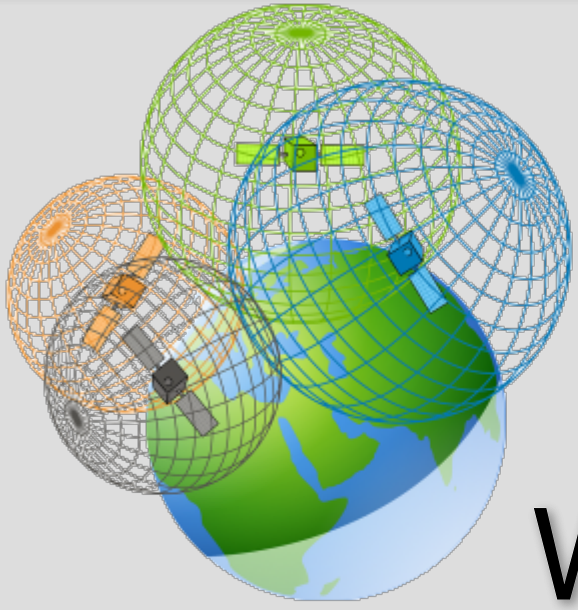
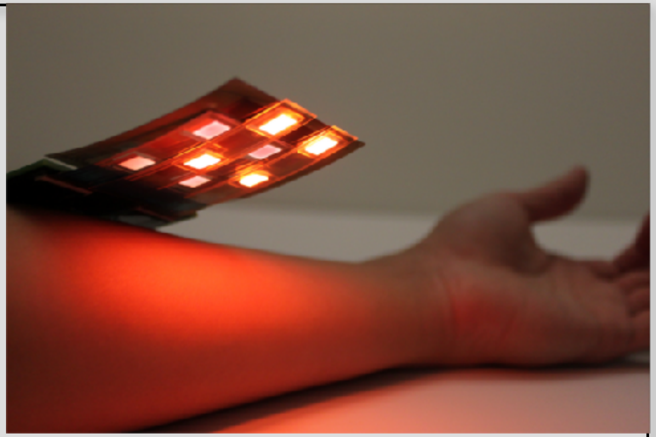




Welcome to EECS 16A!

Designing Information Devices and Systems I

Ana Arias and Miki Lustig

Fa 2022

Lecture 2A

Vectors, Matrices, Multiplications

Announcements

- Last time:
  - Gaussian Elimination
- Today:
  - vectors
  - Matrix-Matrix and Matrix-vector Multiplications
  - Matrix-Vector Multiplications as linear set of equations

Gaussian Elimination Summary

- Reduce to row-echelon form, from left-to-right by using:
    1. Multiply an equation with *nonzero* scalar
    2. Adding a scalar constant multiple of one equation to another
    3. Swapping equations

Single solution

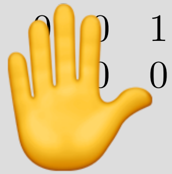
$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 1 & * \end{array}\right]$$

Infinite solutions

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

No solution

$$\left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & * \end{array}\right]$$



- Back substitute to reduced row-echelon form, from right-to-left

Single solution

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & * \\ 0 & 1 & 0 & 0 & * \\ 0 & 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 & * \end{array}\right]$$

Pivots

Infinite solutions

$$\left[\begin{array}{cccc|c} 1 & 0 & * & 0 & * \\ 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Basic variables

Free variables

Data: Augmented matrix $A \in \mathbb{R}^{m\times(n+1)}$, for a system of m equations with n variables
Result: Reduced form of augmented matrix
 # Forward elimination procedure:
for *each variable index i from 1 to n* **do**
  if *entry in row i, column i of A is 0* **then**
    if *all entries in column i and row $> i$ of A are 0* **then**
      proceed to next variable index;
    else
      find j, the smallest row index $> i$ of A for which entry in column $i \neq 0$;
      # The following rows implement the "swap" operation:
      old_row_j ⟵ row j of A;
      row j of A ⟵ row i of A;
      row i of A ⟵ old_row_j;
    end
  end
  divide row i of A by entry in row i, column i of A;
  for *each row index k from $i+1$ to m* **do**
    scaled_row_i ⟵ row i of A times entry in row k, column i of A;
    row k of A ⟵ row k of A − scaled_row_i;
  end
end
 # Back substitution procedure:
for *each variable index u from $n-1$ to 1* **do**
  if *entry in row u, column u of A $\neq 0$* **then**
    for *each row v from $u-1$ to 1* **do**
      scaled_row_u ⟵ row u of A times entry in row v, column u of A;
      row v of A ⟵ row v of A − scaled_row_u;
    end
  end
end

Algorithm 1: The Gaussian elimination algorithm.

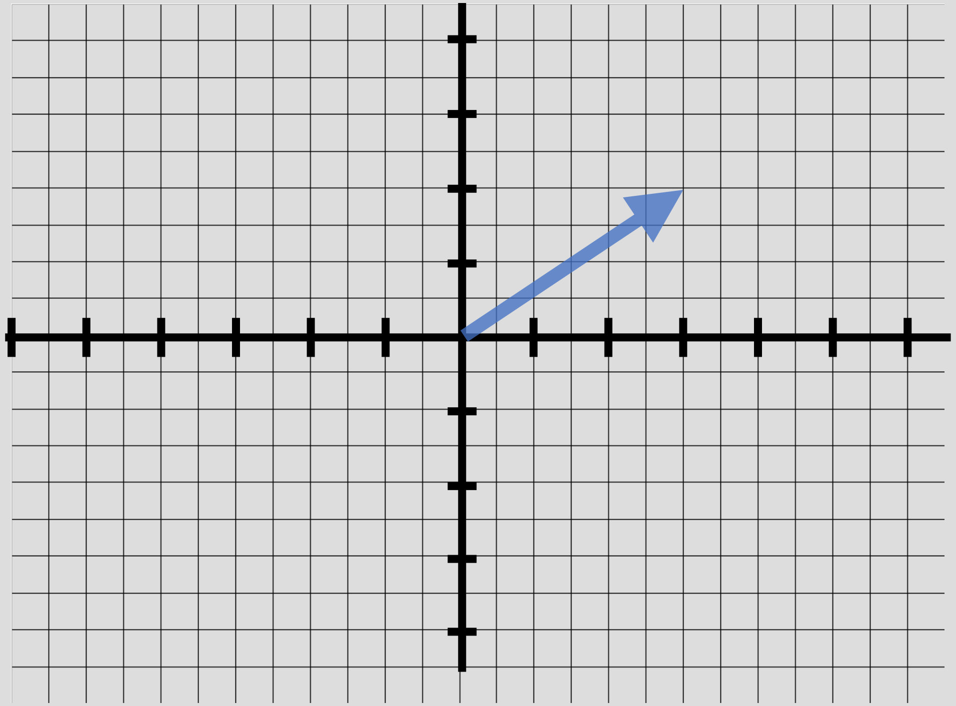
Vectors

- An array of N numbers
  - Represents coordinates in an N-dimensional space

$$\vec{x} = \begin{bmatrix} x_1 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}, \qquad \vec{x} \in \mathbb{R}^N$$

- For example:

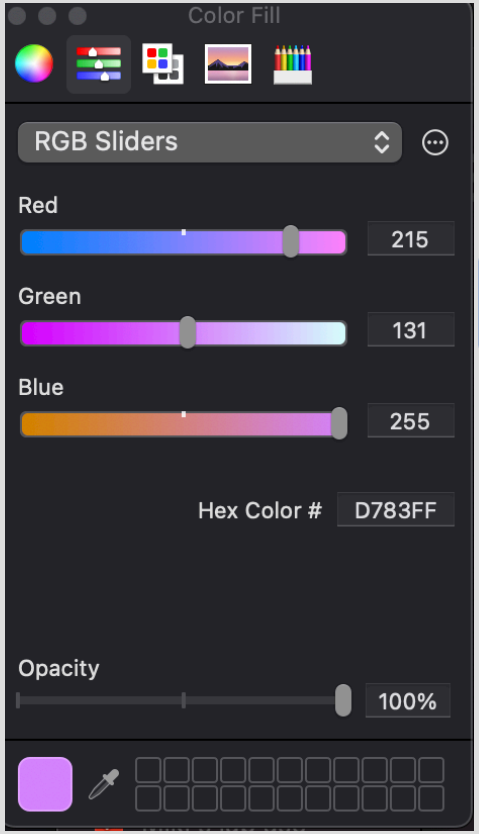
$$\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \qquad \vec{x} \in \mathbb{R}^2$$



Vectors

- Since it's an array of numbers, it can represent other things….

pixel color



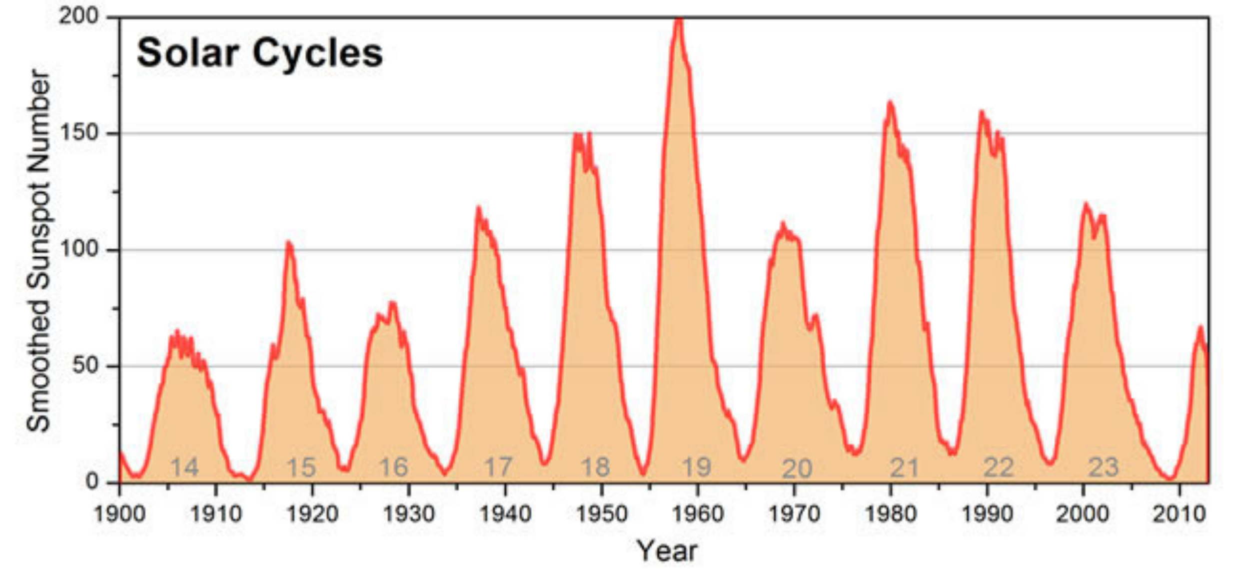
$$\vec{x} = \begin{bmatrix} 215 \\ 131 \\ 25 \end{bmatrix}$$

pixel values in an image

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Vectors

Data



$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{120} \end{bmatrix}$$

Special Vectors

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \quad \vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad \vec{e}_N = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Matrices

