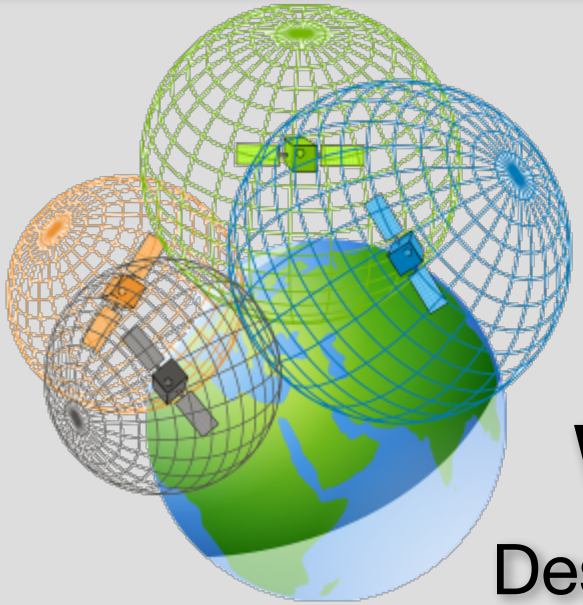
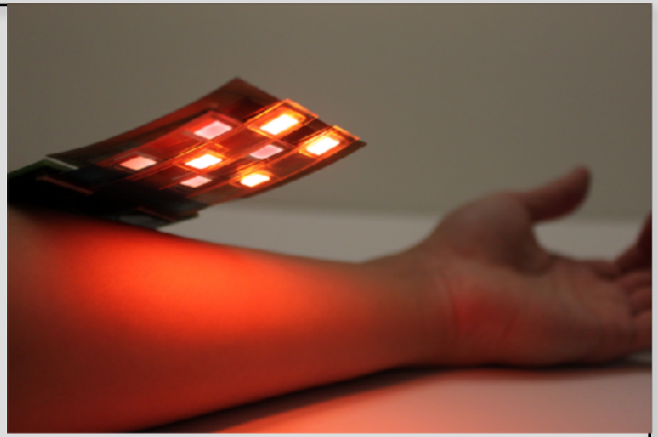




Welcome to EECS 16A!

Designing Information Devices and Systems I

Ana Arias and Miki Lustig

Sp 2022

Lecture 1A
Gaussian Elimination
Vectors

Previously on EECS16A

Seems like there were issues with the audio for the 1st 10min of the video in prev lecture.

Announcements

- Last time:
  - Tomography
  - Linear equations

- Today:
  - Solving sets of linear equations
    - Gaussian Elimination
  - Vectors

What is linear algebra?

- The study of linear functions and linear equations, typically using vectors and matrices
- Linearity is not always applicable, but can be a good first-order approximation
- There exist good fast algorithms to solve these problems

Linear Equations

Consider: $f(x) : \mathbb{R} \rightarrow \mathbb{R}$

$f(x) = bx^2$ $\qquad$ $f(x) = b^2x$

f is *linear if the following identity holds:*

(1) Homogeneity:

$$f(\alpha x) = \alpha f(x)$$

$$f(ax) = ba^2x^2 \qquad f(ax) = b^2ax$$

$$af(x) = bax^2 \qquad af(x) = b^2ax$$

(2) Super Position (distributivity): if $x = y + z$*, then*

$$f(y+z) = f(y) + f(z)$$

$$f(y+z) = b(y+z)^2 \qquad f(y+z) = b^2y + b^2z$$

$$f(y) + f(z) = by^2 + bz^2 \qquad f(y) + f(z) = b^2y + b^2z$$

Linear Equations

- Definition:

  Consider: $f(x_1, x_2, \cdots, x_N) : \mathbb{R}^n \to \mathbb{R}$

  f is *linear if the following identity holds:*

  (1) Homogeneity:

$$f(\alpha x_1, \cdots, \alpha x_N) = \alpha f(x_1, \ldots, x_N)$$

  (2) Super Position (distributivity): if $x_i = y_i + z_i$, *then*

$$f(y_1 + z_1, \cdots, y_N + z_N) = f(y_1, \cdots, y_N) + f(z_1, \cdots, z_N)$$

  Claim: linear functions can always be expressed as:

$$f(x_1, x_2, \cdots, x_N) = c_1 x_1 + c_2 x_2 + \cdots + c_N x_N$$

Proof for $\mathbb{R}^2$

- $f(x_1, x_2) : \mathbb{R}^2 \Rightarrow \mathbb{R}$ is linear. Need to prove: $f(x_1, x_2) = c_1x_1 + c_2x_2$

  Trick:

$$x_1 = 1 \cdot x_1 + 0 \cdot x_2$$
$$x_2 = 0 \cdot x_1 + 1 \cdot x_2$$

So,

$$\begin{aligned}
f(x_1, x_2) &= f(x_1 \cdot 1 + x_2 \cdot 0,\ x_1 \cdot 0 + x_2 \cdot 1) \\
&= f(x_1 \cdot 1, x_1 \cdot 0) + f(x_2 \cdot 0, x_2 \cdot 1) \\
&= f(\mathbf{x_1} \cdot 1, \mathbf{x_1} \cdot 0) + f(\mathbf{x_2} \cdot 0, \mathbf{x_2} \cdot 1) \\
&= x_1 \underbrace{f(1,0)}_{= c_1} + x_2 \underbrace{f(0,1)}_{= c_2} \\
&= c_1x_1 + c_2x_2
\end{aligned}$$

Linear Set of Equations

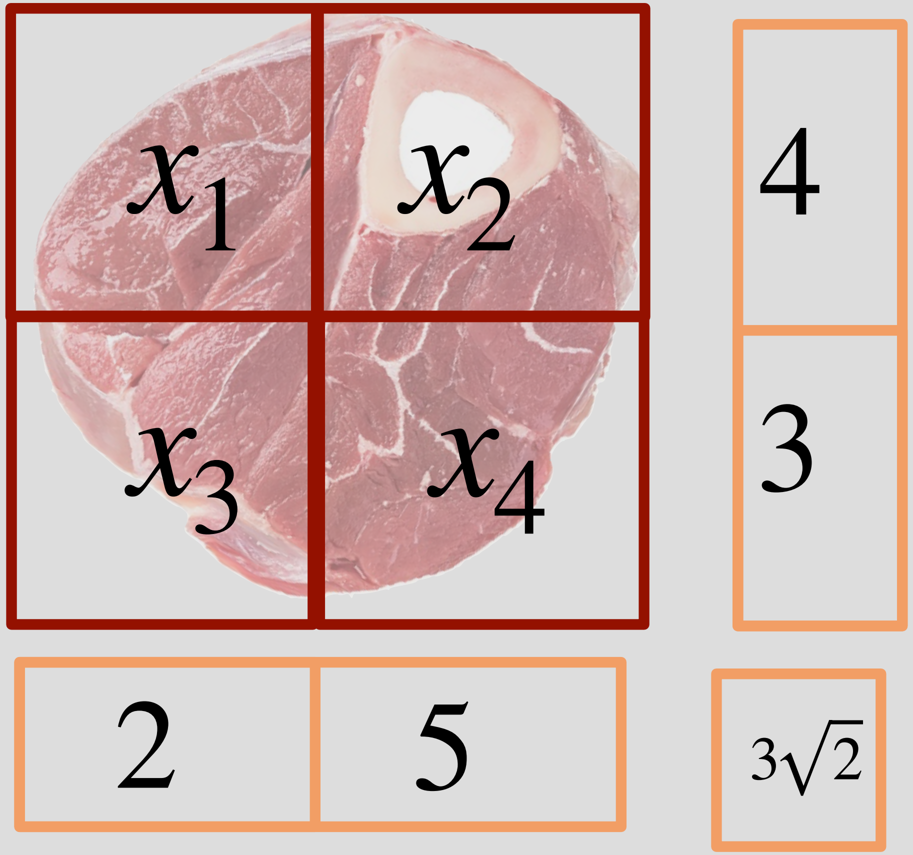
- Consider the set of M linear equations with N variables:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N &= b_2 \\
\vdots \qquad\qquad\qquad & \vdots \\
a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N &= b_M
\end{aligned}
$$

- Can be written compactly using augmented matrix:

