Welcome to EECS 16A!
Designing Information Devices and Systems I

Ana Arias and Miki Lustig

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Lecture 1A
Tomography and Linear Equations
Module 1: Imaging
Image

Merriam-Webster: A visual representation of something

Imaging

Merriam-Webster: the action or the process of producing an image
Different Images

- Thermal Image
  - Infra-red
- Radiograph
  - X-Ray
- Computed Tomography
  - X-Ray
- Camera
  - Visible light Photons
- MRI
  - magnetic-fields
  - Radio waves
- Cosmic-Microwave background Radiation
- Ultrasound
- PET
  - Gamma radiation
Imaging Systems in General

Energy source

Imaged body

Energy detection

Imaging System
(electronics, control, computing, algorithms, visualization…)}
“Medical imaging” circa 1632
“The Anatomy Lesson of Dr. Nicolaes Tulp”, Rembrandt
Mauritshuis, The Hague
Projection Xray
Projection Xray
Tomography

‘tomo’ – slice
‘graphy’ – to write

Assume it is not desirable to slice open leg. How does tomography visualizes cross-sectional slices?
From Projections

Projections

Axial Slices

Sagittal Slices
3D Rendering from Slices
Computed Tomography

Detector array

x-ray source

Sinogram

Compute

cross-section
Computed Tomography

http://www.youtube.com/watch?v=4gklQHM19aY&feature=related
Modeling Tomography

\[ e^{-x_1} e^{-x_2} e^{-x_3} e^{-x_4} e^{-x_5} e^{-x_6} \hat{y} = 1 \cdot e^{-(x_1 + x_2 + x_3 + x_4 + x_5 + x_6)} = \hat{y} \]

\[ \log\{e^{-(x_1 + x_2 + x_3 + x_4 + x_5 + x_6)}\} = \log\{\hat{y}\} \]

\[ y = -\log\{\hat{y}\} \]

\[ x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = y \]

.... or y is the sum of x-ray attenuation coefficients along a line
Modeling Tomography

\[ y = 1 \cdot x \]

\[ x = y \]
Modeling Tomography

\[ y = x_1 + x_2 + x_3 + x_4 \]

1 equation 4 unknowns!
Modeling Tomography

\[ y_1 = x_1 + x_2 \]
\[ y_2 = x_3 + x_4 \]

2 equation 4 unknowns!
Modeling Tomography

Can we solve this?

\begin{align*}
y_1 &= x_1 + x_2 \\
y_2 &= x_3 + x_4 \\
y_3 &= x_1 + x_3 \\
y_4 &= + x_2 + x_4
\end{align*}
Modeling Tomography

\[
\begin{align*}
y_1 &= x_1 + x_2 \\
y_2 &= x_3 + x_4 \\
y_3 &= x_1 + x_3 \\
y_4 &= x_2 + x_4 \\
y_5 &\approx \sqrt{2}x_1 + \sqrt{2}x_4
\end{align*}
\]

May be able to solve this!
Modeling Tomography

Possible reconstruction

Blurred version of:
All our measurements are (converted to) **linear**

What does that mean? Each variable (x) is multiplied by a scalar to contribute to the measurement

\[
\begin{align*}
y_1 &= x_1 + x_2 \\
y_2 &= x_3 + x_4 \\
y_3 &= x_1 + x_3 \\
y_4 &= x_2 + x_4 \\
y_5 &= \sqrt{2}x_1 + \sqrt{2}x_4
\end{align*}
\]

This is called a **system of linear equations**

**Linear Algebra** is what we need to solve it!
Camera Model

Lens maps image onto sensor
Each pixel is sensed separately

\[ y_i = 1 \cdot x_i \]
Single Pixel Scanner

• What if we had only a single sensor?
• How can we create an image?

$y_i = 1 \cdot x_i$

https://www.youtube.com/watch?v=U5PwsVqHT8Y
Non-moving Single Pixel Camera

• Use a projector to illuminate pixels
• Sense reflected light with a sensor

\[ y_i = 1 \cdot x_i \]
Non-moving Single Pixel Camera

- Use a projector to illuminate several pixels!
- Sense reflected light with a sensor
- Make many measurements and solve the equations!

\[ y_1 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \]

Intensity=1

Similar math as Tomography!
Non-moving Single Pixel Camera

- Use a projector to illuminate several pixels!
- Sense reflected light with a sensor
- Make many measurements and solve the equations!

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Intensity=1

Similar math as Tomography!
Imaging Lab #1 Setup
Imaging Lab #1

Sensor → Analog Circuit → Analog to Digital → Post Processing

Solar Cell

5V

100 kΩ

1 µF

IP[y]: IPython
Non-moving Single Pixel Camera

- How many measurements do you need?
- What are the best patterns?
What is linear algebra?