- A collection of numbers in a rectangular form

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1M} \\ x_{21} & x_{22} & \cdots & x_{2M} \\ \vdots & \vdots & \ldots & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{NM} \end{bmatrix}, \quad X \in \mathbb{R}^{\mathbb{N} \times \mathbb{M}}$$

- Or a collection of M, N-length vectors

$$X = \begin{bmatrix} \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_M \\ \\ \end{bmatrix}, \quad X \in \mathbb{R}^{\mathbb{N} \times \mathbb{M}}$$

Vectors as Matrices

- A vector is a degenerate matrix

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}, \quad \vec{x} \in \mathbb{R}^{N \times 1}$$

- A scalar is a degenerate vector or matrix

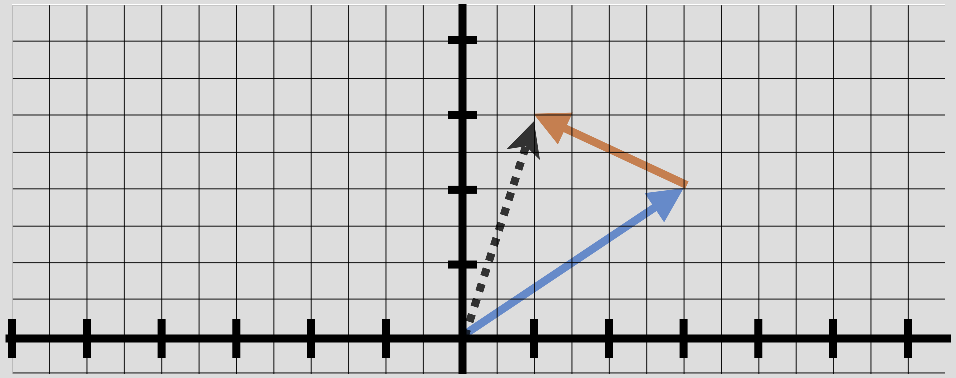
$$a \in \mathbb{R}^{1 \times 1}$$

Vector addition

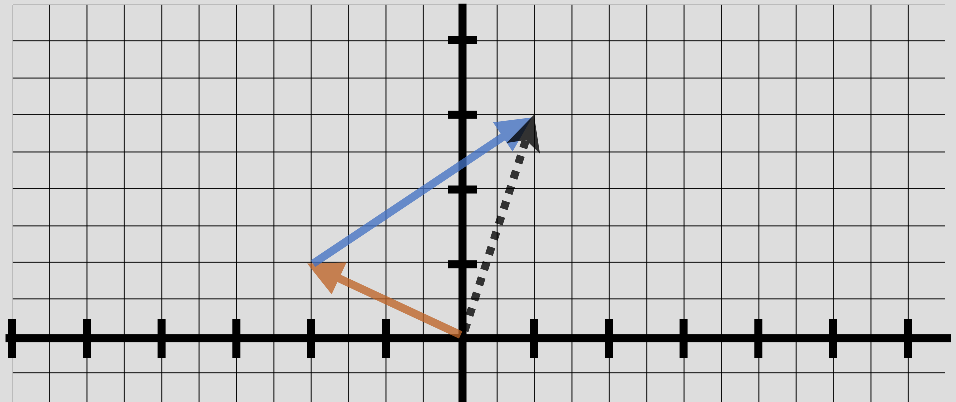
- Two vectors of the same length can be added
    - Addition is element-wise

$$\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \ \vec{y} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$\vec{x} + \vec{y} =$



$\vec{y} + \vec{x} =$



Properties of vector addition

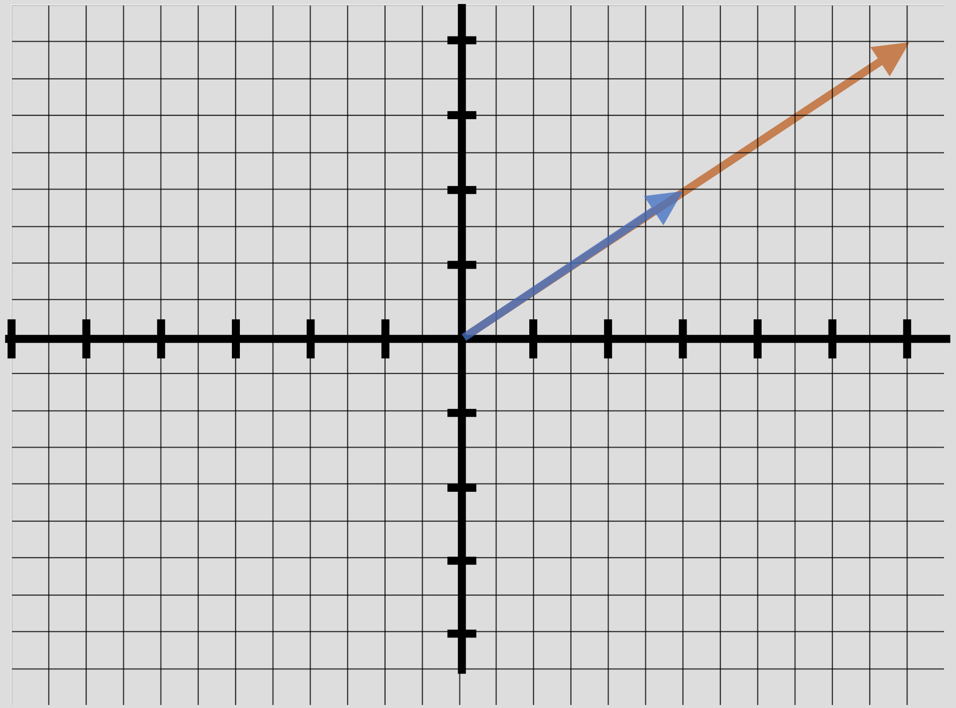
- Commutativity: $\overrightarrow{x} + \overrightarrow{y} = \overrightarrow{y} + \overrightarrow{x}$
- Associativity: $(\overrightarrow{x} + \overrightarrow{y}) + \vec{z} = \overrightarrow{x} + (\overrightarrow{y} + \vec{z})$
- Additive negative: $\overrightarrow{x} + (-\overrightarrow{x}) = \overrightarrow{0}$
- Additive identity: $\overrightarrow{x} + \overrightarrow{0} = \overrightarrow{x}$

Scalar Vector Multiplication

- Multiplying with a scalar result in multiplying each element.

$$a\vec{x} = \begin{bmatrix} ax_1 \\ ax_1 \\ \vdots \\ ax_{N \times 1} \end{bmatrix}$$

$$2 \cdot \vec{x} = 2 \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$



Vector Transpose

- $\overrightarrow{x}^T$ is the transpose of $\overrightarrow{x}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad \vec{x} \in \mathbb{R}^{N \times 1} \qquad \vec{x}^T = \begin{bmatrix} x_1 & x_1 & \cdots & x_N \end{bmatrix}, \quad \vec{x}^T \in \mathbb{R}^{1 \times N}$$

- $\overrightarrow{x}$ is always a column vector
- To represent a row vector, write: $\overrightarrow{x}^T$

Matrix Addition

- When matrices are the same size, they can be added element-wise

$$X + Y = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

- Scalar multiplication — by all elements

$$2X = 2\begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Matrix Addition

- When matrices are the same size, they can be added element-wise

$$X + Y = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

diagonal matrix!

- Scalar multiplication — by all elements

$$2X = 2\begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -2 & -4 \end{bmatrix}$$

Vector Transpose

- $\overrightarrow{x}^T$ is the transpose of $\overrightarrow{x}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad \vec{x} \in \mathbb{R}^{N \times 1}$$

$$\vec{x}^T = \begin{bmatrix} x_1 & x_1 & \cdots & x_N \end{bmatrix}, \quad \vec{x}^T \in \mathbb{R}^{1 \times N}$$

- $\overrightarrow{x}$ is always a column vector
- To represent a row vector, write: $\overrightarrow{x}^T$

Matrix Transpose

If the elements of the matrix $A \in \mathbb{R}^{N \times M}$ are a_{ij}

The elements of $A^T \in \mathbb{R}^{M \times N}$ are a_{ji}

Matrix transpose is not (generally) an inverse!