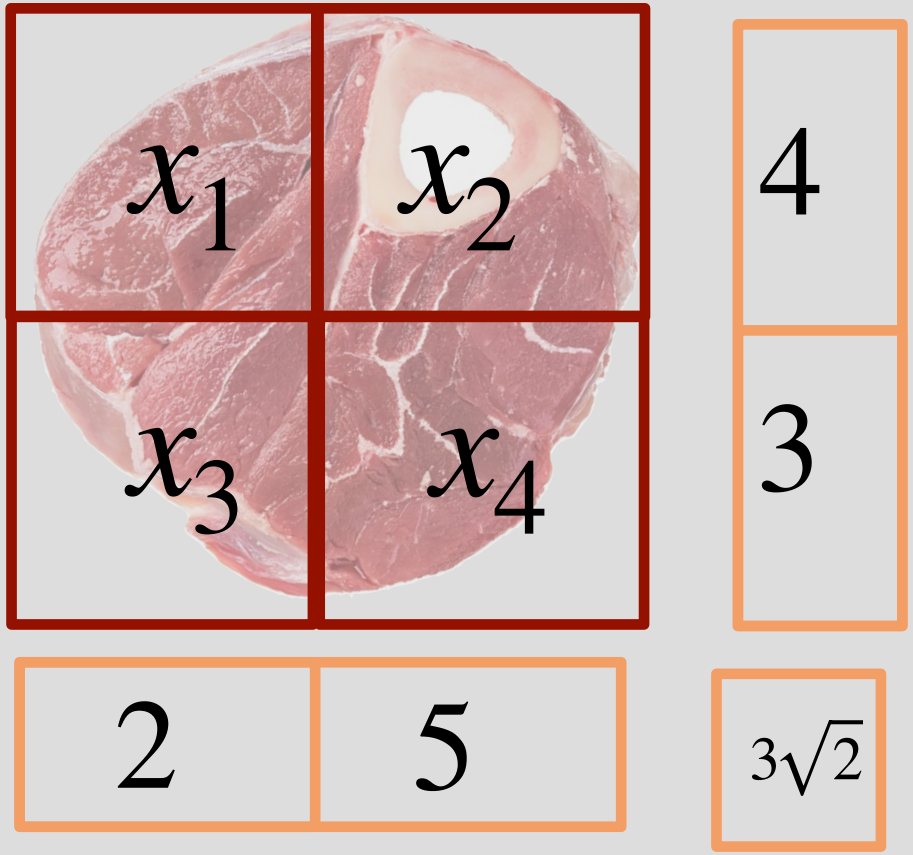
$$
\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1N} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2N} & b_2 \\
\vdots & & \vdots & & \vdots \\
a_{M1} & a_{M2} & \cdots & a_{MN} & b_M
\end{array}\right]
$$

Back to Tomography



$$1 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 4$$
$$0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 1 \cdot x_4 = 3$$
$$1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 = 2$$
$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 5$$
$$\sqrt{2}x_1 + 0 \cdot x_2 + 0 \cdot x_3 + \sqrt{2}x_4 = 3\sqrt{2}$$

Back to Tomography



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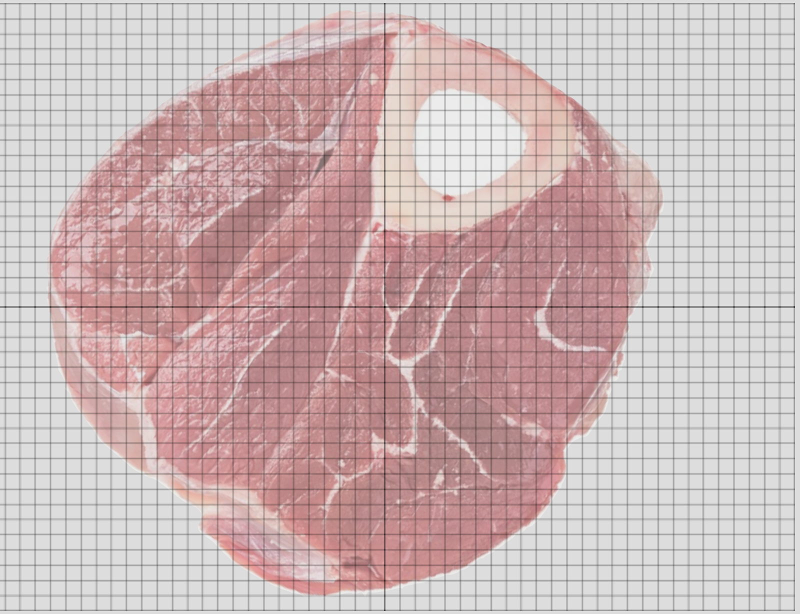
$$\sqrt{2}x_1 + 0 \cdot x_2 + 0 \cdot x_3 + \sqrt{2}x_4 = 3\sqrt{2}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \\ \sqrt{2} & 0 & 0 & \sqrt{2} & 3\sqrt{2} \end{array}\right]$$

How do we solve it?

Back to Tomography

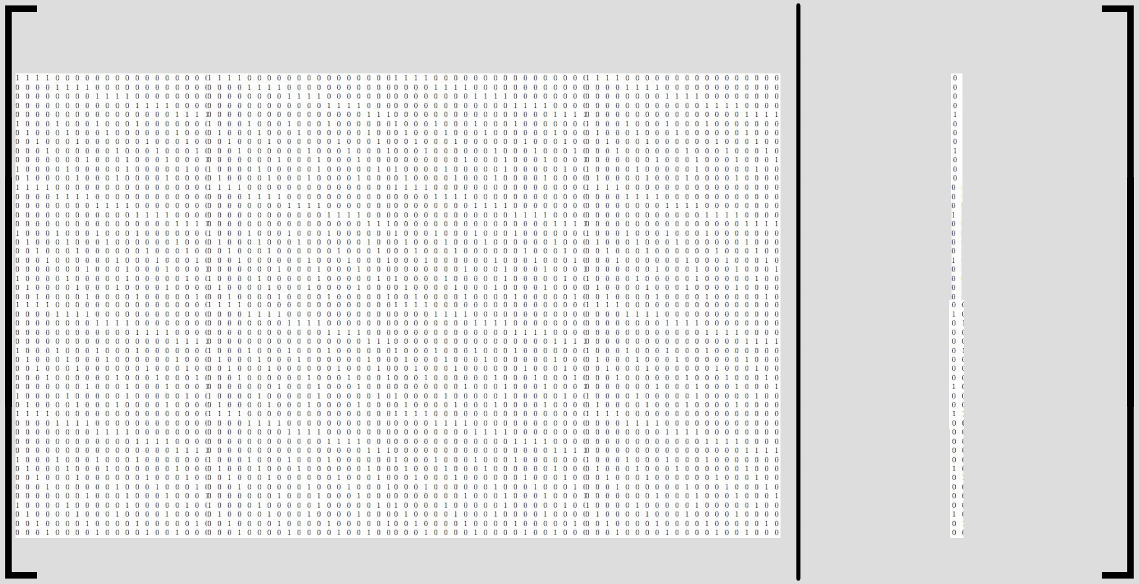


$$x_1 + x_2 + \dots + x_N = b_1$$

$$= b_2$$

$$\vdots$$

$$= b_m$$



How do we systematically solve it?

Algorithm for solving linear equations

- Three basic operations that don’t change a solution:
    1. Multiply an equation with *nonzero* scalar
    2. Adding a scalar constant multiple of one equation to another
    3. Swapping equations

Algorithm for solving linear equations

- Three basic operations that don't change a solution:
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  3. Swapping equations

$$\begin{array}{lrcrcl} (1) & x & + & y & = & 2 \\ (2) & 3x & + & 2y & = & 5 \end{array}$$

and

$$\begin{array}{lrcrcl} (1) & 3x & + & 2y & = & 5 \\ (2) & x & + & y & = & 2 \end{array}$$

Have the same solution

Proof: Pretty obvious!

Algorithm for solving linear equations

- Three basic operations that don't change a solution:
    1. Multiply an equation with *nonzero* scalar

$2x + 3y = 4$ has the same solution as: $4x + 6y = 8$

Concept of proof: look at explicit solution, show they are the same (next time)

Also show the reverse — by applying the reverse operations

Algorithm for solving linear equations

- Three basic operations that don't change a solution:
  1. Multiply an equation with *nonzero* scalar
  2. Adding a scalar constant multiple of one equation to another

$$\begin{array}{rrcrcr} (1) & x & + & y & = & 2 \\ (2) & 3x & + & 2y & = & 5 \end{array}$$

and

$$\begin{array}{rrcrcr} (1) & x & + & y & = & 2 \\ 3 \times (1) + (2) & 6x & + & 5y & = & 11 \end{array}$$

Have the same solution

Concept of proof: look at explicit solution, show they are the same

Also show the reverse — by applying the reverse operations

Upper Triangular Systems

- Consider the following equations:

$$
\begin{array}{rcrcrcl}
x & - & y & + & 2z & = & 1 \\
  &   & y & - & z & = & 2 \\
  &   &   &   & z & = & 1
\end{array}
\qquad
\left[\begin{array}{ccc|c}
 & & & \\
 & & & \\
 & & &
\end{array}\right]
$$

Upper Triangular Systems

- Consider the following equations:

$$\begin{aligned} x - y + 2z &= 1 \\ y - z &= 2 \\ z &= 1 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

- Why are they easy to solve?

