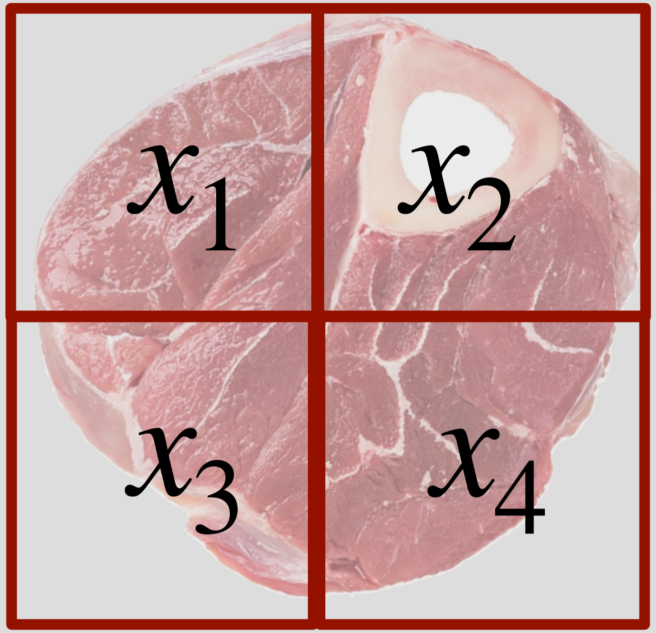
# Modeling Tomography

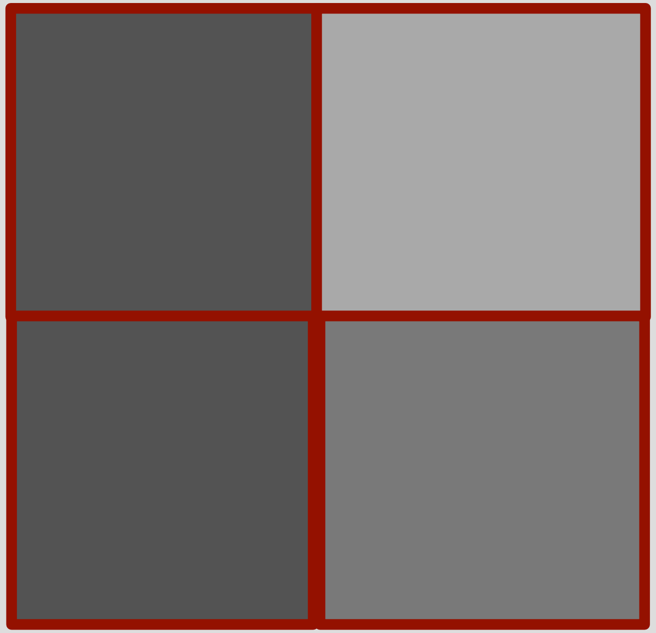

$$
\begin{aligned}
y_1 &= x_1 + x_2 \\
y_2 &= x_3 + x_4 \\
y_3 &= x_1 + x_3 \\
y_4 &= x_2 + x_4 \\
y_5 &\approx \sqrt{2}x_1 + \sqrt{2}x_4
\end{aligned}
$$

May be able to solve this!

# Modeling Tomography




Possible reconstruction



Blurred version of :

# All our measurements are (converted to) *linear*

What does that mean?

Each variable (x) is multiplied by a scalar to contribute to the measurement

$$
\begin{aligned}
y_1 &= x_1 + x_2 \\
y_2 &= x_3 + x_4 \\
y_3 &= x_1 + x_3 \\
y_4 &= x_2 + x_4
\end{aligned}
$$