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_m \end{bmatrix}$$

$A \in \mathbb{R}^{N \times M}$

$$\begin{bmatrix} \vec{a}_1^{\,T} \\ \vec{a}_2^{\,T} \\ \vdots \\ a_m^T \end{bmatrix}$$

$A^T \in \mathbb{R}^{M \times N}$

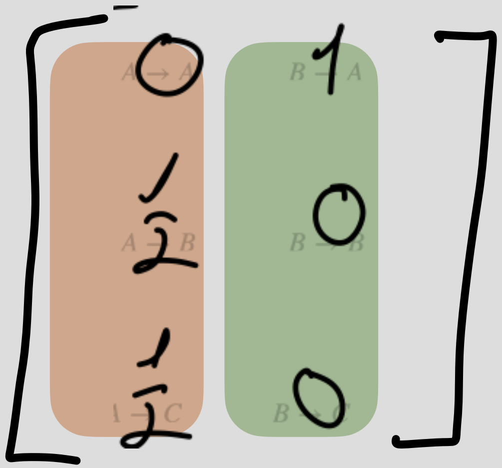
Matrix Transpose

If the elements of the matrix $A \in \mathbb{R}^{N \times M}$ are a_{ij}

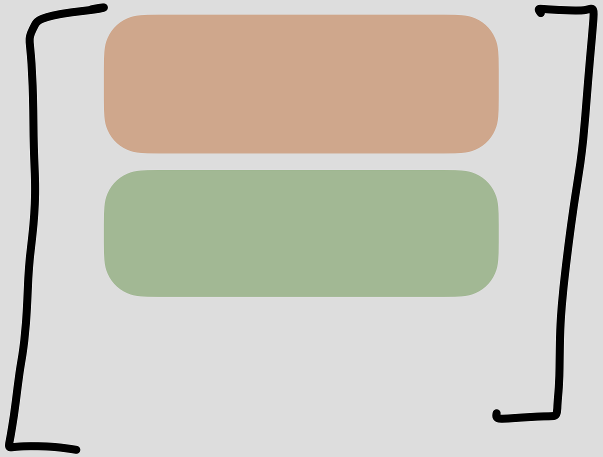
The elements of $A^T \in \mathbb{R}^{M \times N}$ are a_{ji}

Matrix transpose is not (generally) an inverse!

$A \in \mathbb{R}^{N \times M}$



$A^T \in \mathbb{R}^{M \times N}$



Matrix Transpose

If the elements of the matrix $A \in \mathbb{R}^{N \times M}$ are a_{ij}

The elements of $A^T \in \mathbb{R}^{M \times N}$ are a_{ji}

Matrix transpose is not (generally) an inverse!

$A \in \mathbb{R}^{N \times M}$

$$\begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$$

$A^T \in \mathbb{R}^{M \times N}$

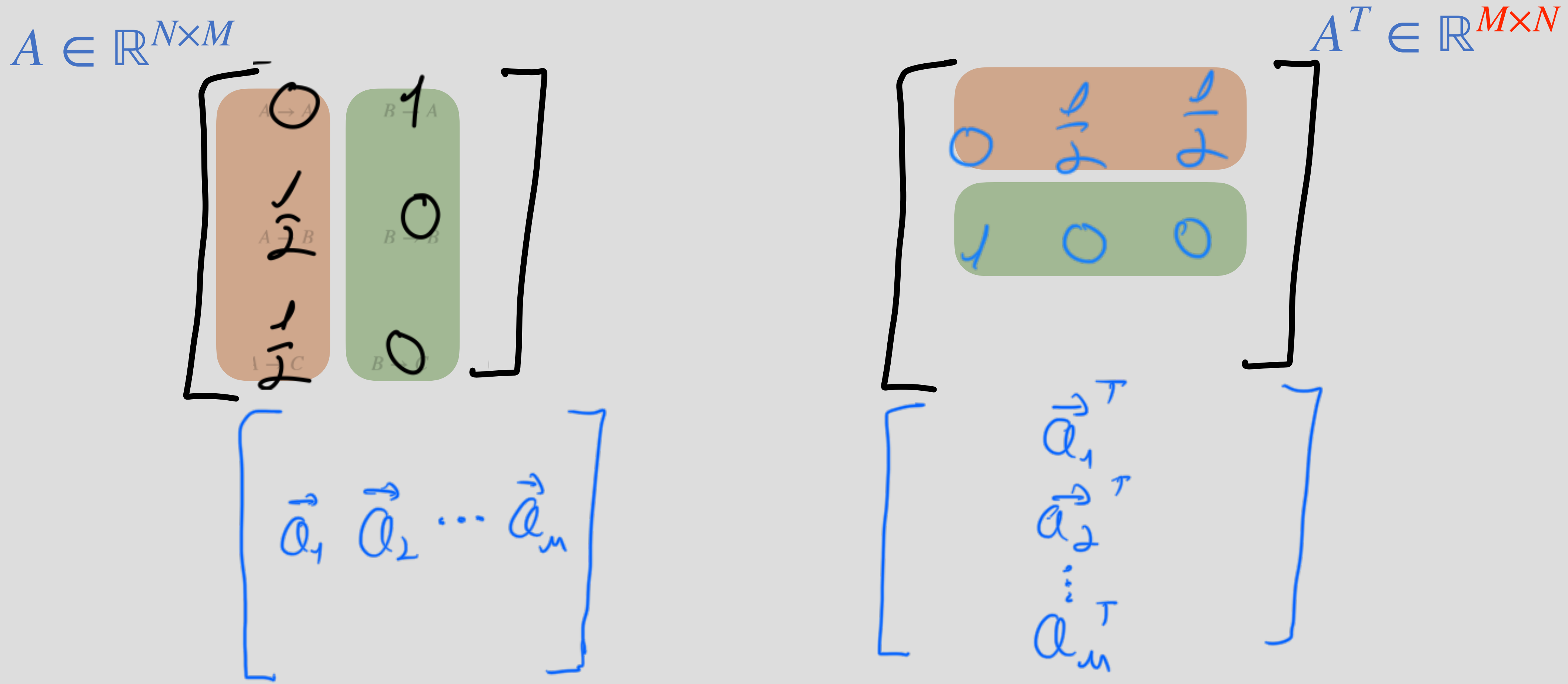
$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}$$

Matrix Transpose

If the elements of the matrix $A \in \mathbb{R}^{N \times M}$ are a_{ij}

The elements of $A^T \in \mathbb{R}^{M \times N}$ are a_{ji}

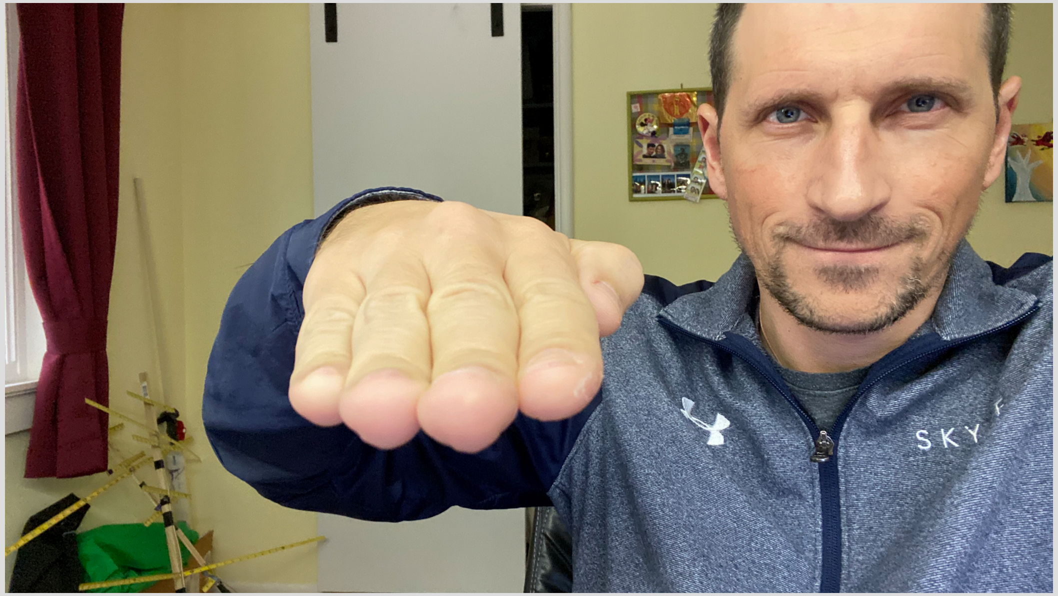
Matrix transpose is not (generally) an inverse!



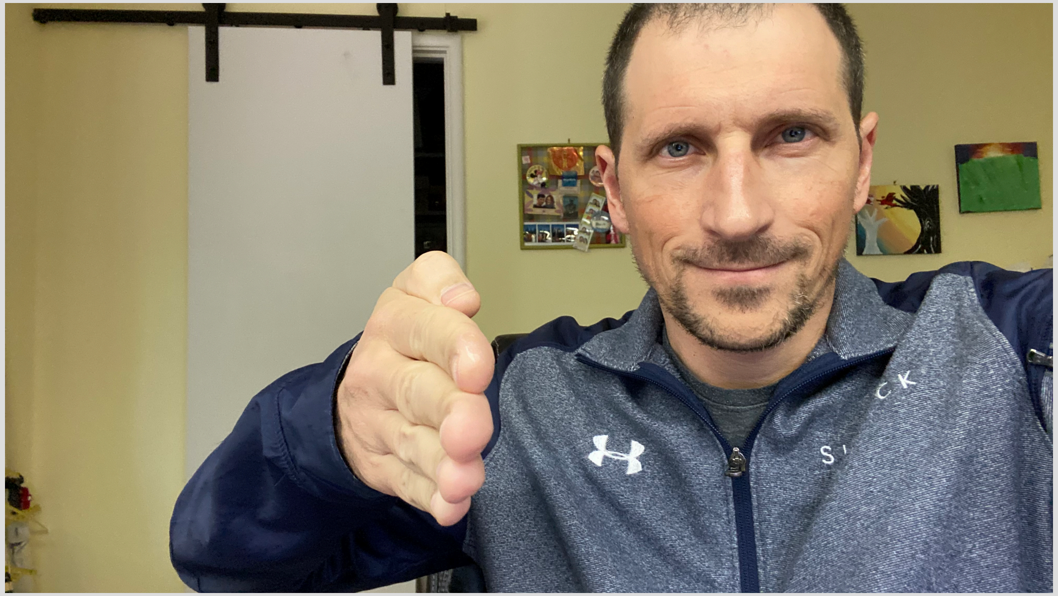
Vector-Vector Multiplication

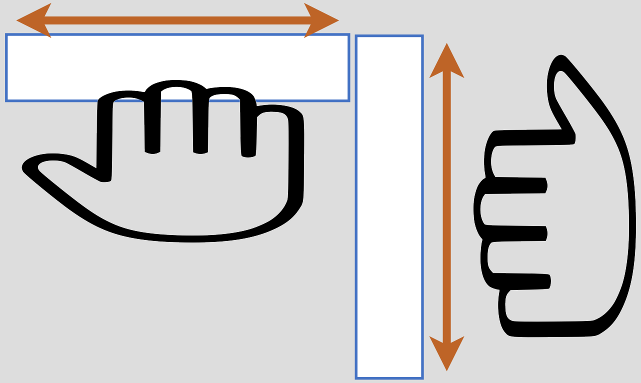
- Multiplication is valid only for specific matching dimensions!
  - Width of the 1st, matches length of the second

Like this....