Upper Triangular matrix \ Row Echelon

$$\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right]$$

Pivots

More general Row Echelon in the notes!

Gaussian Elimination

$$\begin{aligned} x - y + 2z &= 1 \\ 2x + y + z &= 8 \\ -4x + 5y &= 7 \end{aligned}$$

- Row-reduction to upper triang (Row echelon):

Step I

$$\left[\begin{array}{ccc|c} & & & \\ & & & \\ & & & \end{array}\right] \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

Gaussian Elimination

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Step II

$$\begin{array}{r} \\ (2) - 2\times(1) \\ (3) + 4\times(1) \end{array}\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ & & & \\ & & & \end{array}\right]\begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

Gaussian Elimination

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Step II

$$\begin{array}{r} \\ (2) - 2 \times (1) \\ (3) + 4 \times (1) \end{array} \left[\begin{array}{rrr|r} 1 & -1 & 2 & 1 \\ 0 & 3 & -3 & 6 \\ 0 & 1 & 8 & 11 \end{array}\right] \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

Step III

$$\begin{array}{r} \\ (2)/3 \\ \end{array} \left[\begin{array}{rrr|r} 1 & -1 & 2 & 1 \\ & & & \\ & & & \end{array}\right] \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

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$$\begin{array}{r} \\ (2)-2\times(1) \\ (3)+4\times(1) \end{array}\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 3 & -3 & 6 \\ 0 & 1 & 8 & 11 \end{array}\right]\begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

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Step IV

$$\begin{array}{r} \\ \\ (3)-(2) \end{array}\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ & & & \end{array}\right]\begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

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Step III

$$\begin{array}{r} \\ (2)/3 \\ \\ \end{array} \left[\begin{array}{rrr|r} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & 8 & 11 \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

Step IV

$$\begin{array}{r} \\ \\ (3) - (2) \end{array} \left[\begin{array}{rrr|r} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 9 & 9 \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

Step V

$$\begin{array}{r} \\ \\ (3)/9 \end{array} \left[\begin{array}{rrr|r} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ & & & \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

Gaussian Elimination

$$\begin{aligned} x &- y + 2z = 1 \\ 2x &+ y + z = 8 \\ -4x &+ 5y = 7 \end{aligned}$$

- Row-reduction to upper triang (Row echelon):

Step I

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 1 & 1 & 8 \\ -4 & 5 & 0 & 7 \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

Step II

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Step III

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Step IV

$$\begin{array}{r} \\ \\ (3)-(2) \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 9 & 9 \end{array}\right] \begin{array}{c} (1) \\ (2) \\ (3) \end{array}$$

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Pivots

Gaussian Elimination Cont.

$$\begin{array}{rcrcrcl} x & - & y & + & 2z & = & 1 \\ 2x & + & y & + & z & = & 8 \\ -4x & + & 5y & & & = & 7 \end{array}$$

- Back substitution:

Step V

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]\begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

Step VI

$$\begin{array}{r} (1) - 2 \times (3) \\ (2) + (3) \\ \\ \end{array}\left[\begin{array}{ccc|c} & & & \\ & & & \\ 0 & 0 & 1 & 1 \end{array}\right]\begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

Gaussian Elimination Cont.

$$\begin{array}{rcrcrcl} x & - & y & + & 2z & = & 1 \\ 2x & + & y & + & z & = & 8 \\ -4x & + & 5y & & & = & 7 \end{array}$$

- Back substitution:

Step V

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

Step VI

$$\begin{array}{r} (1) - 2 \times (3) \\ (2) + (3) \\ \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

Step VII

$$\begin{array}{r} (1) + (2) \\ \\ \end{array} \left[\begin{array}{ccc|c} & & & \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

Gaussian Elimination Cont.

$$\begin{aligned} x &- y &+ 2z &= 1 \\ 2x &+ y &+ z &= 8 \\ -4x &+ 5y & &= 7 \end{aligned}$$

- Back substitution:

Step V

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]\begin{array}{l}(1)\\(2)\\(3)\end{array}$$

Step VI

$$\begin{array}{r}(1)-2\times(3)\\(2)+(3)\\ \\ \end{array}\left[\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]\begin{array}{l}(1)\\(2)\\(3)\end{array}$$

Step VII

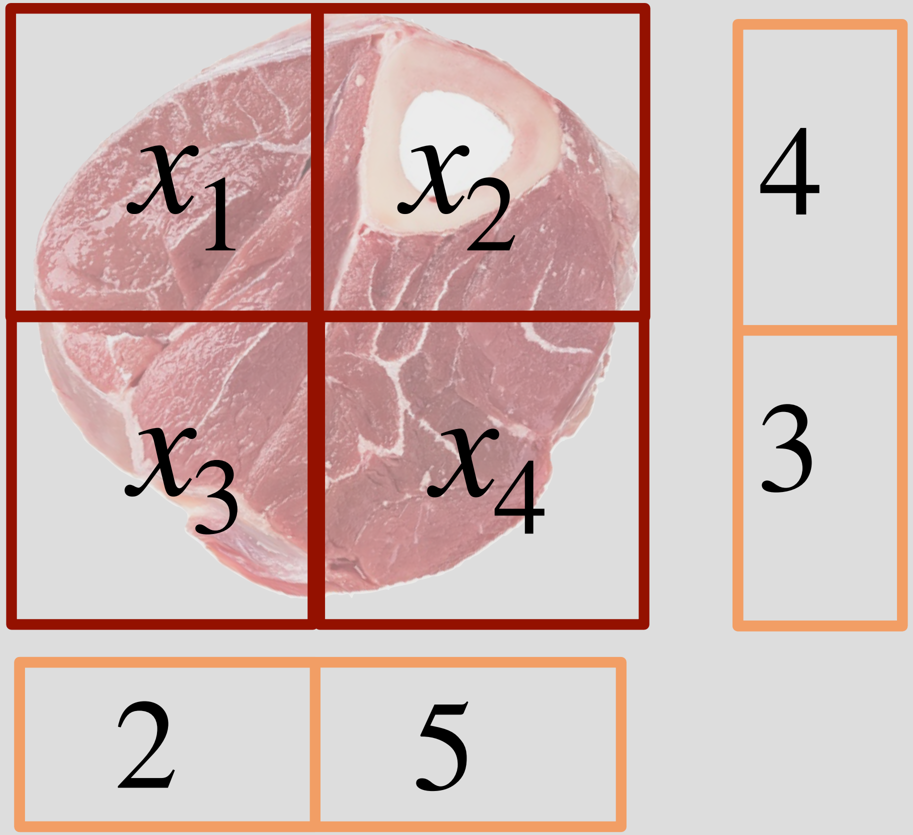
$$\begin{array}{r}(1)+(2)\\ \\ \\ \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right]\begin{array}{l}(1)\\(2)\\(3)\end{array}$$

$$\begin{array}{ccc} x & y & z \end{array}$$

Diagonal/identity matrix
(reduced Row-Echelon form)

$$\begin{aligned} x &= 2 \\ y &= 3 \\ z &= 1 \end{aligned}$$

Back to Tomography



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$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 5$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \end{array}\right]$$

How do we solve it?

Back to Tomography

Step I

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \end{array}\right]\begin{array}{l} (1) \\ (2) \\ (3) \\ (4) \end{array}$$

Step II

$$\begin{array}{r} \\ \\ (3)-(1) \\ \\ \end{array}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ & & & & \\ & & & & \end{array}\right]\begin{array}{l} (1) \\ (2) \\ (3) \\ (4) \end{array}$$

Step III

$$\begin{array}{r} \\ (4) \\ \\ (2) \end{array}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ & & & & \\ & & & & \\ & & & & \end{array}\right]\begin{array}{l} (1) \\ (2) \\ (3) \\ (4) \end{array}$$

Step IV

$$\begin{array}{r} \\ \\ (3)+(2) \\ \\ \end{array}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 5 \\ & & & & \\ & & & & \end{array}\right]\begin{array}{l} (1) \\ (2) \\ (3) \\ (4) \end{array}$$

Step V

$$\begin{array}{r} \\ \\ \\ (4)-(3) \end{array}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 \\ & & & & \end{array}\right]\begin{array}{l} (1) \\ (2) \\ (3) \\ (4) \end{array}$$

Back to Tomography

Step I

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \end{array}\right]\begin{array}{c}(1)\\(2)\\(3)\\(4)\end{array}$$

Step II

$$\begin{array}{r}\\ \\ (3)-(1)\\ \\ \end{array}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 1 & 0 & 1 & 5 \end{array}\right]\begin{array}{c}(1)\\(2)\\(3)\\(4)\end{array}$$

Step III

$$\begin{array}{r}\\ (4)\\ \\ (2)\end{array}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right]\begin{array}{c}(1)\\(2)\\(3)\\(4)\end{array}$$

Step IV

$$\begin{array}{r}\\ \\ (3)+(2)\\ \\ \end{array}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right]\begin{array}{c}(1)\\(2)\\(3)\\(4)\end{array}$$

Step V

$$\begin{array}{r}\\ \\ \\ (4)-(3)\end{array}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]\begin{array}{c}(1)\\(2)\\(3)\\(4)\end{array}$$

Infinite solutions!