**This is called a** ***system of linear equations***

$$y_5 = \sqrt{2}x_1 + \sqrt{2}x_4$$

***Linear Algebra*** **is what we need to solve it!**

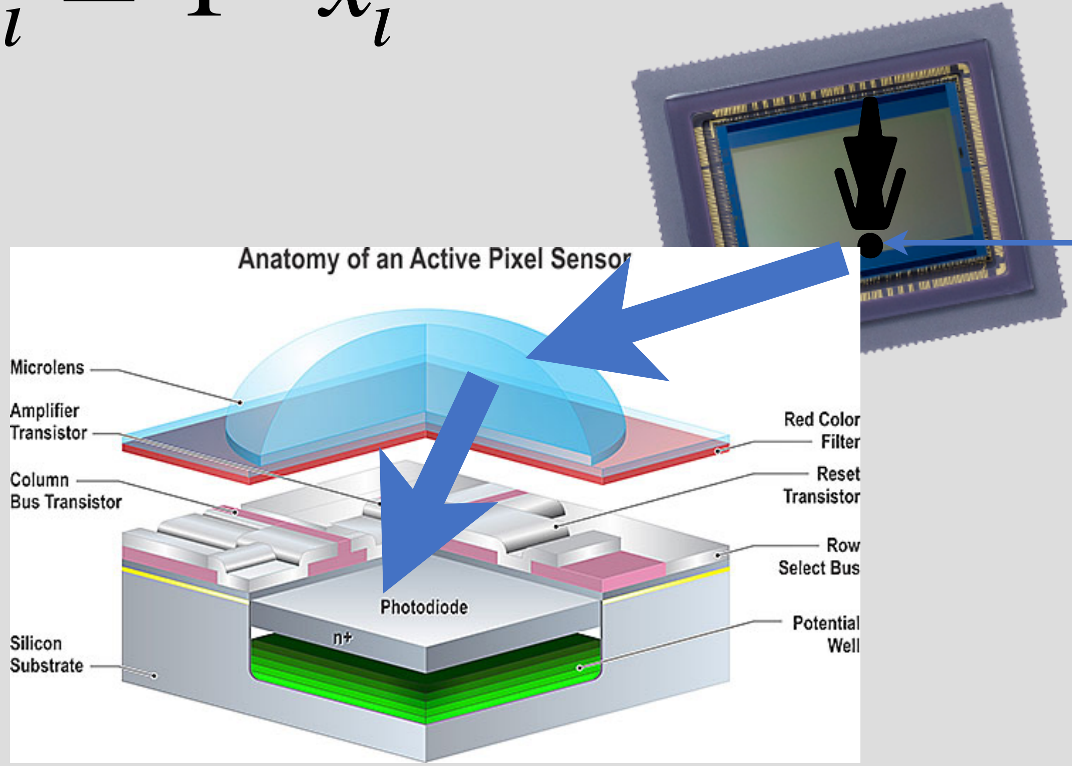
# Camera Model

Lens maps image onto sensor

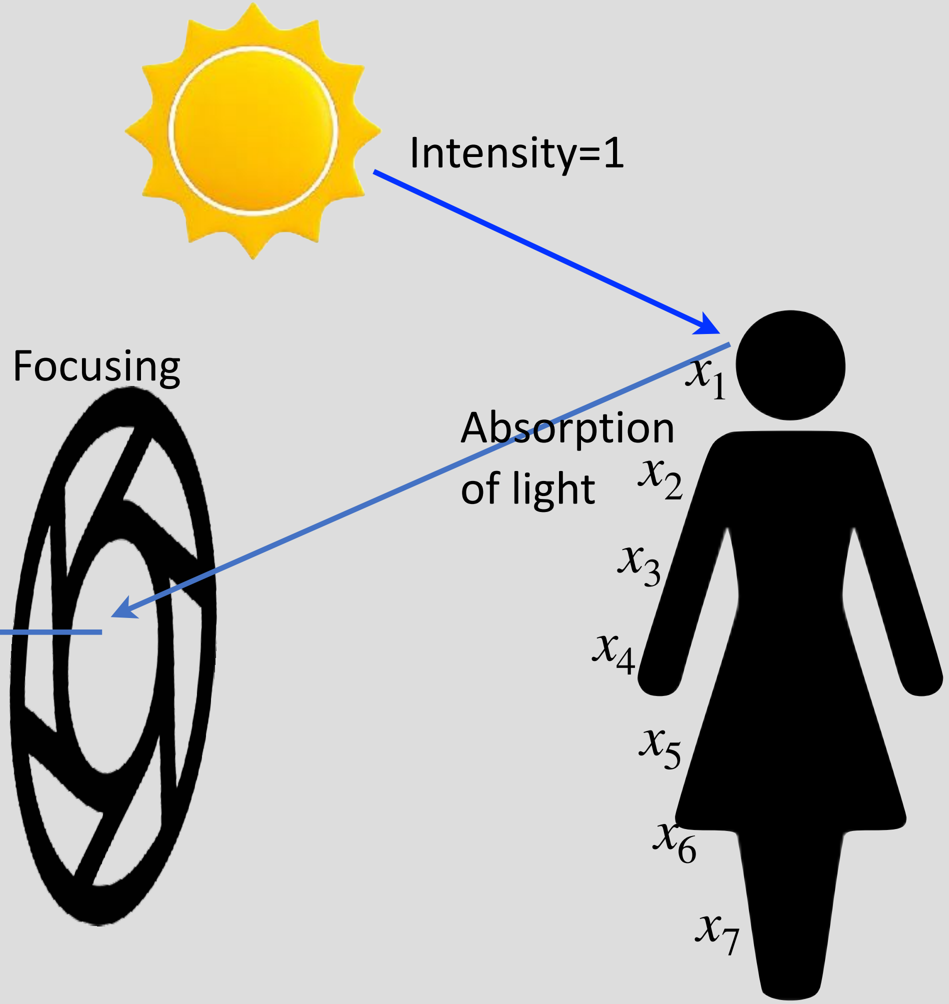
Each pixel is sensed separately

$$y_i = 1 \cdot x_i$$

All pixels sensed in parallel



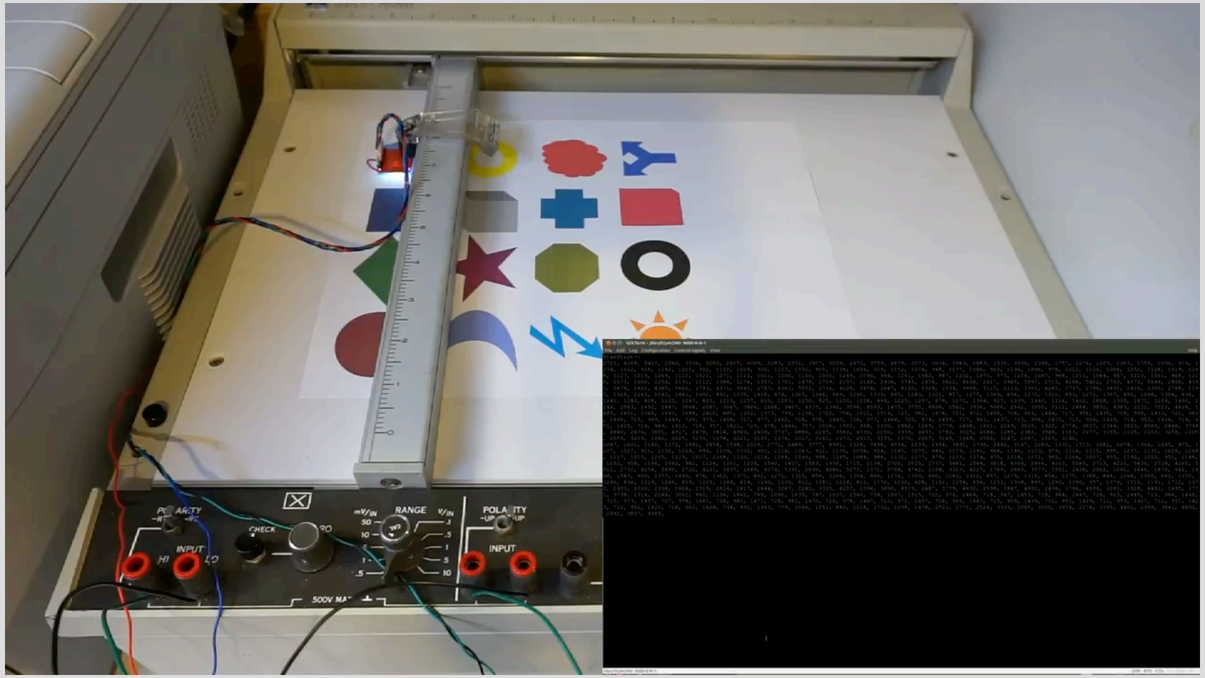

Focusing

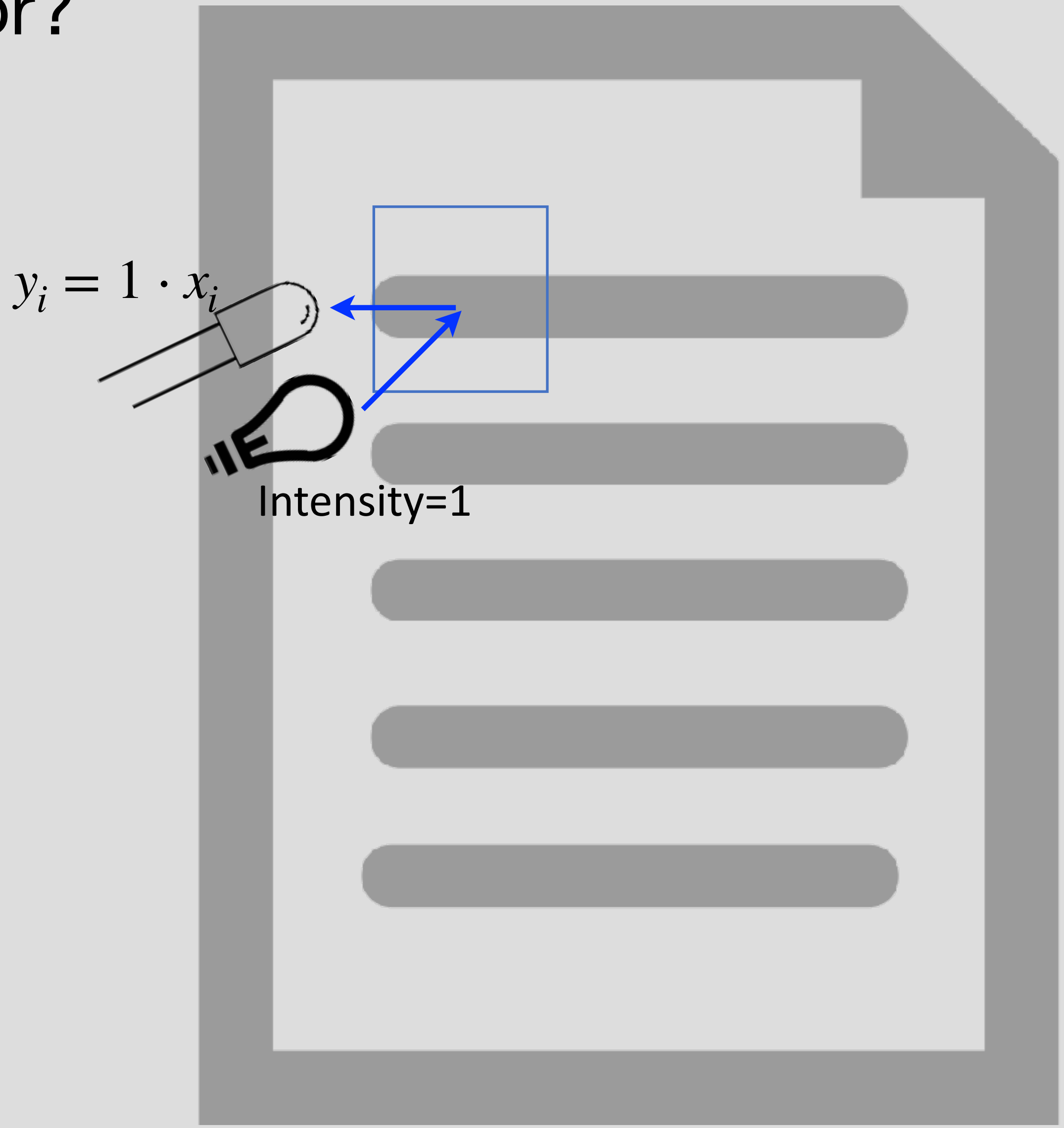


# Single Pixel Scanner

- What if we had only a single sensor?
- How can we create an image?