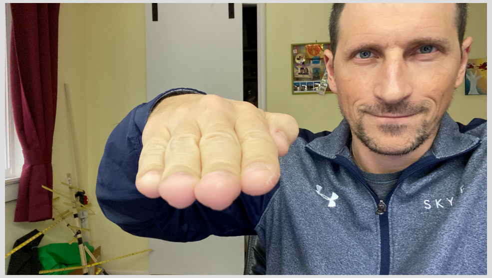
and like that!





Vector Vector Multiplication

Like this….

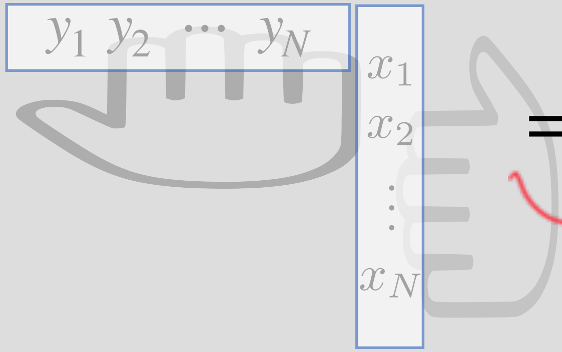


and like that!

$\vec{x}, \vec{y} \in \mathbb{R}^{N \times 1}$

$1 \times N$

$$\vec{y}^T \vec{x} = \begin{bmatrix} y_1 & y_2 & \cdots & y_N \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = y_1 x_1 + y_2 x_2 + y_3 x_3 + \cdots + y_N x_N = \square$$



$N \times 1$

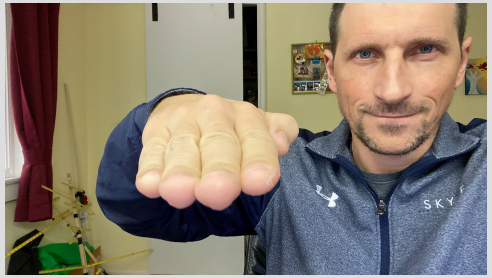
scalar 1×1

1×1

Also known as “inner product” or “dot product”

Matrix-Vector Multiplication

Like this....



and like that!

$\vec{x}, \vec{y} \in \mathbb{R}^{N\times 1}$

$1 \times N$

$$\vec{y}^T\vec{x} = \begin{bmatrix} y_1 & y_2 & \cdots & y_N \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = y_1x_1 + y_2x_2 + y_3x_3 + \cdots + y_Nx_N = \square$$

$N \times 1$

scalar 1×1

1×1

Also known as "inner product" or "dot product"

What about this case....

$A \in R^{M\times N}, \vec{x} \in \mathbb{R}^{N\times 1}$

N

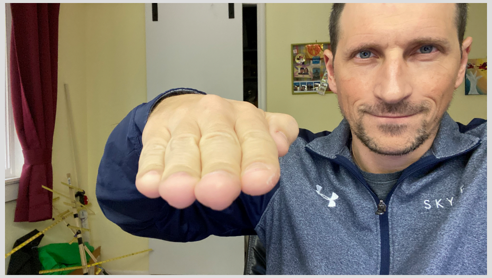
$$A\vec{x} = \underbrace{\begin{bmatrix} \\ \\ A \\ \\ \\ \end{bmatrix}}_{M} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = ?$$

M

$N \times 1$

Matrix-Vector Multiplication

Like this....



and like that!

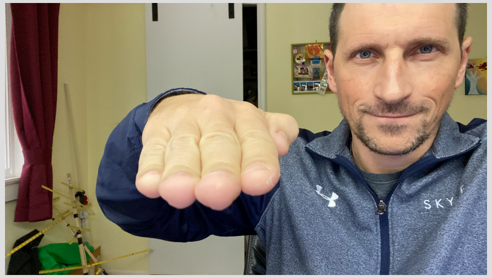
$A \in R^{M\times N}, \vec{x} \in \mathbb{R}^{N\times 1}$

$$A\vec{x} = \overbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ & \vdots & & \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix}}^{N}_{M} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}}_{N\times 1} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N \\ \\ \\ \end{bmatrix}$$

$$[a_{11}\ a_{12}\ \cdots\ a_{1N}]\vec{x} = \vec{y}_1^T\vec{x}$$

Matrix-Vector Multiplication

Like this....



and like that!

$A \in R^{M \times N}, \vec{x} \in \mathbb{R}^{N \times 1}$

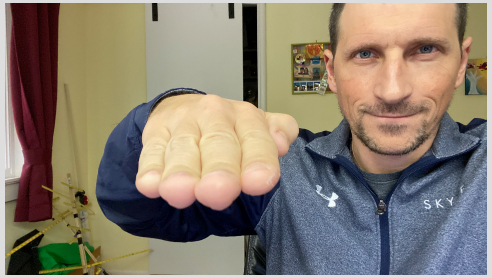
$$A\vec{x} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ & \bullet & & \\ & \bullet & & \\ & \bullet & & \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix}}_{M \times N} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}_{N \times 1} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N \\ \\ \\ \end{bmatrix}$$

$$[a_{11}\ a_{12}\ \cdots\ a_{1N}]\vec{x} = \vec{y}_1^T\vec{x}$$

$$[a_{21}\ a_{22}\ \cdots\ a_{2N}]\vec{x} = \vec{y}_2^T\vec{x}$$

Matrix-Vector Multiplication

Like this....



and like that!

$A \in R^{M\times N}, \vec{x} \in \mathbb{R}^{N\times 1}$

$$A\vec{x} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ & & \vdots & \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix}}_{M \times N} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}}_{N\times 1} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N \\ \vdots \\ a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N \end{bmatrix} = \underbrace{\begin{bmatrix} \\ \\ \\ \end{bmatrix}}_{M\times 1}$$

$$[a_{11}\ a_{12}\ \cdots\ a_{1N}]\vec{x} = \vec{y}_1^T\vec{x}$$

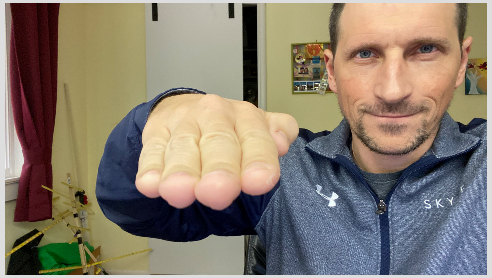
$$[a_{21}\ a_{22}\ \cdots\ a_{2N}]\vec{x} = \vec{y}_2^T\vec{x}$$

$$\vdots$$

$$[a_{M1}\ a_{M2}\ \cdots\ a_{MN}]\vec{x} = \vec{y}_M^T\vec{x}$$

Matrix-Vector Multiplication

Like this….



and like that!

$A \in R^{M\times N}, \vec{x} \in \mathbb{R}^{N\times 1}$

$$A\vec{x} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ & \vdots & & \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix}}_{M \times N} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}_{N\times 1} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N \\ \vdots \\ a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N \end{bmatrix} = \left[\ \right]_{M\times 1}$$

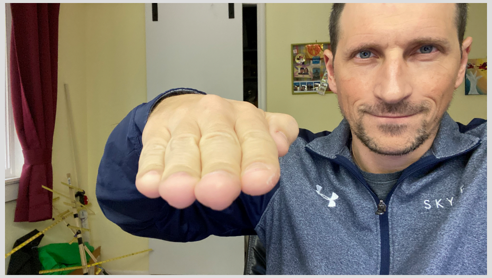
What about this case….

$A \in R^{M\times N}, B \in \mathbb{R}^{N\times L}$

$$AB = \underbrace{A}_{M\times N}\ \underbrace{B}_{N\times L} = ?$$

Matrix-Matrix Multiplication

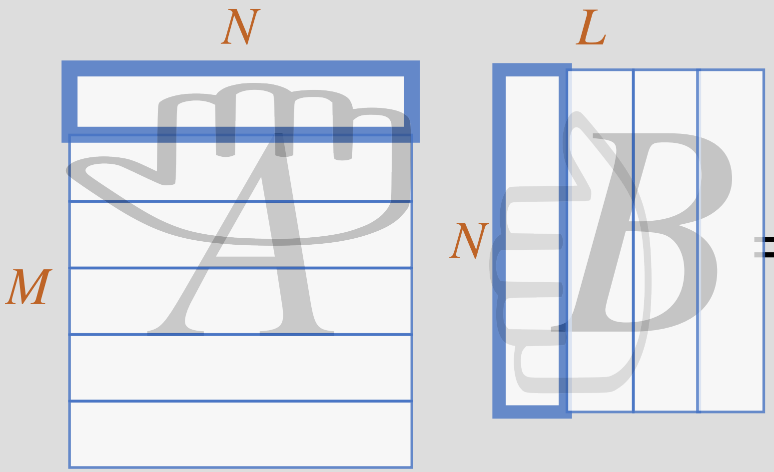
Like this....



and like that!