Back to Tomography

Back substitution:

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$(1)-(2)\left[\begin{array}{cccc|c} & & & & \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Row Echelon

Pivots

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Reduced Row Echelon

Pivots

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & * & * \\ 0 & 1 & 0 & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

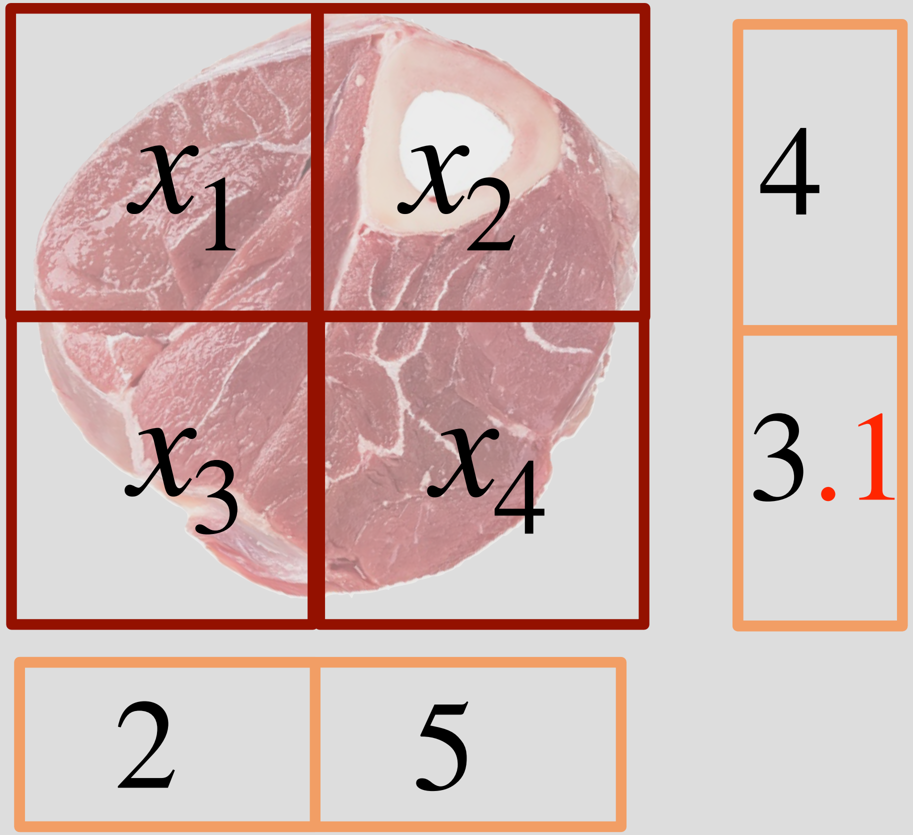
Basic variables

Free variables

Question?

Back to Tomography

Perturbations in the measurements



$$1 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 4$$
$$0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 1 \cdot x_4 = 3.1$$
$$1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 = 2$$
$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 5$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3.1 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \end{array}\right]$$

How do we solve it?

Back to Tomography

Step I

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3.1 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \end{array}\right]$$

Step II

$$\begin{array}{r} \\ \\ (3)-(1) \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3.1 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 1 & 0 & 1 & 5 \end{array}\right]$$

Step III

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 3.1 \end{array}\right]$$

Step IV

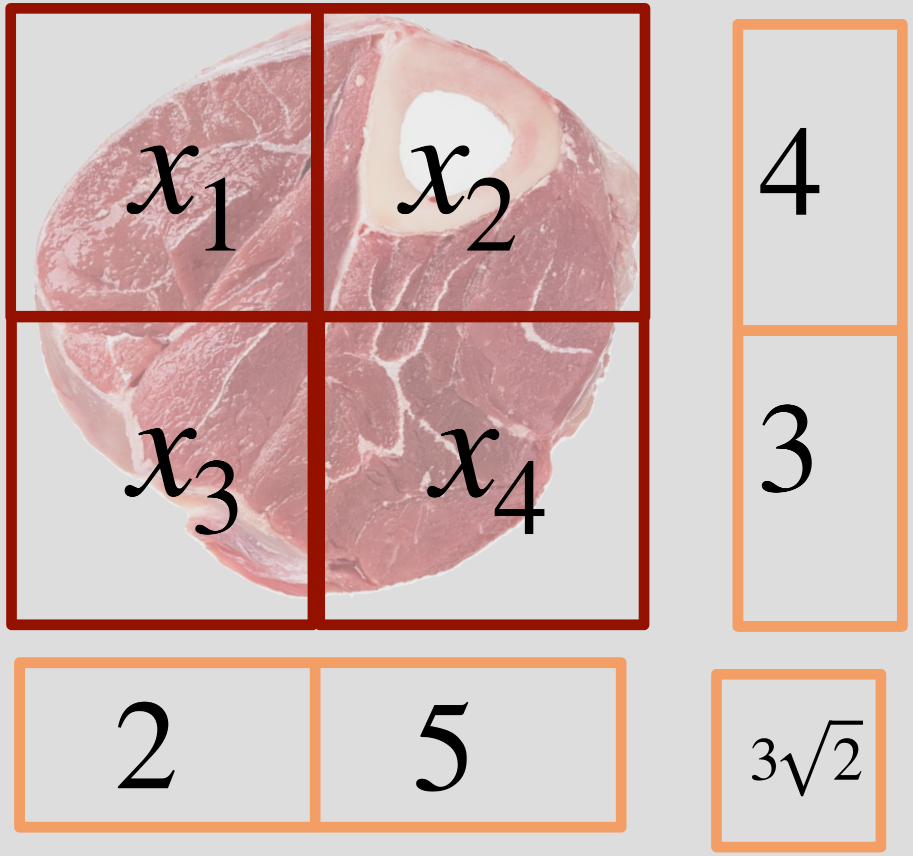
$$\begin{array}{r} \\ \\ (3)+(2) \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & 3.1 \end{array}\right]$$

Step V

$$\begin{array}{r} \\ \\ \\ (4)-(3) \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 \\ & & & & \end{array}\right]$$

No Solution!

Back to Tomography



$$1 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 4$$
$$0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 1 \cdot x_4 = 3$$
$$1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 = 2$$
$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 5$$
$$\sqrt{2}x_1 + 0 \cdot x_2 + 0 \cdot x_3 + \sqrt{2}x_4 = 3\sqrt{2}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \\ \sqrt{2} & 0 & 0 & \sqrt{2} & 3\sqrt{2} \end{array}\right]$$

How do we solve it?

Gaussian Elimination

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \\ \sqrt{2} & 0 & 0 & \sqrt{2} & 3\sqrt{2} \end{array}\right]$$

$(3) - (1)$

$(5) - \sqrt{2} \times (1)$

$-(5)/\sqrt{(}2)$

(2)

$(3) + (2)$

$(4) - (2)$

$(5) - (3)$

$(4)/2$

$(5) - (4)$

$(2) + (4)$

$(3) + (4)$

$(1) - (2)$

Gaussian Elimination

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \\ \sqrt{2} & 0 & 0 & \sqrt{2} & 3\sqrt{2} \end{array}\right]$$

$$\begin{array}{r} \\ \\ (3)-(1) \\ \\ (5)-\sqrt{2}\times(1) \end{array}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & -\sqrt{2} & 0 & \sqrt{2} & -\sqrt{2} \end{array}\right]$$

$$\begin{array}{r} \\ -(5)/\sqrt{(2)} \\ \\ \\ (2) \end{array}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 & -2 \\ 0 & 1 & 0 & 1 & 5 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right]$$

$$\begin{array}{r} \\ (3)+(2) \\ (4)-(2) \\ \\ \\ \end{array}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right]$$