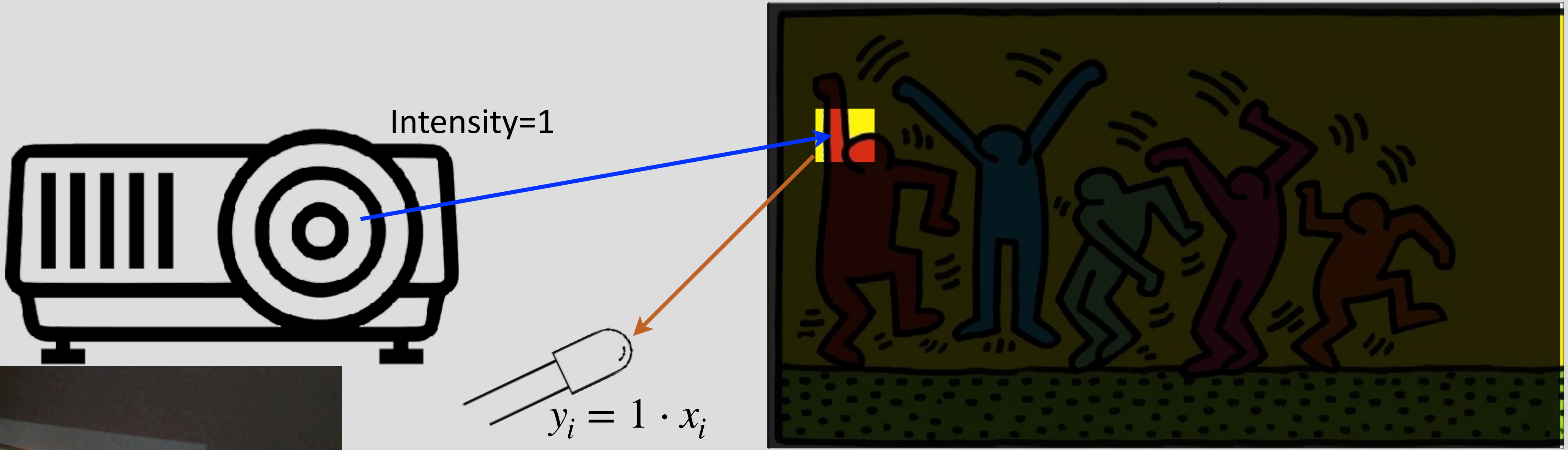
https://www.youtube.com/watch?v=U5PwsVqHT8Y

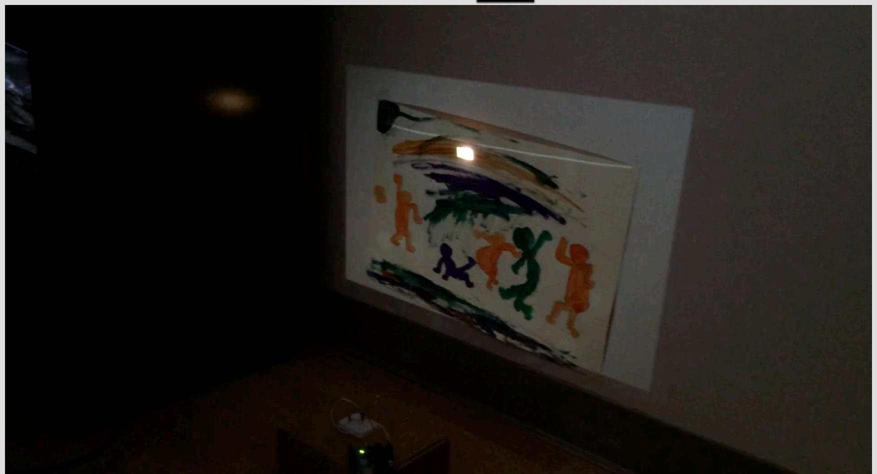




# Non-moving Single Pixel Camera

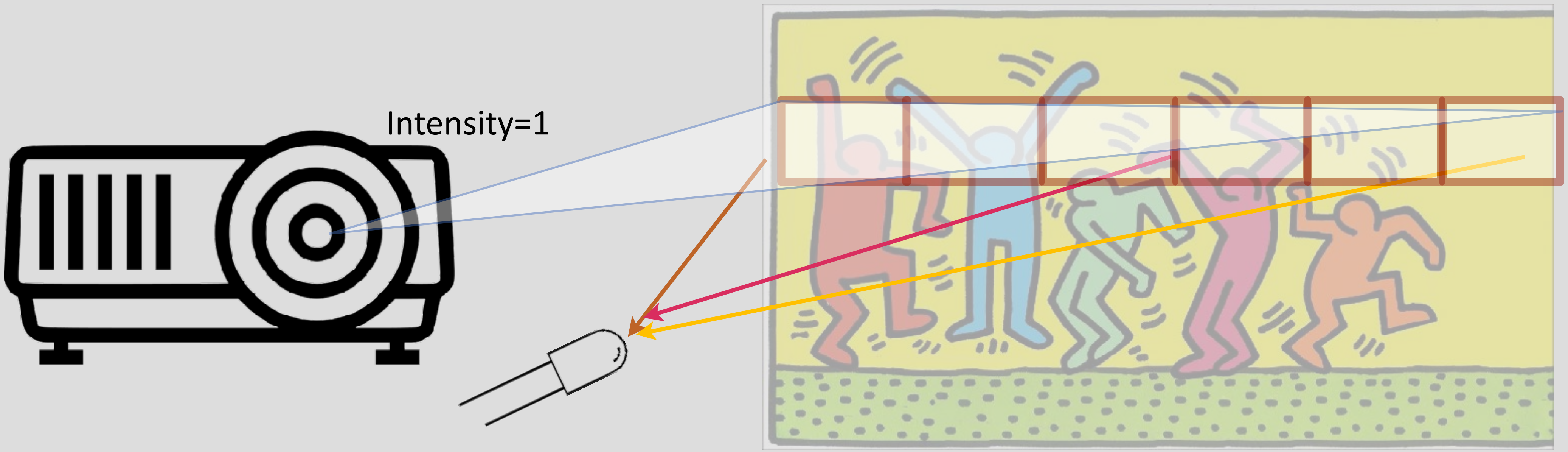
- Use a projector to illuminate pixels
- Sense reflected light with a sensor



# Non-moving Single Pixel Camera

- Use a projector to illuminate several pixels!
- Sense reflected light with a sensor
- Make many measurements and solve the equations!




$$y_1 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

Similar math as Tomography!

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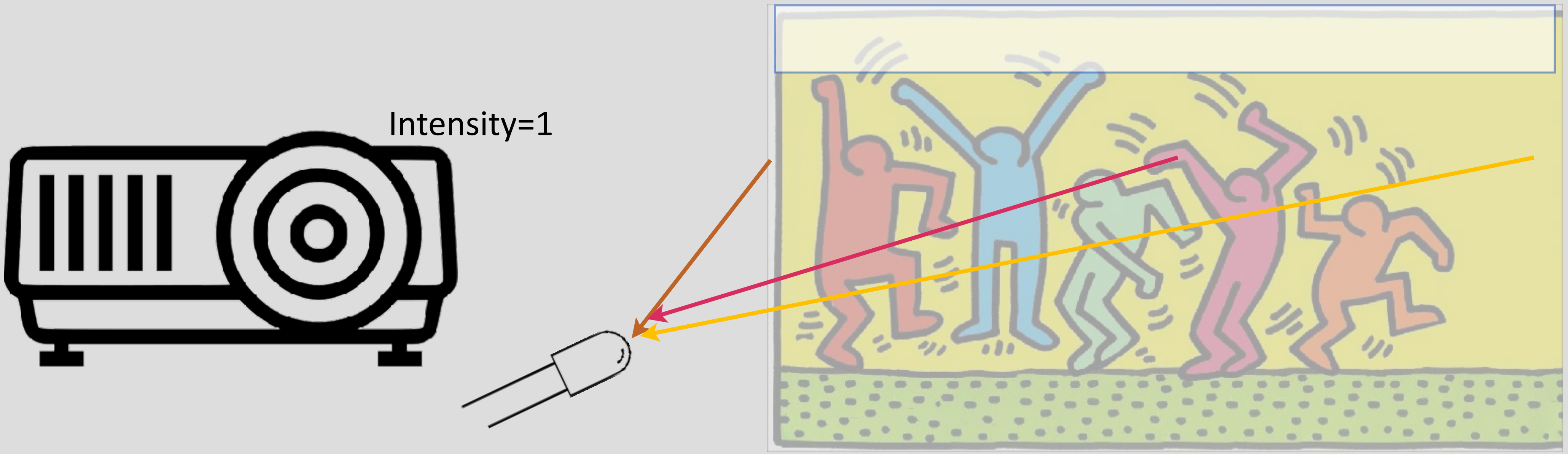