$A \in R^{M \times N}, B \in \mathbb{R}^{N \times L}$

$AB =$

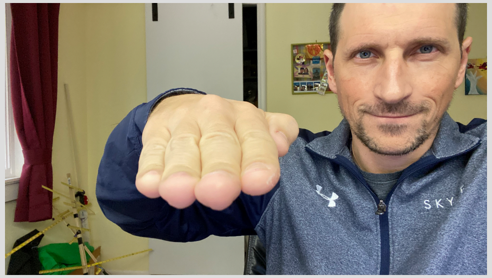


$= ?$

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1N}b_{N1} & & \\ & & \\ & & \end{bmatrix}$$

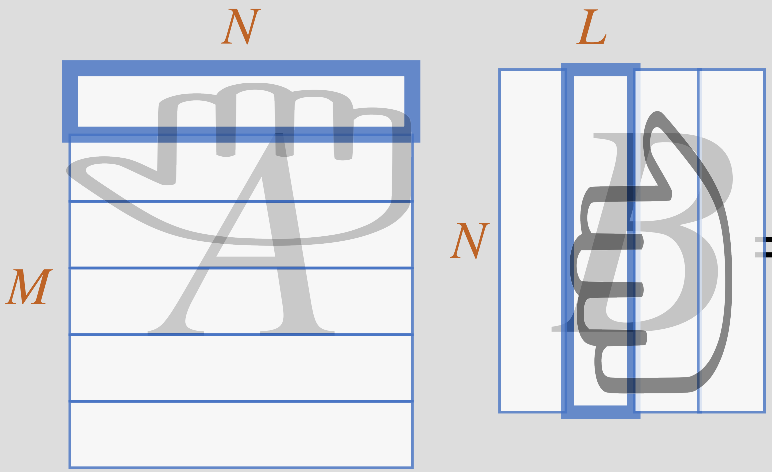
Matrix-Matrix Multiplication

Like this.... and like that!



$A \in R^{M\times N}, B \in \mathbb{R}^{N\times L}$

$AB =$

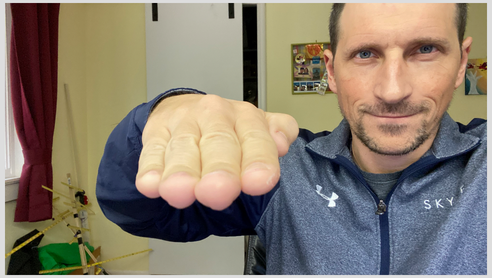


$= ?$

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1N}b_{N1} & a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1N}b_{N2} \end{bmatrix}$$

Matrix-Matrix Multiplication

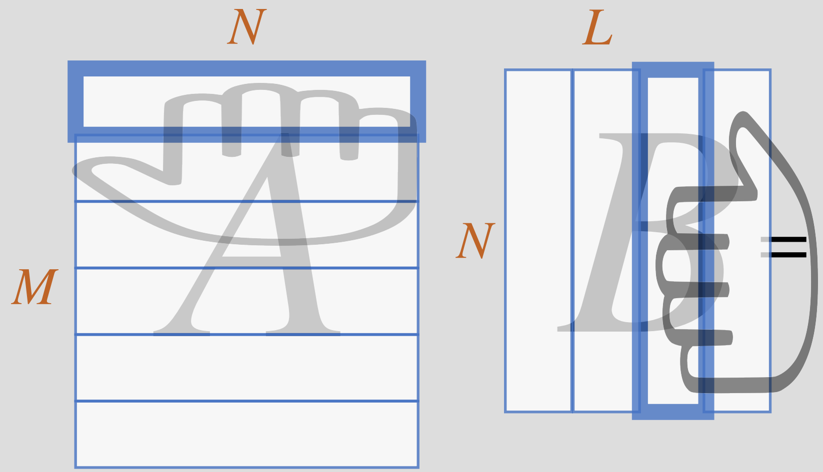
Like this....



and like that!

$A \in R^{M \times N}, B \in \mathbb{R}^{N \times L}$

$AB =$

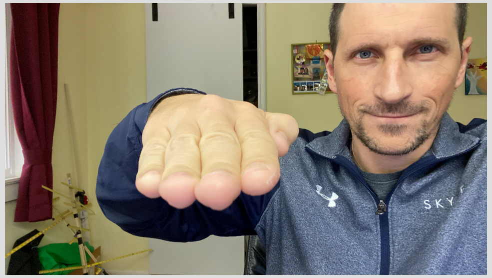


$= ?$

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1N}b_{N1} & a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1N}b_{N2} & \cdots \\ & & \end{bmatrix}$$

Matrix-Matrix Multiplication

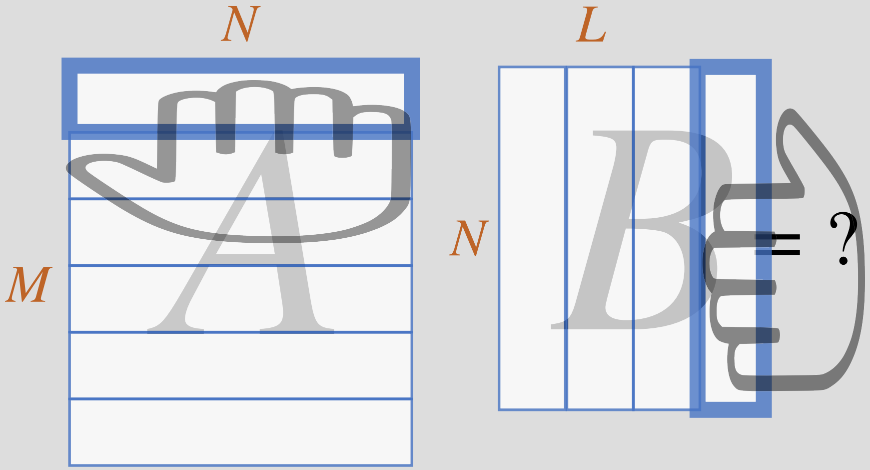
Like this....



and like that!

$A \in R^{M\times N}, B \in \mathbb{R}^{N\times L}$

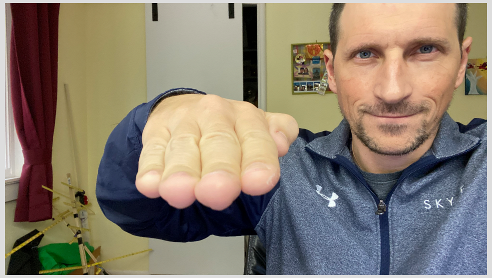
$AB =$



$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1N}b_{N1} & a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1N}b_{N2} & \cdots & a_{11}b_{1L} + a_{12}b_{2L} + \cdots + a_{1N}b_{NL} \\ & & & \\ & & & \end{bmatrix}$$

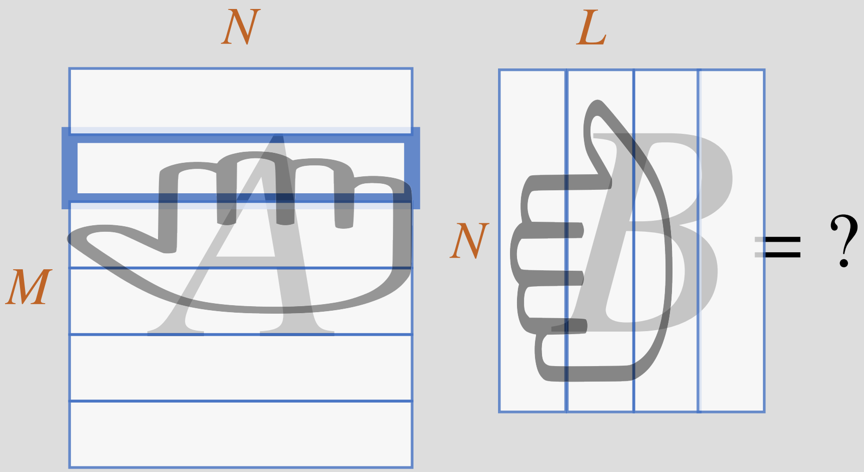
Matrix-Matrix Multiplication

Like this.... and like that!



$A \in R^{M\times N}, B \in \mathbb{R}^{N\times L}$

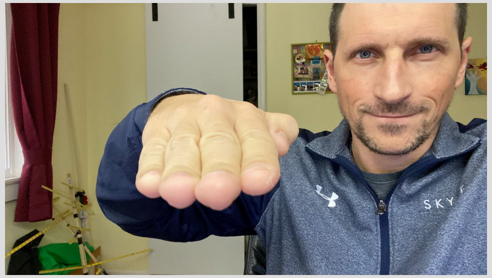
$AB =$



$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1N}b_{N1} & a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1N}b_{N2} & \cdots & a_{11}b_{1L} + a_{12}b_{2L} + \cdots + a_{1N}b_{NL} \\ a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2N}b_{N1} & a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2N}b_{N2} & \cdots & a_{21}b_{1L} + a_{22}b_{2L} + \cdots + a_{2N}b_{NL} \end{bmatrix}$$

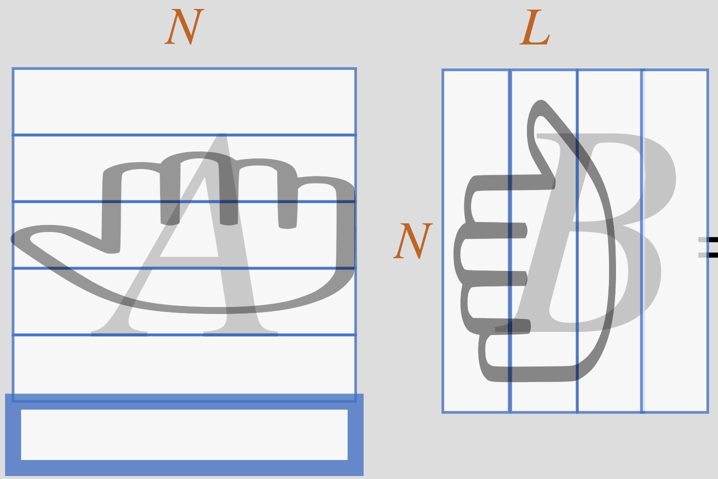
Matrix-Matrix Multiplication

Like this.... and like that!