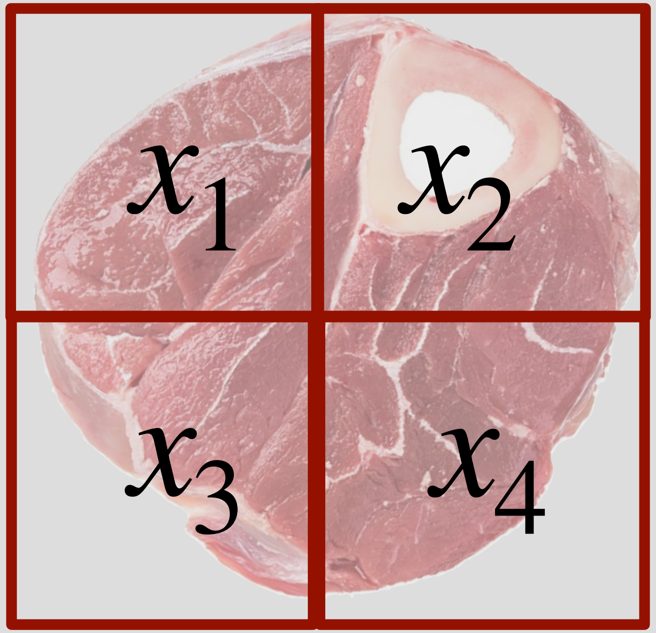
$$\begin{array}{r} \\ \\ \\ \\ (5)-(3) \end{array}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 2 & 4 \end{array}\right]$$

$$\begin{array}{r} \\ \\ \\ (4)/2 \\ (5)-(4) \end{array}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

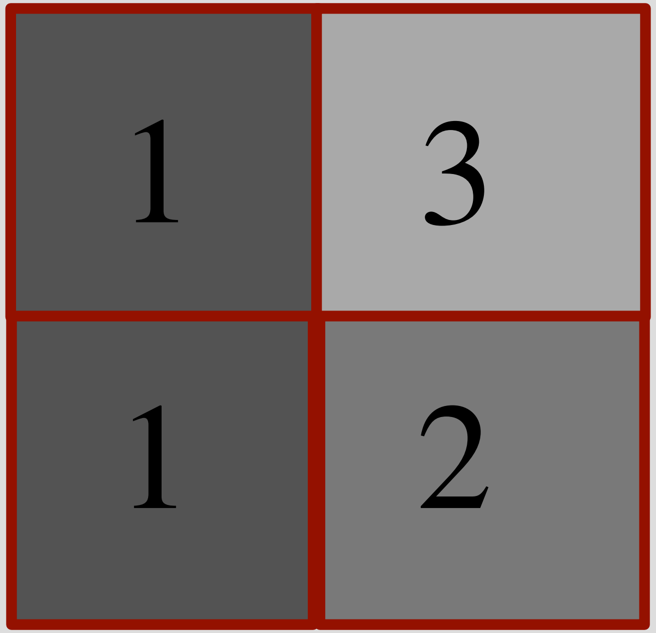
$$\begin{array}{r} \\ (2)+(4) \\ (3)+(4) \\ \\ \\ \end{array}\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{r} (1)-(2) \\ \\ \\ \\ \\ \end{array}\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Tomography Solved!



Possible reconstruction



$$\begin{array}{c} \begin{matrix} \;\;x_1 & x_2 & x_3 & x_4 \;\;\;\;\;\;\;\; \end{matrix} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Blurred version of :

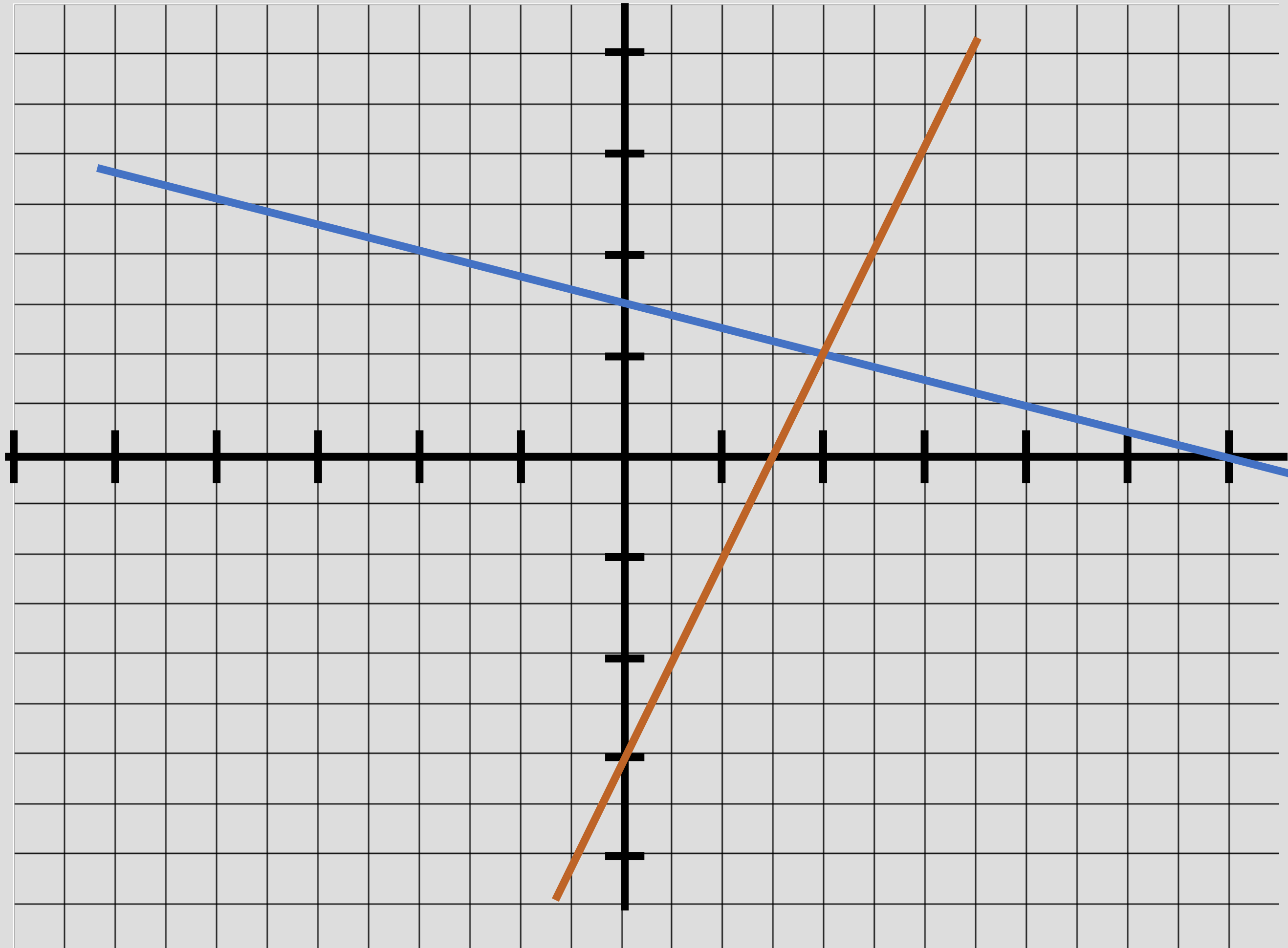
Geometric Interpretation

(1) $x + 4y = 6$ $\quad x = 0 \Rightarrow y = 1.5 \quad y = 0 \Rightarrow x = 6$

(2) $2x - y = 3$ $\quad x = 0 \Rightarrow y = -3 \quad y = 0 \Rightarrow x = 1.5$

Single Solution!