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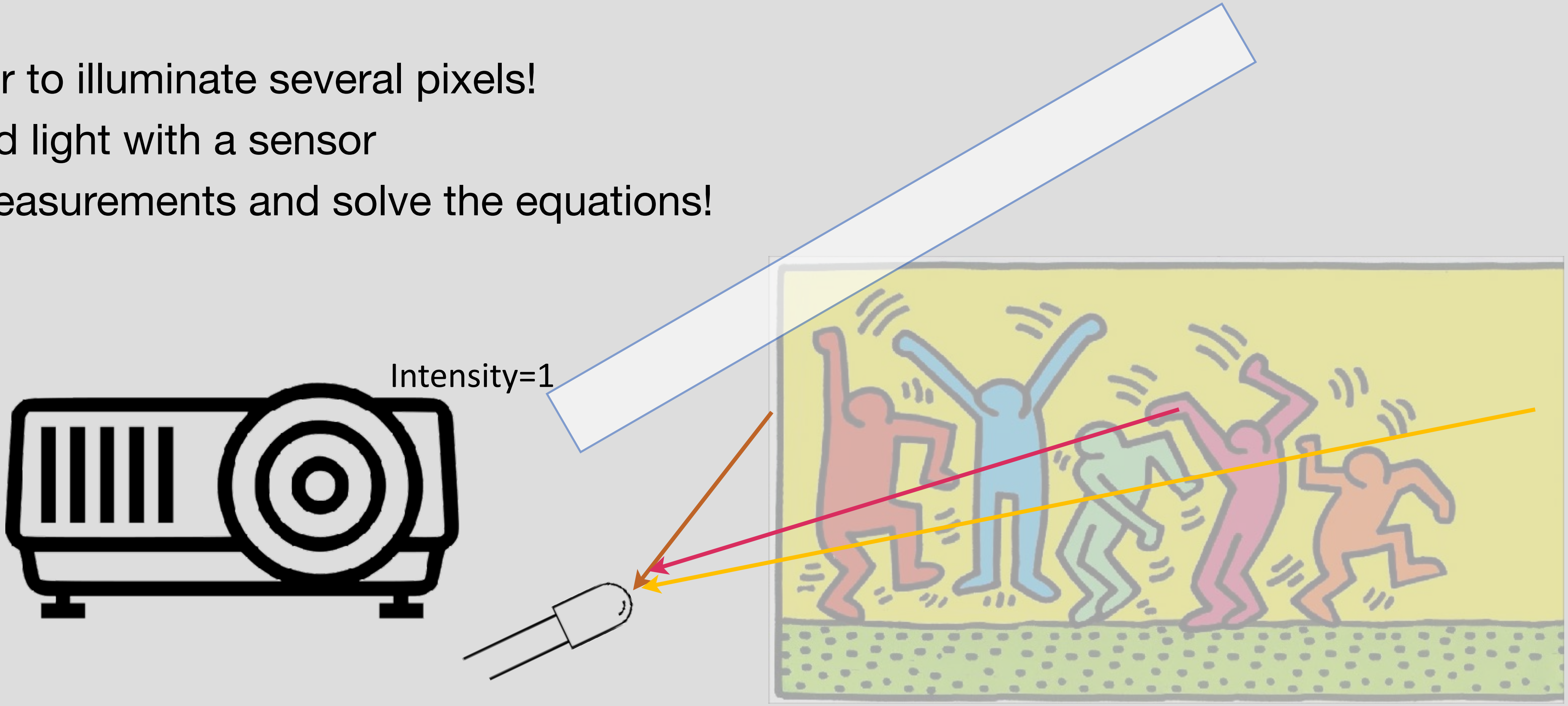


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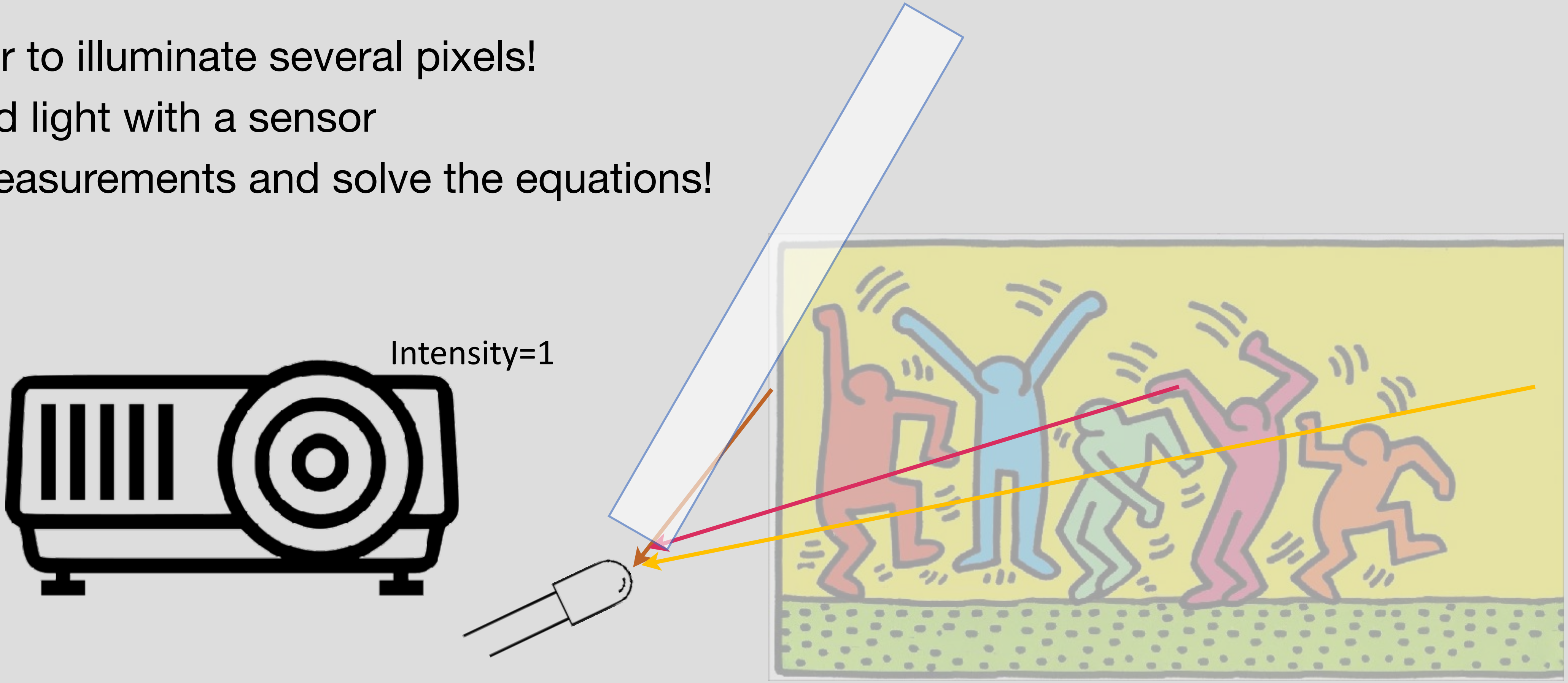