• The study of linear functions and linear equations, typically using vectors and matrices

• Linearity is not always applicable, but can be a good first-order approximation

• There exist good fast algorithms to solve these problems
Linear Equations

• Definition:

Consider: \( f(x_1, x_2, \ldots, x_N) : \mathbb{R}^n \to \mathbb{R} \)

\( f \) is linear if the following identity holds:

(1) Homogeneity:

\[
\alpha f(x_1, \ldots, x_N) = f(\alpha x_1, \ldots, \alpha x_N)
\]

(2) Super Position (distributivity): if \( x_i = y_i + z_i \), then

\[
f(y_1 + z_1, \ldots, y_N + z_N) = f(y_1, \ldots, y_N) + f(z_1, \ldots, z_N)
\]

Claim: linear functions can always be expressed as:

\[
f(x_1, x_2, \ldots, x_N) = c_1 x_1 + c_2 x_2 + \cdots + c_N x_N
\]
Proof for \( \mathbb{R}^2 \)

- \( f(x_1, x_2) : \mathbb{R}^2 \Rightarrow \mathbb{R} \) is linear. Need to prove: \( f(x_1, x_2) = c_1 x_1 + c_2 x_2 \)

**Trick:**

\[
\begin{align*}
x_1 &= 1 \cdot x_1 + 0 \cdot x_2 \\
x_2 &= 0 \cdot x_1 + 1 \cdot x_2
\end{align*}
\]

So,

\[
\begin{align*}
f(x_1, x_2) &= f(x_1 \cdot 1 + x_2 \cdot 0, x_1 \cdot 0 + x_2 \cdot 1) \\
&= f(x_1 \cdot 1, x_1 \cdot 0) + f(x_2 \cdot 0, x_2 \cdot 1) \\
&= f(x_1 \cdot 1, x_1 \cdot 0) + f(x_2 \cdot 0, x_2 \cdot 1) \\
&= x_1 f(1,0) + x_2 f(0,1) \\
&= c_1 x_1 + c_2 x_2
\end{align*}
\]
Linear Set of Equations

• Consider the set of $M$ linear equations with $N$ variables:

\[
\begin{align*}
    a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N &= b_1 \\
    a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N &= b_2 \\
    \vdots & \vdots \\
    a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N &= b_M
\end{align*}
\]

• Can be written compactly using augmented matrix:

\[
\begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1N} & b_1 \\
    a_{21} & a_{22} & \cdots & a_{2N} & b_2 \\
    \vdots & \vdots & & \vdots & \vdots \\
    a_{M1} & a_{M2} & \cdots & a_{MN} & b_M
\end{bmatrix}
\]
Back to Tomography

\[
\begin{align*}
1 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 &= 4 \\
0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 1 \cdot x_4 &= 3 \\
1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 &= 2 \\
0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 &= 5 \\
\sqrt{2}x_1 + 0 \cdot x_2 + 0 \cdot x_3 + \sqrt{2}x_4 &= 3\sqrt{2}
\end{align*}
\]
How do we solve it?

1 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 4
0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 1 \cdot x_4 = 3
1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 = 2
0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 5
\sqrt{2}x_1 + 0 \cdot x_2 + 0 \cdot x_3 + \sqrt{2}x_4 = 3\sqrt{2}
Back to Tomography

How do we systematically solve it?
Algorithm for solving linear equations

• Three basic operations that don’t change a solution:
  1. Multiply an equation with *nonzero* scalar
  2. Adding a scalar constant multiple of one equation to another
  3. Swapping equations
Algorithm for solving linear equations

• Three basic operations that don’t change a solution:
  1. Multiply an equation with nonzero scalar
  2. Adding a scalar constant multiple of one equation to another
  3. Swapping equations

\[
\begin{align*}
(1) & \quad x + y = 2 \\
(2) & \quad 3x + 2y = 5
\end{align*}
\]

and

\[
\begin{align*}
(1) & \quad 3x + 2y = 5 \\
(2) & \quad x + y = 2
\end{align*}
\]

Have the same solution

Proof: Pretty obvious!
Algorithm for solving linear equations

• Three basic operations that don’t change a solution:

  1. Multiply an equation with *nonzero* scalar

     \[ 2x + 3y = 4 \text{ has the same solution as: } 4x + 6y = 8 \]

Proof for \( N=2 \):

Let \( ax + by = c \), with solution \( x_0, y_0 \)
\[ ax_0 + by_0 = c \]

Show that \( \beta ax + \beta by = \beta c \), has the same solution.

Substitute \( x_0, y_0 \) for \( x, y \):

\[ \beta ax_0 + \beta by_0 = \beta c \]
\[ \beta(ax_0 + by_0) = \beta c \]
\[ \beta c = \beta c \]

But is it the only solution?

Since \( \beta \neq 0 \):  
\[ \beta ax_1 + \beta by_1 = \beta c \Rightarrow ax_1 + by_1 = c \]

SOLUTION OF ONE, IMPLIES THE OTHER AND VICE-VERSA!