$x = 2,\ y = 1$



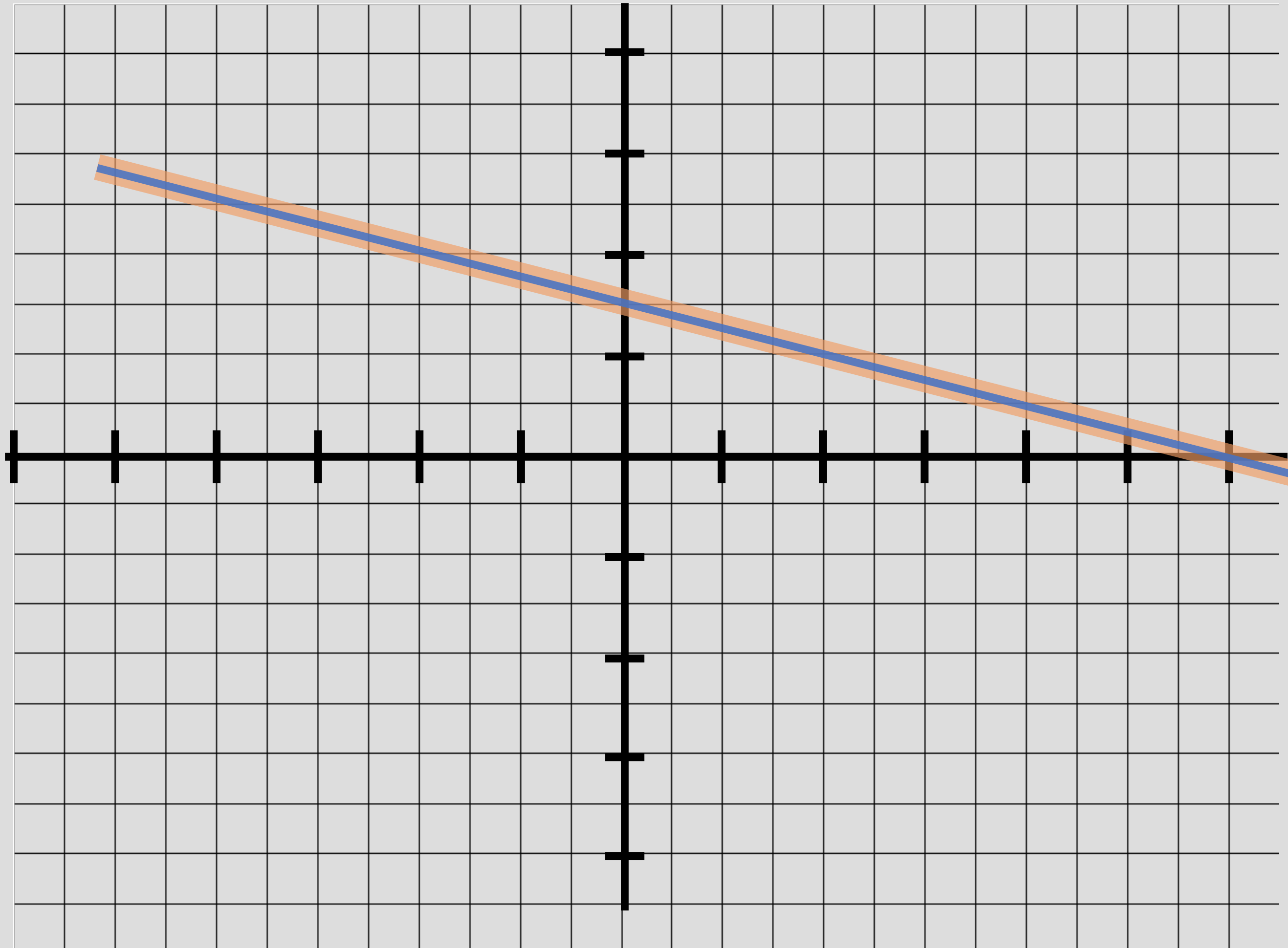
Geometric Interpretation

(1) $x + 4y = 6$ $\quad x = 0 \Rightarrow y = 1.5 \quad y = 0 \Rightarrow x = 6$

(2) $2x + 8y = 12$ $\quad x = 0 \Rightarrow y = 1.5 \quad y = 0 \Rightarrow x = 6$

Infinite Solutions!
anything that satisfies:

$$x = 6 - 4y_0$$



Geometric Interpretation

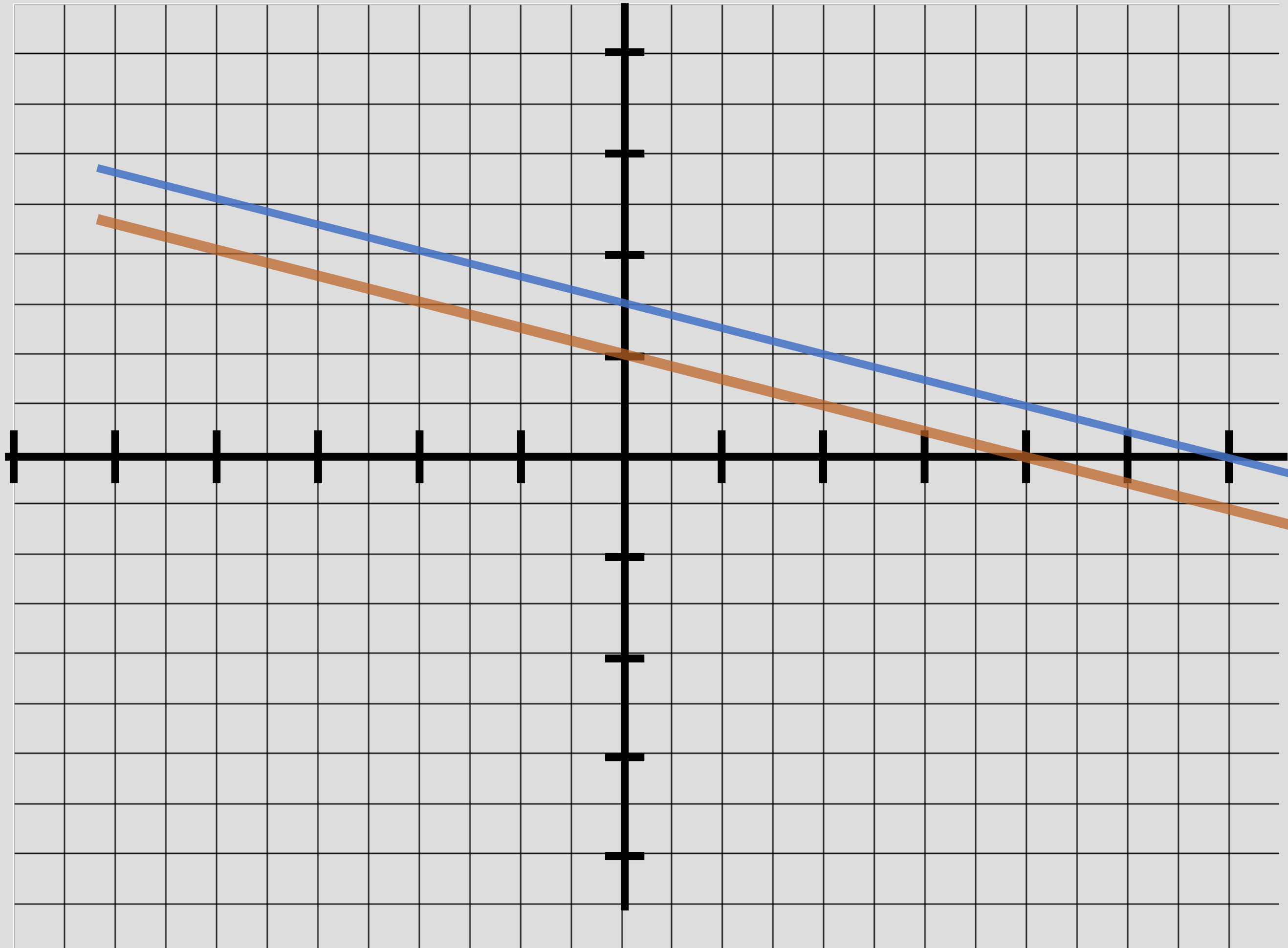
(1) $x + 4y = 6$ $\quad x = 0 \Rightarrow y = 1.5 \quad y = 0 \Rightarrow x = 6$

(2) $2x + 8y = \underline{8}$ $\quad x = 0 \Rightarrow y = 1 \quad y = 0 \Rightarrow x = 4$

No Solutions!

Parallel lines do not intersect!

$\varnothing$



Gaussian Elimination Summary

- Reduce to row-echelon form, from left-to-right by using:
  1. Multiply an equation with *nonzero* scalar
  2. Adding a scalar constant multiple of one equation to another
  3. Swapping equations

Single solution

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \end{array}\right]$$

Infinite solutions

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

No solution

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & * \end{array}\right]$$

- Back substitute to reduced row-echelon form, from right-to-left

Single solution

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \end{array}\right]$$

Pivots

Infinite solutions

$$\left[\begin{array}{cccc|c} 1 & 0 & * & 0 & * \\ 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Basic variables Free variables

Data: Augmented matrix $A \in \mathbb{R}^{m\times(n+1)}$, for a system of m equations with n variables
Result: Reduced form of augmented matrix
 # Forward elimination procedure:
for *each variable index i from 1 to n* **do**
  if *entry in row i, column i of A is 0* **then**
    if *all entries in column i and row $>$ i of A are 0* **then**
      proceed to next variable index;
    else
      find j, the smallest row index $> i$ of A for which entry in column $i \neq 0$;
      # The following rows implement the "swap" operation:
      `old_row_j` $\longleftarrow$ row j of A;
      row j of $A \longleftarrow$ row i of A;
      row i of $A \longleftarrow$ `old_row_j`;
    end
  end
  divide row i of A by entry in row i, column i of A;
  for *each row index k from* $i+1$ *to m* **do**
    `scaled_row_i` $\longleftarrow$ row i of A times entry in row k, column i of A;
    row k of $A \longleftarrow$ row k of A $-$ `scaled_row_i`;
  end
end
 # Back substitution procedure:
for *each variable index u from* $n-1$ *to* 1 **do**
  if *entry in row u, column u of A* $\neq 0$ **then**
    for *each row v from* $u-1$ *to 1* **do**
      `scaled_row_u` $\longleftarrow$ row u of A times entry in row v, column u of A;
      row v of $A \longleftarrow$ row v of A $-$ `scaled_row_u`;
    end
  end
end

Algorithm 1: The Gaussian elimination algorithm.