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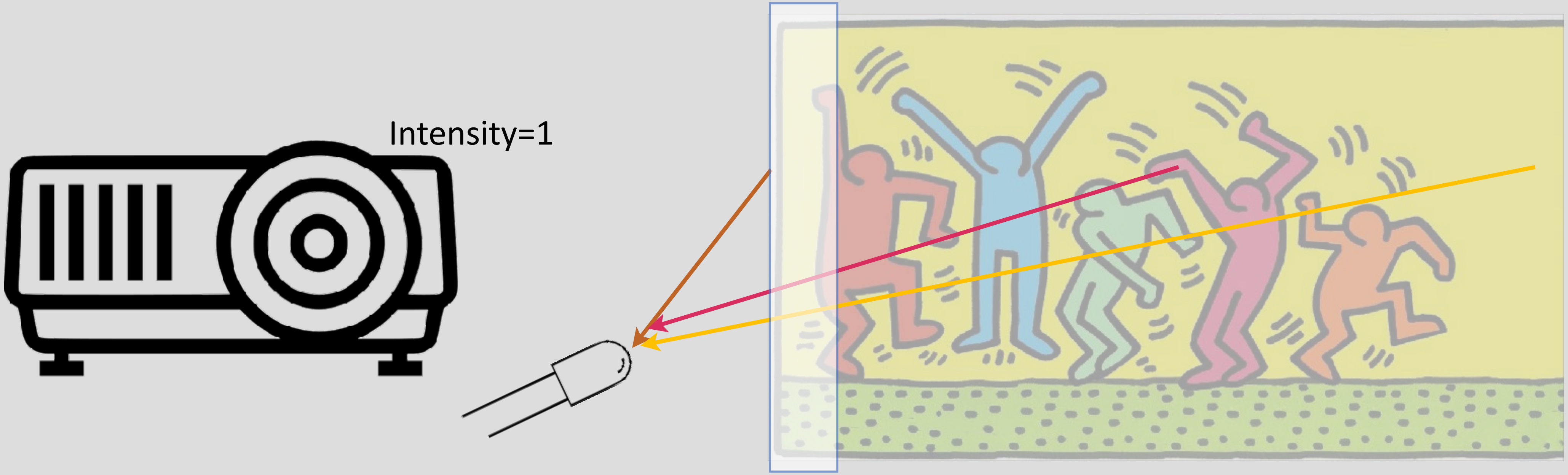


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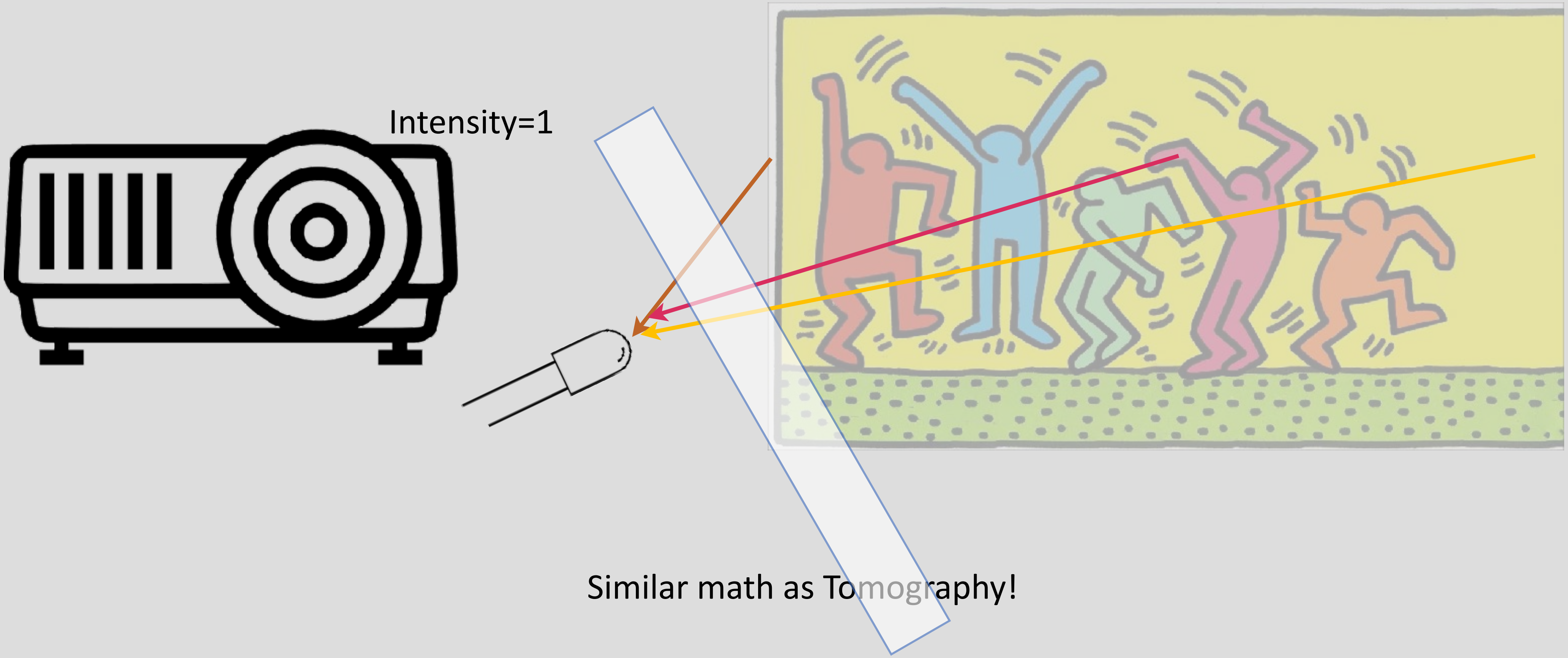