$A \in R^{M\times N}, B \in \mathbb{R}^{N\times L}$

$AB =$

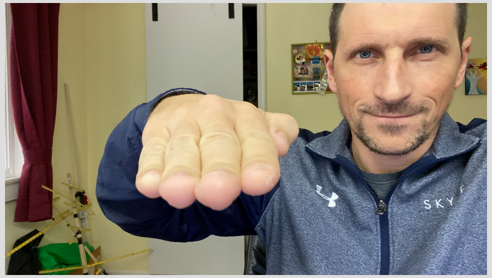


$$= ? \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1N}b_{N1} & a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1N}b_{N2} & \cdots & a_{11}b_{1L} + a_{12}b_{2L} + \cdots + a_{1N}b_{NL} \\ a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2N}b_{N1} & a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2N}b_{N2} & \cdots & a_{21}b_{1L} + a_{22}b_{2L} + \cdots + a_{2N}b_{NL} \\ \vdots & \vdots & & \vdots \\ a_{M1}b_{11} + a_{M2}b_{21} + \cdots + a_{MN}b_{N1} & a_{M1}b_{12} + a_{M2}b_{22} + \cdots + a_{MN}b_{N2} & & a_{M1}b_{1L} + a_{M2}b_{2L} + \cdots + a_{MN}b_{NL} \end{bmatrix}$$

Matrix-Matrix Multiplication

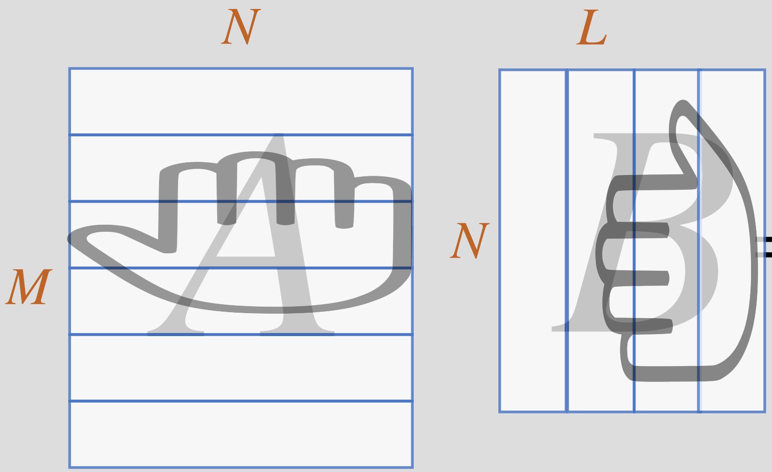
$A \in R^{M \times N}, B \in \mathbb{R}^{N \times L}$

Like this….



and like that!

$AB =$



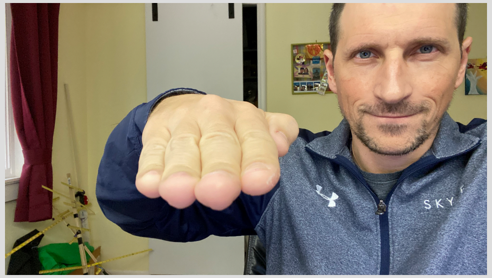
$= ?$

$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \cdots + a_{1N}b_{N1} & a_{11}b_{12} + a_{12}b_{22} + \cdots + a_{1N}b_{N2} & \cdots & a_{11}b_{1L} + a_{12}b_{2L} + \cdots + a_{1N}b_{NL} \\ a_{21}b_{11} + a_{22}b_{21} + \cdots + a_{2N}b_{N1} & a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2N}b_{N2} & \cdots & a_{21}b_{1L} + a_{22}b_{2L} + \cdots + a_{2N}b_{NL} \\ \vdots & \vdots & & \vdots \\ a_{M1}b_{11} + a_{M2}b_{21} + \cdots + a_{MN}b_{N1} & a_{M1}b_{12} + a_{M2}b_{22} + \cdots + a_{MN}b_{N2} & & a_{M1}b_{1L} + a_{M2}b_{2L} + \cdots + a_{MN}b_{NL} \end{bmatrix}$$

Result at location 2x2 = $a_{21}b_{12} + a_{22}b_{22} + \cdots + a_{2N}b_{N2}$

Matrix-Vector Multiplication

Like this....



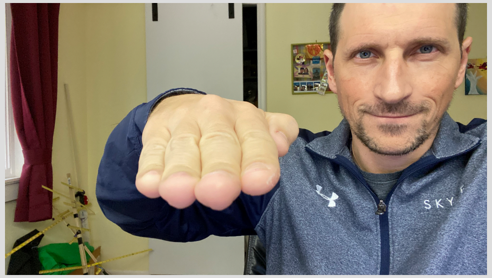
and like that!

$$A \quad B \quad = \quad C$$

$$M \times N \qquad N \times L \qquad M \times L$$

Vector Vector Multiplication

Like this....



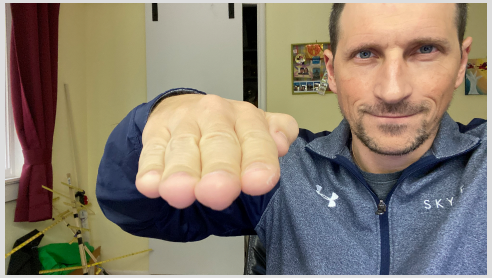
and like that!

$$\vec{x}, \vec{y} \in \mathbb{R}^{N\times 1}$$

$$\vec{x}\,\vec{y}^{T} = \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}}_{N\times 1} \overbrace{\begin{bmatrix} y_1 & y_2 & \cdots & y_N \end{bmatrix}}^{1\times N} = \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_N \\ x_2y_1 & x_2y_2 & \cdots & x_2y_N \\ \vdots & \vdots & \ldots & \vdots \\ x_Ny_1 & x_Ny_2 & \cdots & x_Ny_N \end{bmatrix} = \underbrace{\square}_{N\times N}$$

Vector Vector Multiplication

Like this....



and like that!

$\vec{x}, \vec{y} \in \mathbb{R}^{N\times 1}$

$\vec{y}^T\vec{x} =$

$= y_1x_1 + y_2x_2 + y_3x_3 + \cdots + y_Nx_N$

scalar 1×1

Also known as "inner product" or "dot product"

$1 \times N$

$\vec{x}\,\vec{y}^T =$

$N \times 1$

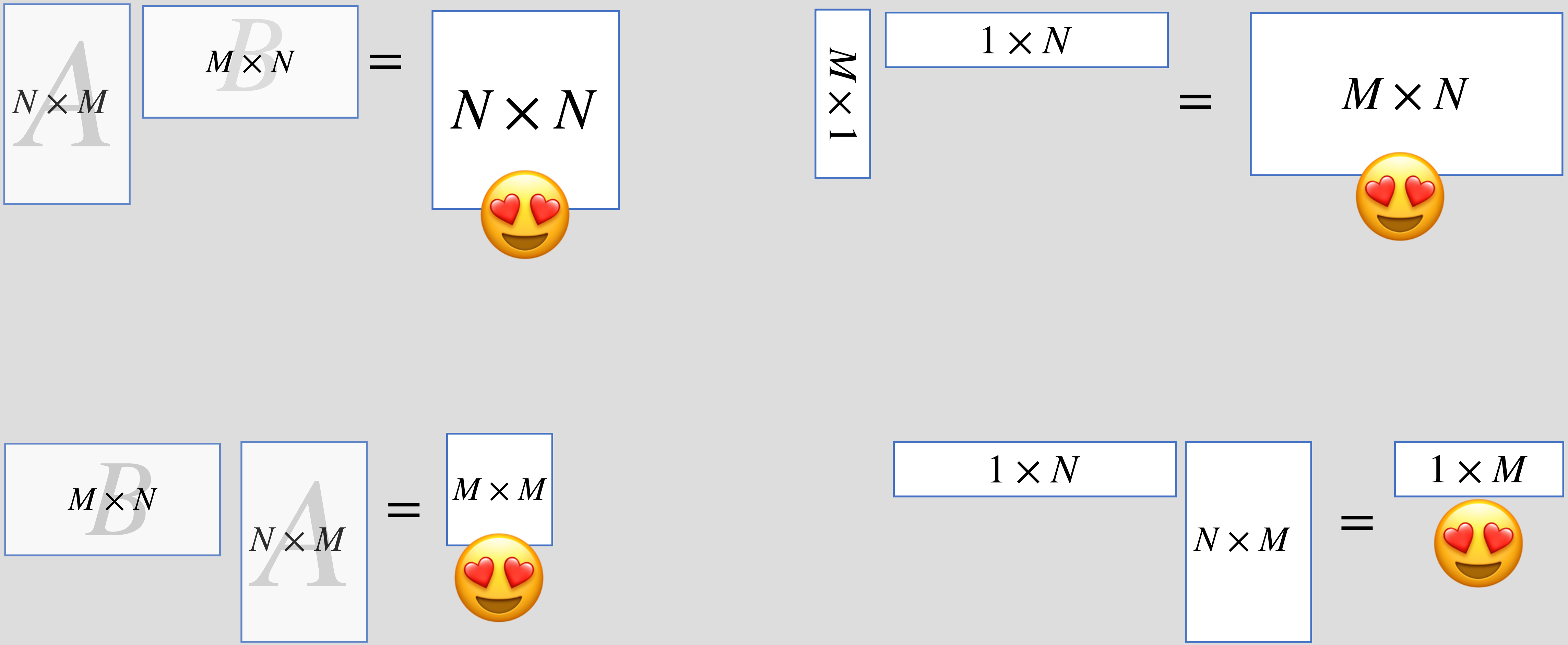
$$= \begin{bmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_N \\ x_2y_1 & x_2y_2 & \cdots & x_2y_N \\ \vdots & \vdots & \ldots & \vdots \\ x_Ny_1 & x_Ny_2 & \cdots & x_Ny_N \end{bmatrix}$$

$N \times N$

Do not commute!