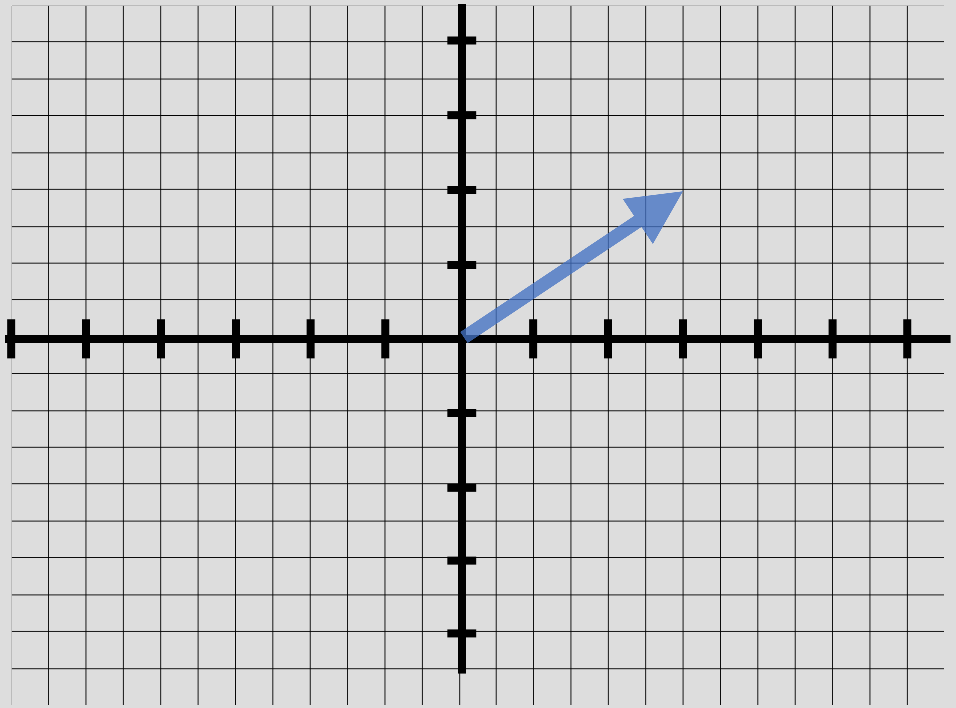
Vectors

- An array of N numbers
  - Represents coordinates in an N-dimensional space

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}, \qquad \vec{x} \in \mathbb{R}^N$$

- For example:

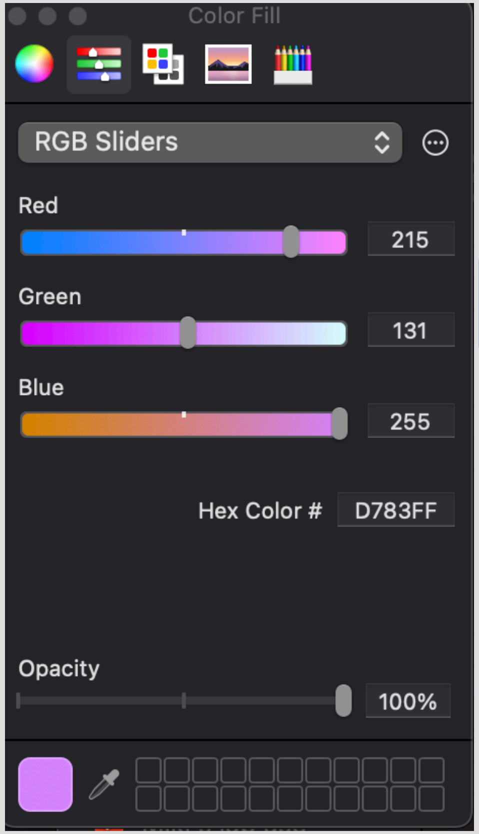
$$\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \qquad \vec{x} \in \mathbb{R}^2$$



Vectors

- Since it's an array of numbers, it can represent other things….

pixel color



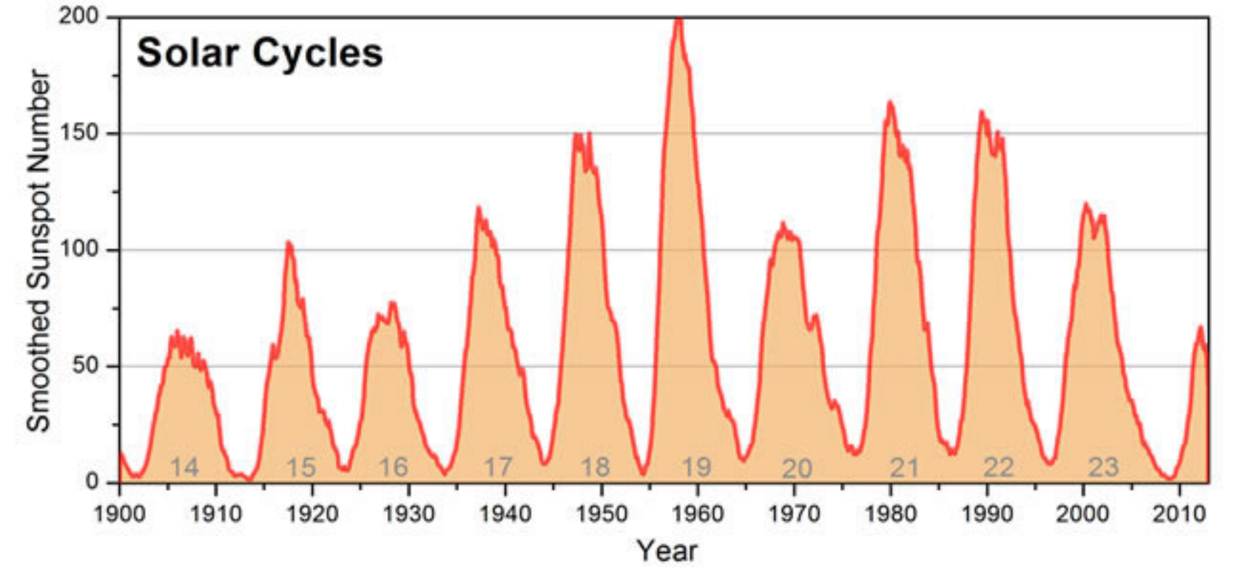
$$\vec{x} = \begin{bmatrix} 215 \\ 131 \\ 25 \end{bmatrix}$$

pixel values in an image

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Vectors

Data



$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{120} \end{bmatrix}$$

Special Vectors

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e}_N = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Matrices

- A collection of numbers in a rectangular form

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1M} \\ x_{21} & x_{22} & \cdots & x_{2M} \\ \vdots & \vdots & \cdots & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{NM} \end{bmatrix}, \quad X \in \mathbb{R}^{\mathbb{N} \times \mathbb{M}}$$

- Or a collection of M, N-length vectors

$$X = \begin{bmatrix} \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_M \\ \\ \end{bmatrix}, \quad X \in \mathbb{R}^{\mathbb{N} \times \mathbb{M}}$$

Vectors as Matrices

- A vector is a degenerate matrix

$$\vec{x} = \begin{bmatrix} x_1 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad \vec{x} \in \mathbb{R}^{N \times 1}$$