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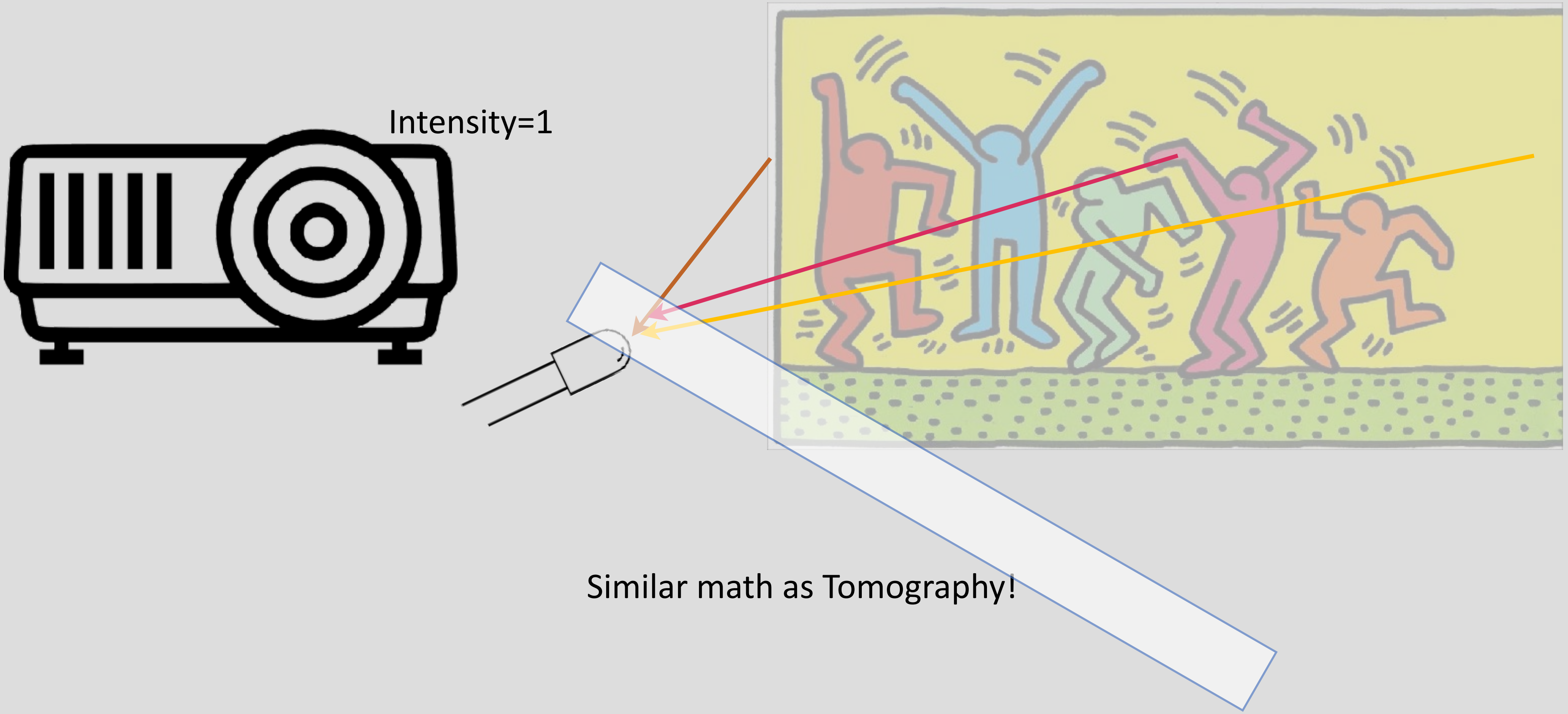
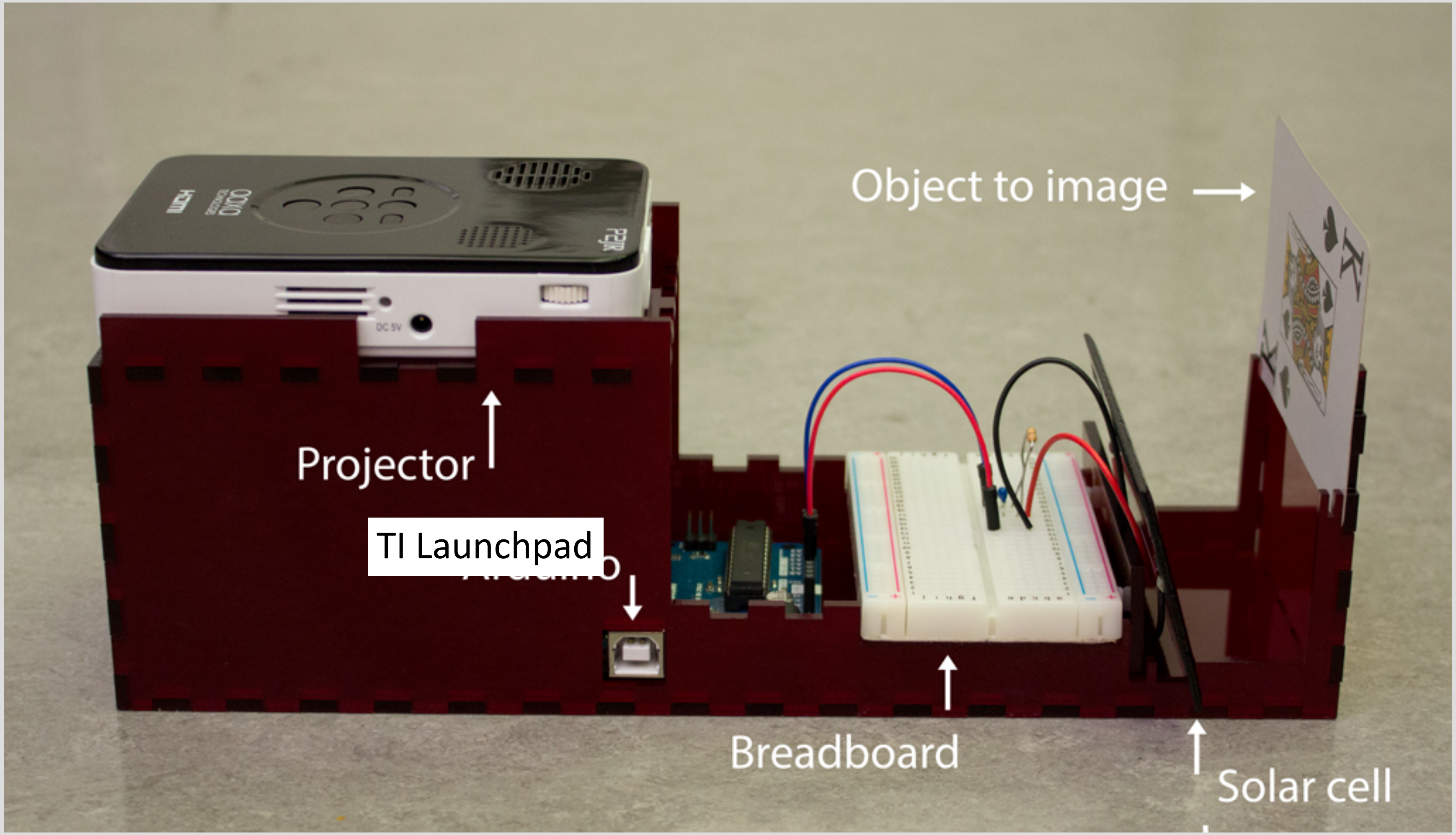
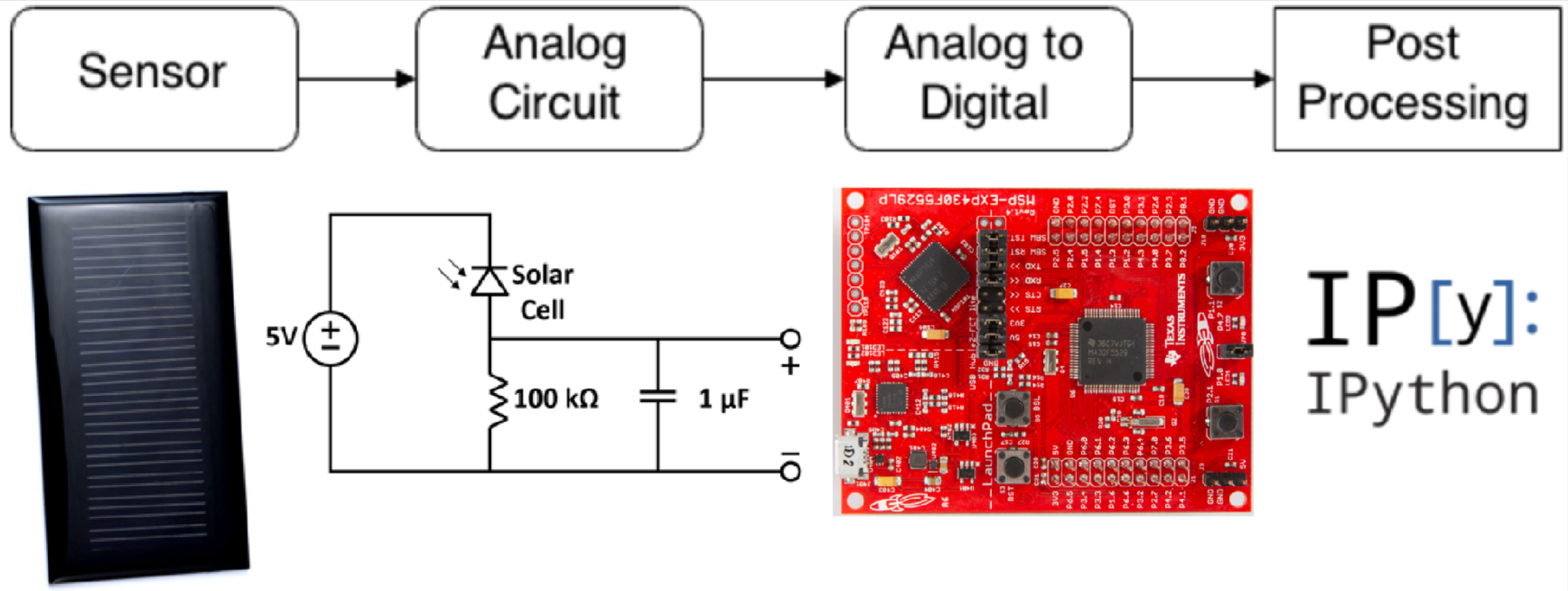



Similar math as Tomography!

# Imaging Lab #1 Setup

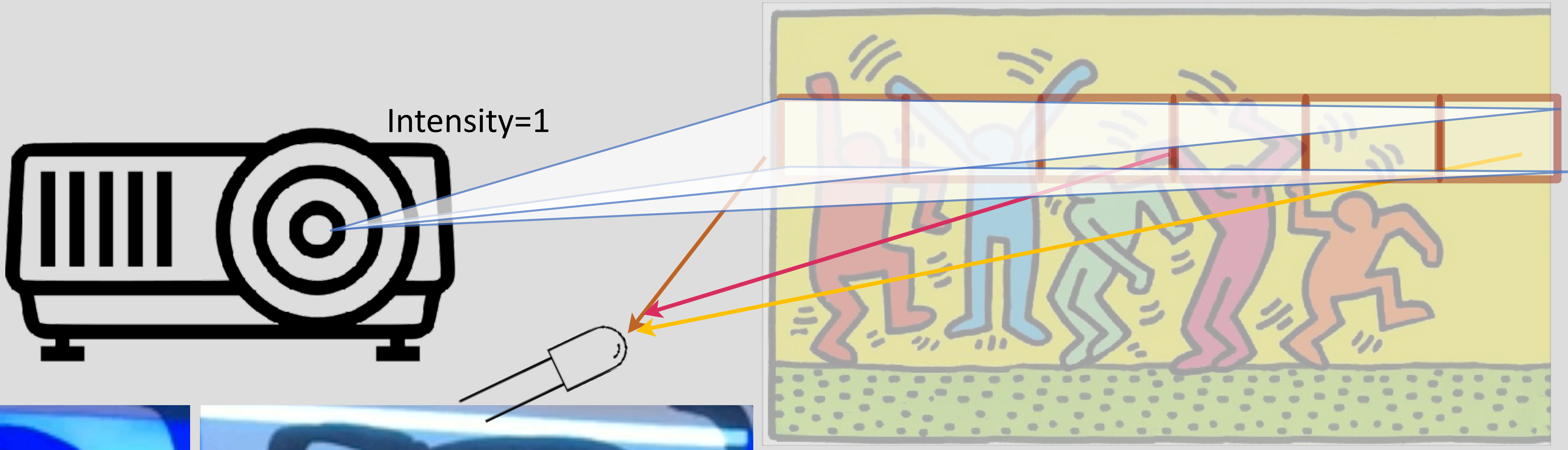


# Imaging Lab #1



# Non-moving Single Pixel Camera

- How many measurements do you need?
- What are the best patterns?