Also known as "outer product"

Matrix Matrix Multiplication



Matrix multiplication does not commute!

Matrix-Vector Form of Systems of Linear Equations

- Consider the matrix equation: $A\vec{x} = \vec{b}$

$$A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ & & \vdots & \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N \\ \vdots \\ a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{bmatrix}$$

$M \times N$ $\quad N \times 1$ $\quad M \times 1$

Same as the Augmented Matrix!

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1N} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2N} & b_2 \\ \vdots & & \vdots & & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} & b_M \end{array}\right]$$

$A\vec{x} = \vec{b}$ is another way to write
A linear set of equations!

Row vs Column Perspective

- Row / Measurement Perspective of $A\overrightarrow{x} = \overrightarrow{b}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} \quad \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

Row vs Column Perspective

- Row / Measurement Perspective of $A\overrightarrow{x} = \overrightarrow{b}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Q: What does a row mean?

A: How each variable affect a particular measurement

Row vs Column Perspective

- Column Perspective of $A\overrightarrow{x} = \overrightarrow{b}$

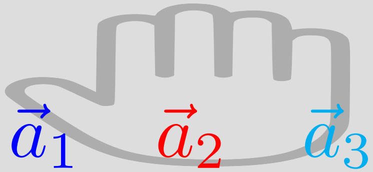
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$

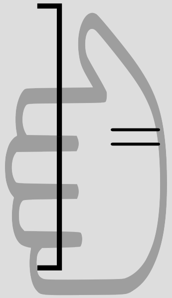
Row vs Column Perspective

- Column Perspective of $A\overrightarrow{x} = \overrightarrow{b}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$



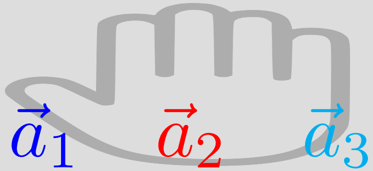
$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$



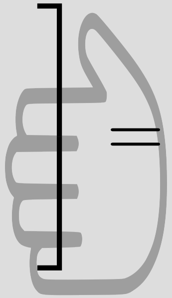
Row vs Column Perspective

- Column Perspective of $A\overrightarrow{x} = \overrightarrow{b}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$



$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \quad x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 =$$



Row vs Column Perspective

- Column Perspective of $A\overrightarrow{x} = \overrightarrow{b}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 =$$

$$= \begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ a_{22}x_2 \end{bmatrix} + \begin{bmatrix} a_{13}x_3 \\ a_{23}x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Q: What does a column mean?

A: How a particular variable affects all measurements.

Linear combination of vectors

- Given set of vectors $\{\overrightarrow{a}_1, \overrightarrow{a}_2, \cdots, \overrightarrow{a}_M\} \in \mathbb{R}^N$, and coefficients $\{\alpha_1, \alpha_2, \cdots, \alpha_M\} \in \mathbb{R}$
- A linear combination of vectors is defined as: $\overrightarrow{b} \triangleq \alpha_1 \overrightarrow{a}_1 + \alpha_2 \overrightarrow{a}_2 + \cdots + \alpha_M \overrightarrow{a}_M$

Recall: $A\overrightarrow{x}$:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3$$

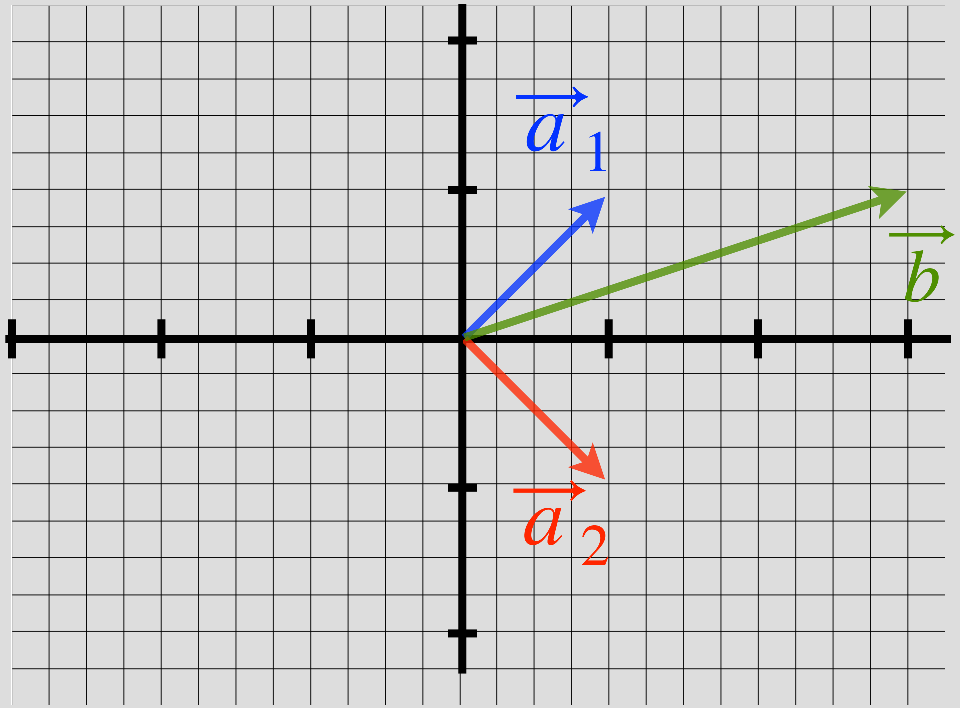
Matrix-vector multiplication is a linear combination of the columns of A!

Linear Set of Equations as a Linear Combination

- Consider the problem: $A\overrightarrow{x} = \overrightarrow{b}$:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$\overrightarrow{a}_1$ $\overrightarrow{a}_2$ $\overrightarrow{b}$



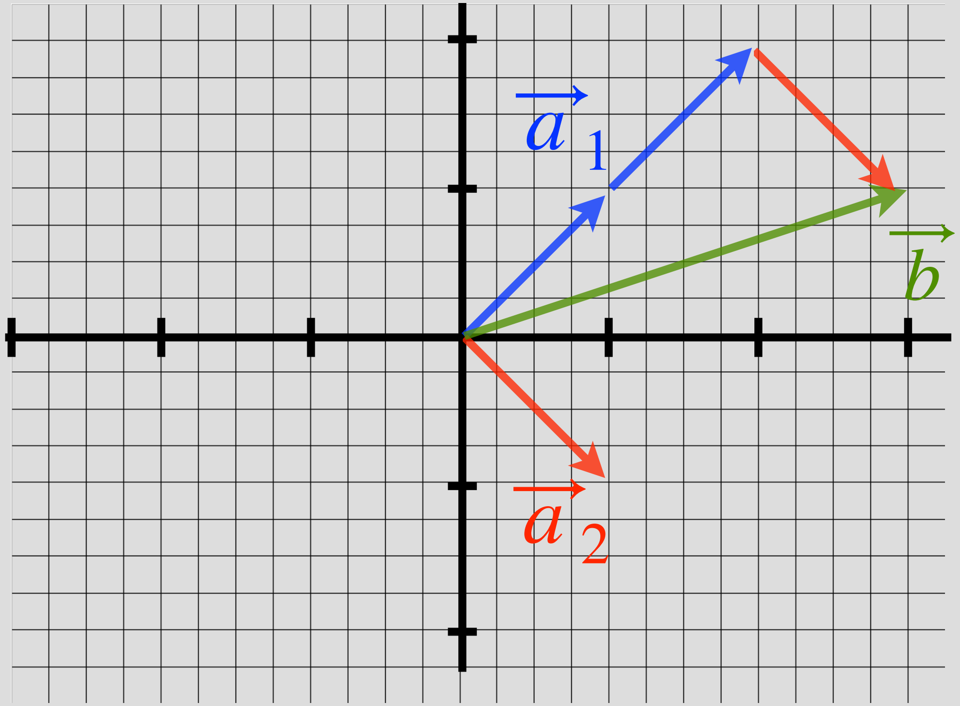
Linear Set of Equations as a Linear Combination

- Consider the problem: $A\overrightarrow{x} = \overrightarrow{b}$:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$\overrightarrow{a}_1$ $\quad$ $\overrightarrow{a}_2$ $\quad$ $\overrightarrow{b}$

Q: What linear combination of $\overrightarrow{a}_1, \overrightarrow{a}_2$ will give $\overrightarrow{b}$?