- A scalar is a degenerate vector or matrix

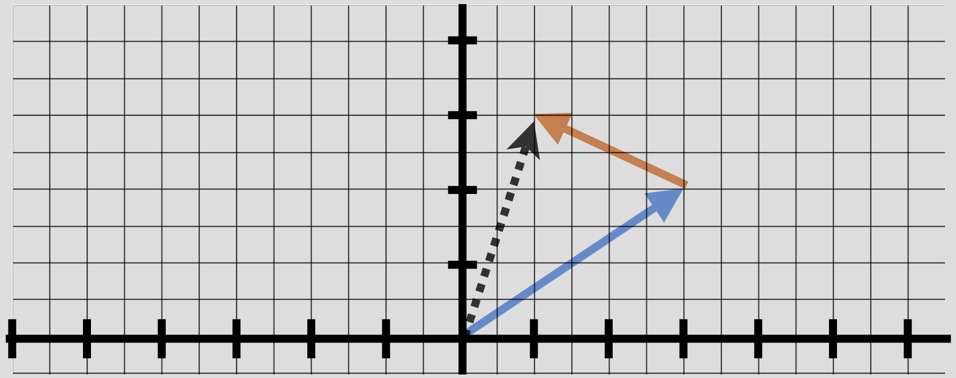
$$a \in \mathbb{R}^{1 \times 1}$$

Vector addition

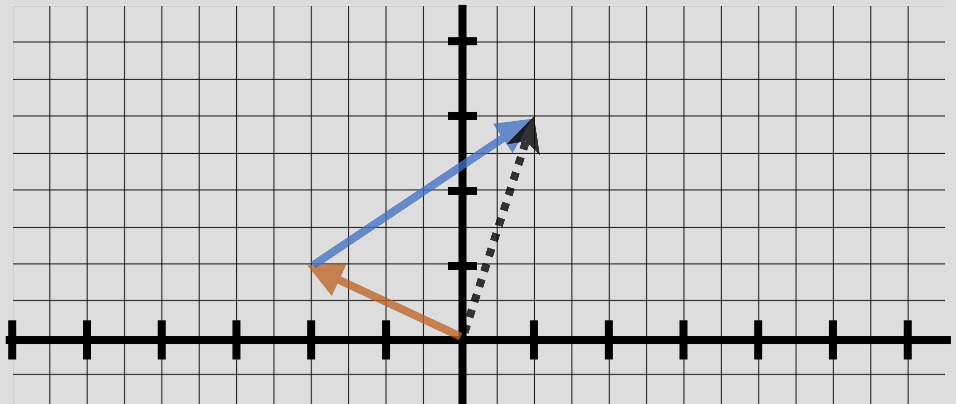
- Two vectors of the same length can be added
    - Addition is element-wise

$$\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \ \vec{y} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$\vec{x} + \vec{y} =$



$\vec{y} + \vec{x} =$



Properties of vector addition

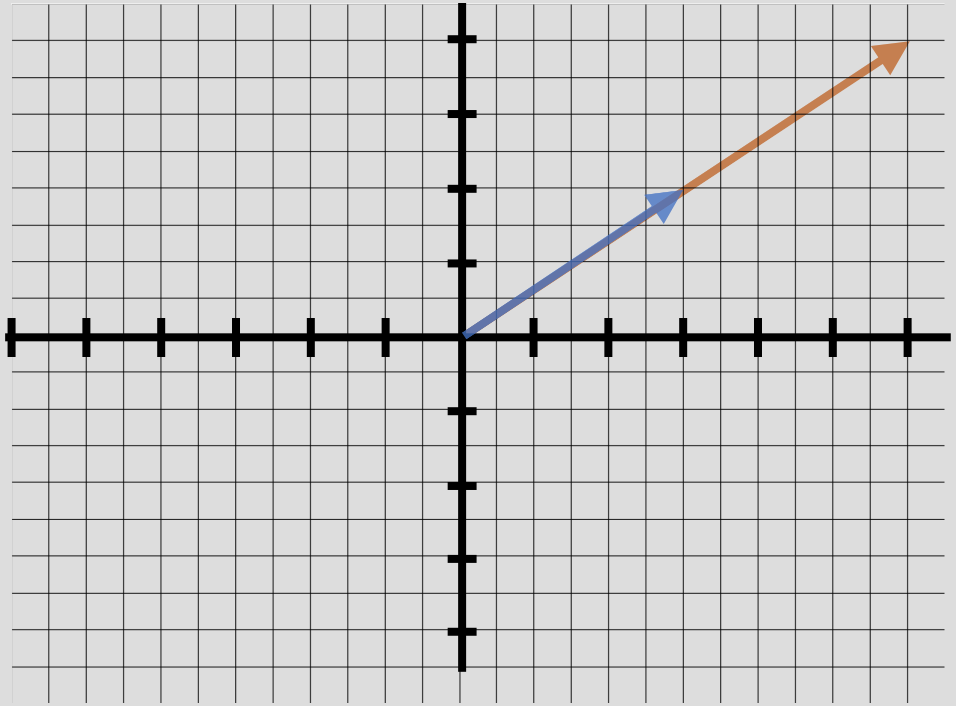
- Commutativity: $\overrightarrow{x} + \overrightarrow{y} = \overrightarrow{y} + \overrightarrow{x}$
- Associativity: $(\overrightarrow{x} + \overrightarrow{y}) + \vec{z} = \overrightarrow{x} + (\overrightarrow{y} + \vec{z})$
- Additive negative: $\overrightarrow{x} + (-\overrightarrow{x}) = \overrightarrow{0}$
- Additive identity: $\overrightarrow{x} + \overrightarrow{0} = \overrightarrow{x}$

Scalar Vector Multiplication

- Multiplying with a scalar result in multiplying each element.

$$a\vec{x} = \begin{bmatrix} ax_1 \\ ax_2 \\ \vdots \\ ax_{N\times 1} \end{bmatrix}$$

$$2 \cdot \vec{x} = 2 \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$



Vector Transpose

- $\overrightarrow{x}^T$ is the transpose of $\overrightarrow{x}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad \vec{x} \in \mathbb{R}^{N \times 1} \qquad \vec{x}^T = \begin{bmatrix} x_1 & x_1 & \cdots & x_N \end{bmatrix}, \quad \vec{x}^T \in \mathbb{R}^{1 \times N}$$

- $\overrightarrow{x}$ is always a column vector
- To represent a row vector, write: $\overrightarrow{x}^T$

Matrix Addition

- When matrices are the same size, they can be added element-wise

$$X + Y = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

- Scalar multiplication — by all elements

$$2X = 2\begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

Matrix Addition

- When matrices are the same size, they can be added element-wise

$$X + Y = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

diagonal matrix!

- Scalar multiplication — by all elements

$$2X = 2\begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -2 & -4 \end{bmatrix}$$

Vector Transpose

- $\overrightarrow{x}^T$ is the transpose of $\overrightarrow{x}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad \vec{x} \in \mathbb{R}^{N \times 1} \qquad\qquad \vec{x}^T = \begin{bmatrix} x_1 & x_1 & \cdots & x_N \end{bmatrix}, \quad \vec{x}^T \in \mathbb{R}^{1 \times N}$$

- $\overrightarrow{x}$ is always a column vector
- To represent a row vector, write: $\overrightarrow{x}^T$

Matrix Transpose

If the elements of the matrix $A \in \mathbb{R}^{N \times M}$ are a_{ij}

The elements of $A^T \in \mathbb{R}^{M \times N}$ are a_{ji}

Matrix transpose is not (generally) an inverse!

$$A \in \mathbb{R}^{N \times M} \quad \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_m \end{bmatrix}$$

$$\begin{bmatrix} \vec{a}_1^{\,T} \\ \vec{a}_2^{\,T} \\ \vdots \\ a_m^T \end{bmatrix} \quad A^T \in \mathbb{R}^{M \times N}$$

Matrix Transpose

If the elements of the matrix $A \in \mathbb{R}^{N \times M}$ are a_{ij}

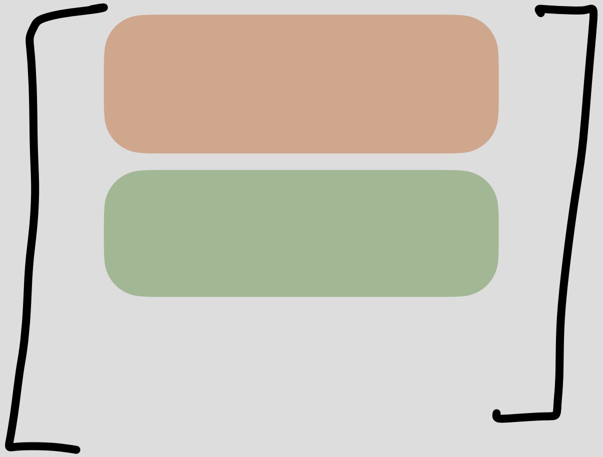
The elements of $A^T \in \mathbb{R}^{M \times N}$ are a_{ji}

Matrix transpose is not (generally) an inverse!

$A \in \mathbb{R}^{N \times M}$

$$\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$$

$A^T \in \mathbb{R}^{M \times N}$



Matrix Transpose

If the elements of the matrix $A \in \mathbb{R}^{N \times M}$ are a_{ij}

The elements of $A^T \in \mathbb{R}^{M \times N}$ are a_{ji}

Matrix transpose is not (generally) an inverse!

$A \in \mathbb{R}^{N \times M}$

$$\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$$

$A^T \in \mathbb{R}^{M \times N}$

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}$$

Matrix Transpose

If the elements of the matrix $A \in \mathbb{R}^{N \times M}$ are a_{ij}

The elements of $A^T \in \mathbb{R}^{M \times N}$ are a_{ji}

Matrix transpose is not (generally) an inverse!

$A \in \mathbb{R}^{N \times M}$

$$\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$$

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_m \end{bmatrix}$$

$A^T \in \mathbb{R}^{M \times N}$

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \vec{a}_1^{\,T} \\ \vec{a}_2^{\,T} \\ \vdots \\ a_m^T \end{bmatrix}$$