# What is linear algebra?

- The study of linear functions and linear equations, typically using vectors and matrices
- Linearity is not always applicable, but can be a good first-order approximation
- There exist good fast algorithms to solve these problems

# Linear Equations

- Definition:

  Consider: $f(x_1, x_2, \cdots, x_N) : \mathbb{R}^n \rightarrow \mathbb{R}$

  $f$ is *linear if the following identity holds:*

  *(1) Homogeneity:*

$$f(\alpha x_1, \cdots, \alpha x_N) = \alpha f(x_1, \ldots, x_N)$$

  *(2) Super Position (distributivity): if* $x_i = y_i + z_i$, *then*

$$f(y_1 + z_1, \cdots, y_N + z_N) = f(y_1, \cdots, y_N) + f(z_1, \cdots, z_N)$$

  *Claim: linear functions can always be expressed as:*

$$f(x_1, x_2, \cdots, x_N) = c_1 x_1 + c_2 x_2 + \cdots + c_N x_N$$

# Proof for $\mathbb{R}^2$

- $f(x_1, x_2) : \mathbb{R}^2 \Rightarrow \mathbb{R}$ is linear. Need to prove: $f(x_1, x_2) = c_1 x_1 + c_2 x_2$

  *Trick:*

$$x_1 = 1 \cdot x_1 + 0 \cdot x_2$$
$$x_2 = 0 \cdot x_1 + 1 \cdot x_2$$

*So,*

$$\begin{aligned} f(x_1, x_2) &= f(x_1 \cdot 1 + x_2 \cdot 0,\ x_1 \cdot 0 + x_2 \cdot 1) \\ &= f(x_1 \cdot 1, x_1 \cdot 0) + f(x_2 \cdot 0, x_2 \cdot 1) \\ &= f(\mathbf{x_1} \cdot 1, \mathbf{x_1} \cdot 0) + f(\mathbf{x_2} \cdot 0, \mathbf{x_2} \cdot 1) \\ &= x_1 f(1,0) + x_2 f(0,1) \end{aligned}$$

$f(1,0) = c_1$, $f(0,1) = c_2$

$$= c_1 x_1 + c_2 x_2$$

# Linear Set of Equations

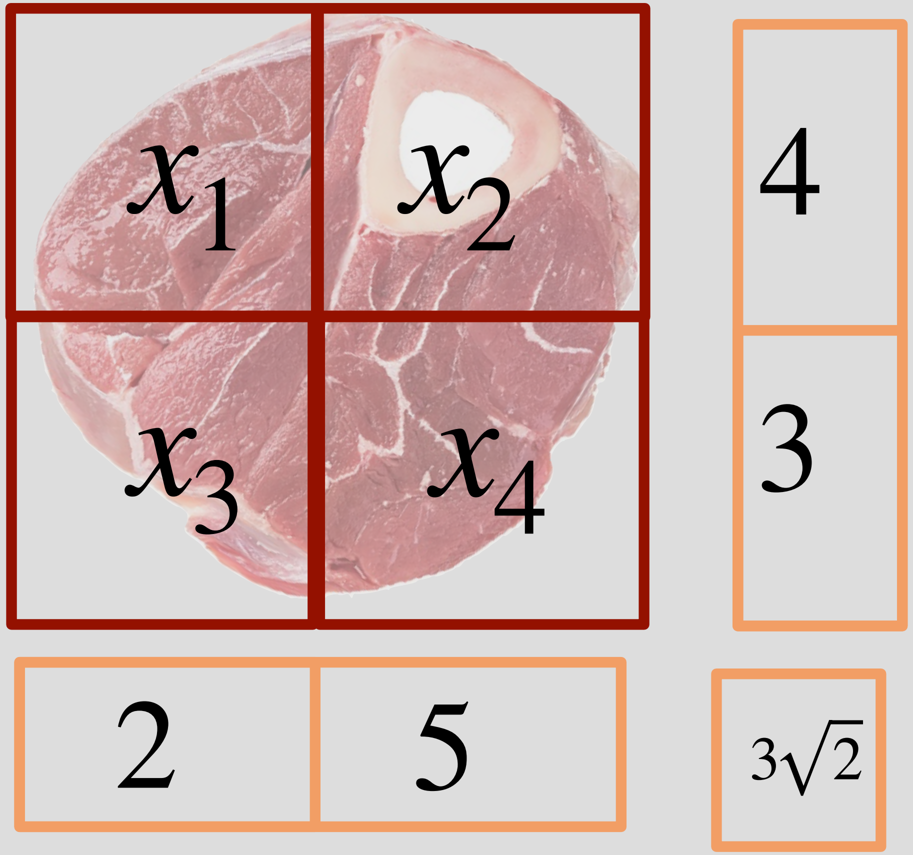
- Consider the set of M linear equations with N variables:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N &= b_2 \\
\vdots \qquad\qquad & \vdots \\
a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N &= b_M
\end{aligned}
$$

- Can be written compactly using augmented matrix:

$$
\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1N} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2N} & b_2 \\
\vdots & & \vdots & & \vdots \\
a_{M1} & a_{M2} & \cdots & a_{MN} & b_M
\end{array}\right]
$$

# Back to Tomography




$$1 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 4$$

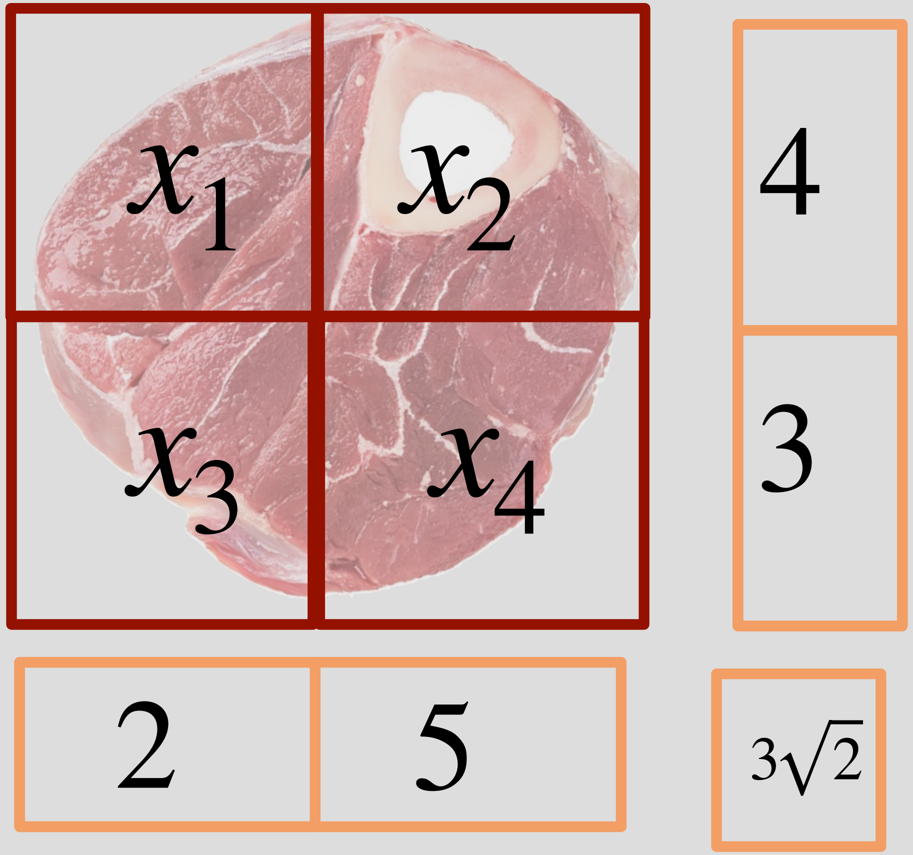
$$0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 1 \cdot x_4 = 3$$

$$1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 = 2$$

$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 5$$

$$\sqrt{2}x_1 + 0 \cdot x_2 + 0 \cdot x_3 + \sqrt{2}x_4 = 3\sqrt{2}$$

# Back to Tomography

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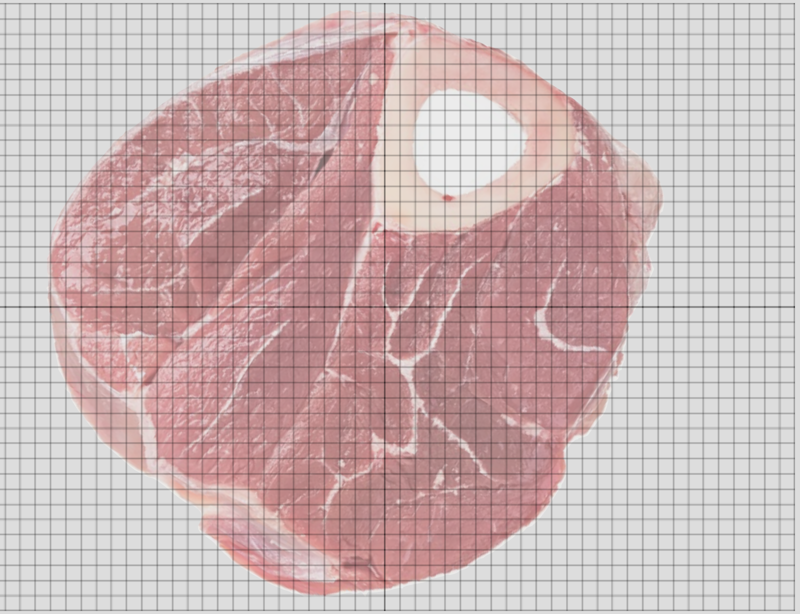
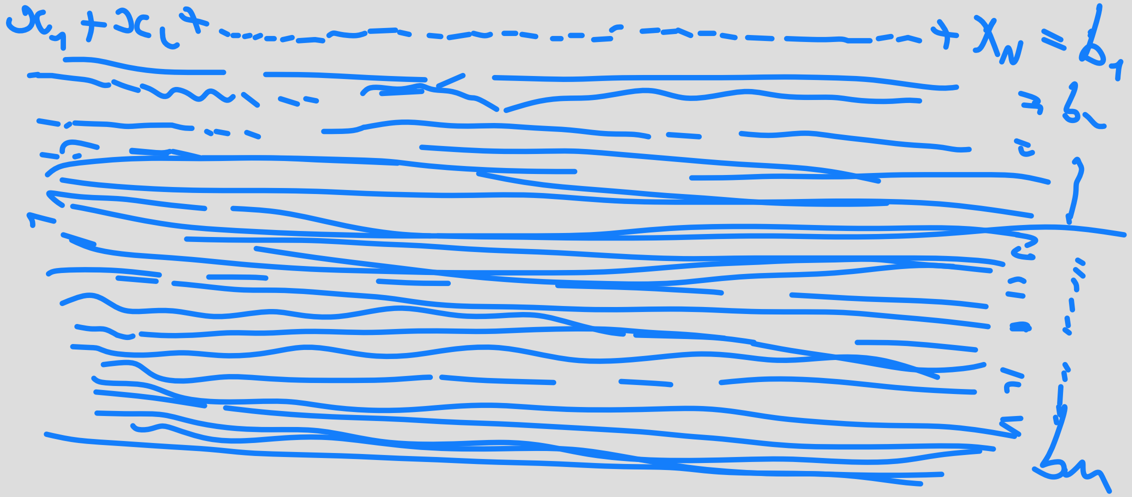
$$\sqrt{2}x_1 + 0 \cdot x_2 + 0 \cdot x_3 + \sqrt{2}x_4 = 3\sqrt{2}$$

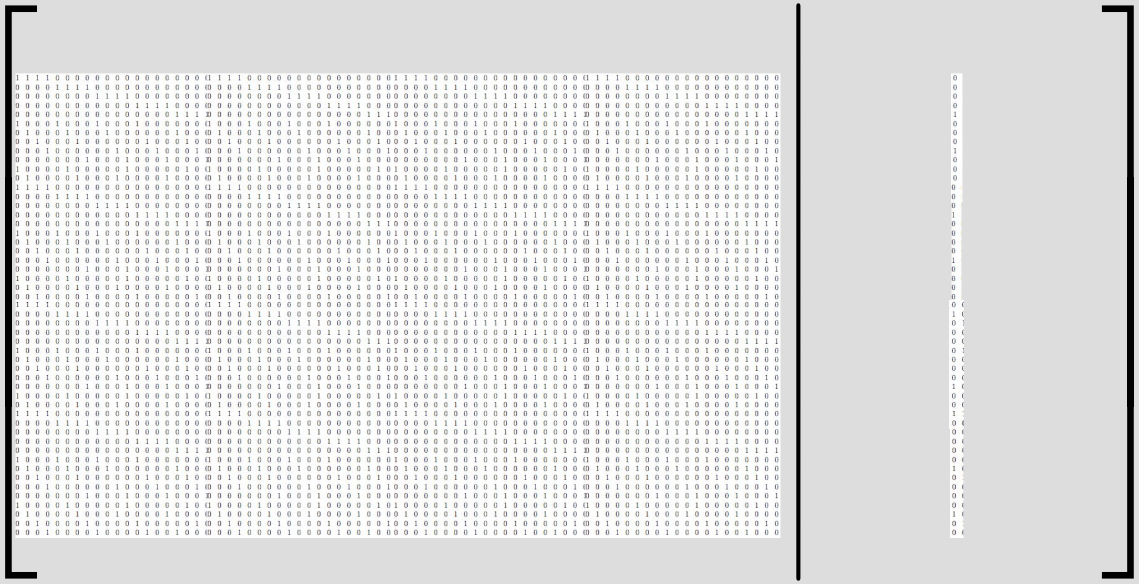
$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \\ \sqrt{2} & 0 & 0 & \sqrt{2} & 3\sqrt{2} \end{array}\right]$$

How do we solve it?

# Back to Tomography





How do we systematically solve it?

# Algorithm for solving linear equations

- Three basic operations that don't change a solution:
  1. Multiply an equation with *nonzero* scalar
  2. Adding a scalar constant multiple of one equation to another
  3. Swapping equations

# Algorithm for solving linear equations

- Three basic operations that don't change a solution:
    1. Multiply an equation with *nonzero* scalar
    2. Adding a scalar constant multiple of one equation to another
    3. Swapping equations

$$\begin{array}{lrcrcl} (1) & x & + & y & = & 2 \\ (2) & 3x & + & 2y & = & 5 \end{array}$$

and

$$\begin{array}{lrcrcl} (1) & 3x & + & 2y & = & 5 \\ (2) & x & + & y & = & 2 \end{array}$$

Have the same solution

Proof: Pretty obvious!

# Algorithm for solving linear equations

- Three basic operations that don't change a solution:
  1. Multiply an equation with *nonzero* scalar

$$2x + 3y = 4 \quad \text{has the same solution as:} \quad 4x + 6y = 8$$

Proof for N=2:

Let $ax + by = c$, with solution $x_0, y_0$
$\Rightarrow ax_0 + by_0 = c$

Show that $\beta ax + \beta by = \beta c$,
has the same solution.

Substitute $x_0, y_0$ for $x, y$:

$$\beta ax_0 + \beta by_0 = \beta c$$

$$\beta(ax_0 + by_0) = \beta c$$

$$\beta c = \beta c$$

But is it the only solution?

$\beta ax + \beta by = \beta c$, with solution: $x_1, y_1$
$\Rightarrow \beta ax_1 + \beta by_1 = \beta c$

Show that $ax + by = c$,
has the same solution.....

Since $\beta \neq 0$....

$$\beta ax_1 + \beta by_1 = \beta c \Rightarrow ax_1 + by_1 = c$$

SOLUTION OF ONE, IMPLIES THE OTHER
AND VICE-VERSA!