Linear Set of Equations as a Linear Combination

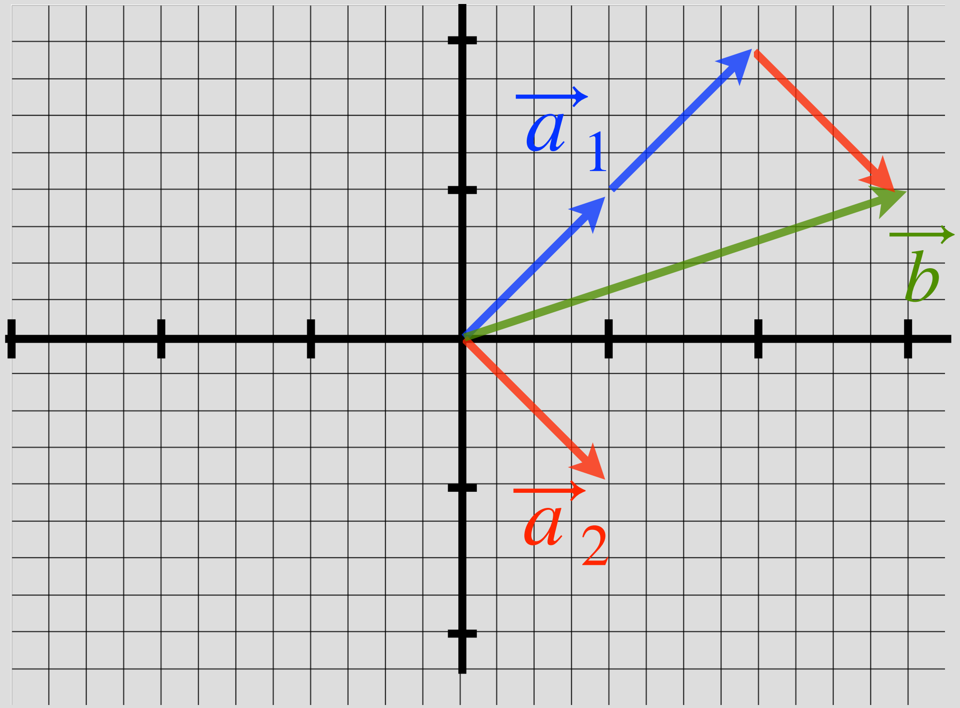
- Consider the problem: $A\overrightarrow{x} = \overrightarrow{b}$:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$\overrightarrow{a}_1$ $\overrightarrow{a}_2$ $\overrightarrow{b}$

Q: What linear combination of $\overrightarrow{a}_1, \overrightarrow{a}_2$ will give $\overrightarrow{b}$?

A: $2\overrightarrow{a}_1 + 1\overrightarrow{a}_2$

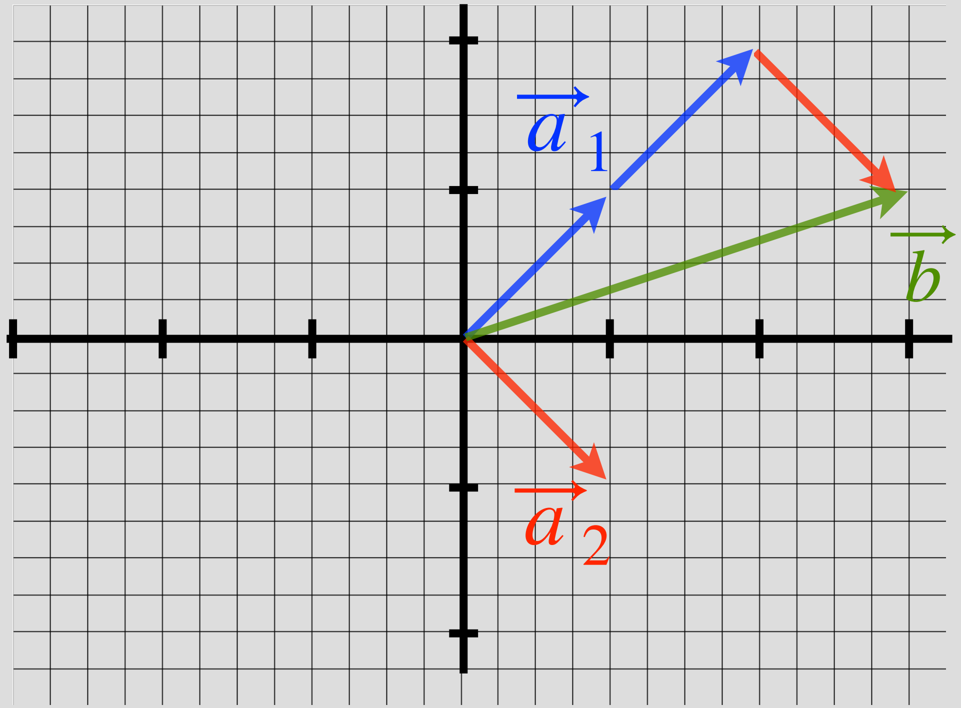


Linear Set of Equations as a Linear Combination

- Consider the problem: $A\vec{x} = \vec{b}$:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$\vec{a}_1$ $\quad$ $\vec{a}_2$ $\quad$ $\vec{b}$



Q: What linear combination of $\vec{a}_1, \vec{a}_2$ will give $\vec{b}$?

A: $2\vec{a}_1 + 1\vec{a}_2$

Gaussian Elimination:

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -1 & 1 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -2 & -2 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 1 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right]$$

$x_1 = 2$

$x_2 = 1$

$\Rightarrow \vec{b} = 2\vec{a}_1 + 1 \cdot \vec{a}_2$

same as

!

Linear Set of Equations as a Linear Combination

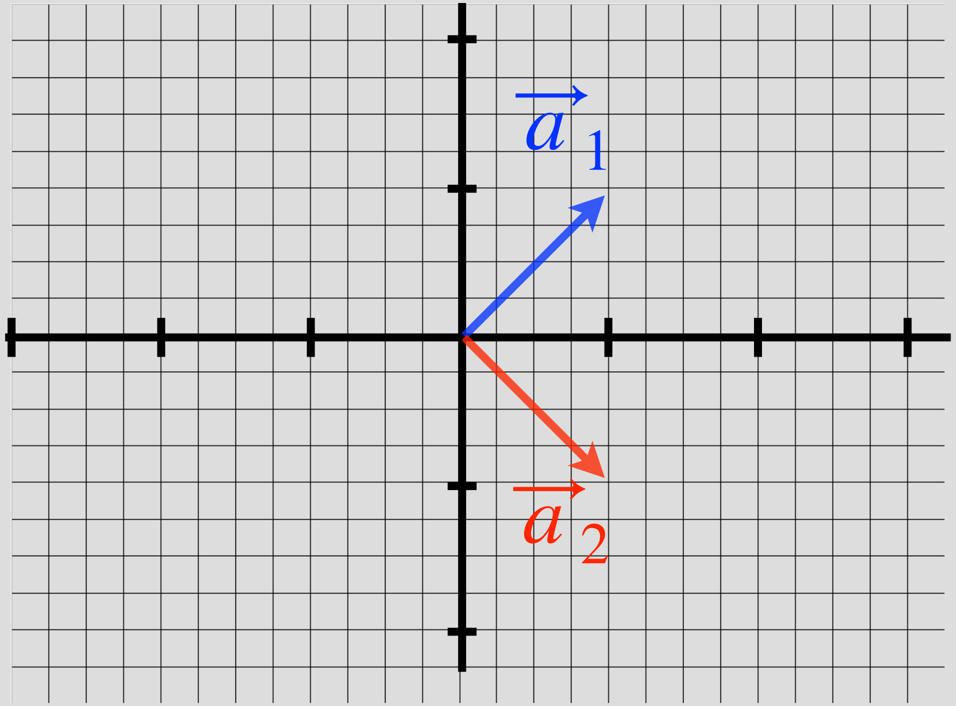
- Consider the problem: $A\overrightarrow{x} = \overrightarrow{b}$:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{b}$$

$\overrightarrow{a}_1$ $\overrightarrow{a}_2$

Q: Can linear combination of $\overrightarrow{a}_1, \overrightarrow{a}_2$ give any $\overrightarrow{b}$?

A: Hmmm….I think so….



Linear Set of Equations as a Linear Combination

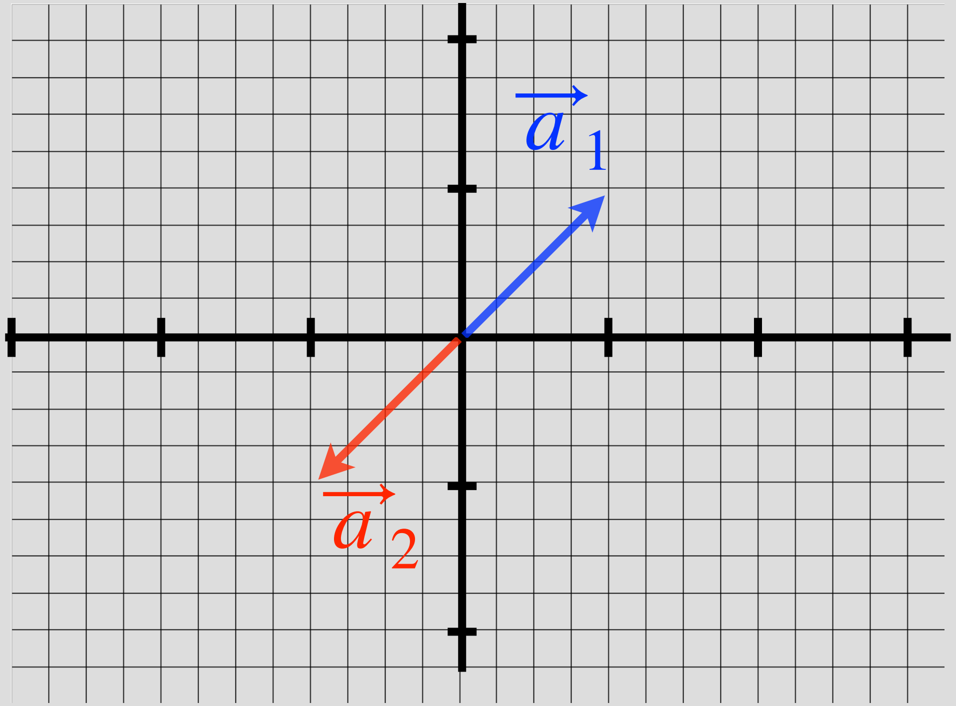
- Consider the problem: $A\overrightarrow{x} = \overrightarrow{b}$:

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{b}$$

$\overrightarrow{a}_1$ $\overrightarrow{a}_2$

Q: Can linear combination of $\overrightarrow{a}_1, \overrightarrow{a}_2$ give any $\overrightarrow{b}$?

A: Hmmm….I don’t think so…. Unless its along the line $\overrightarrow{a}_1$

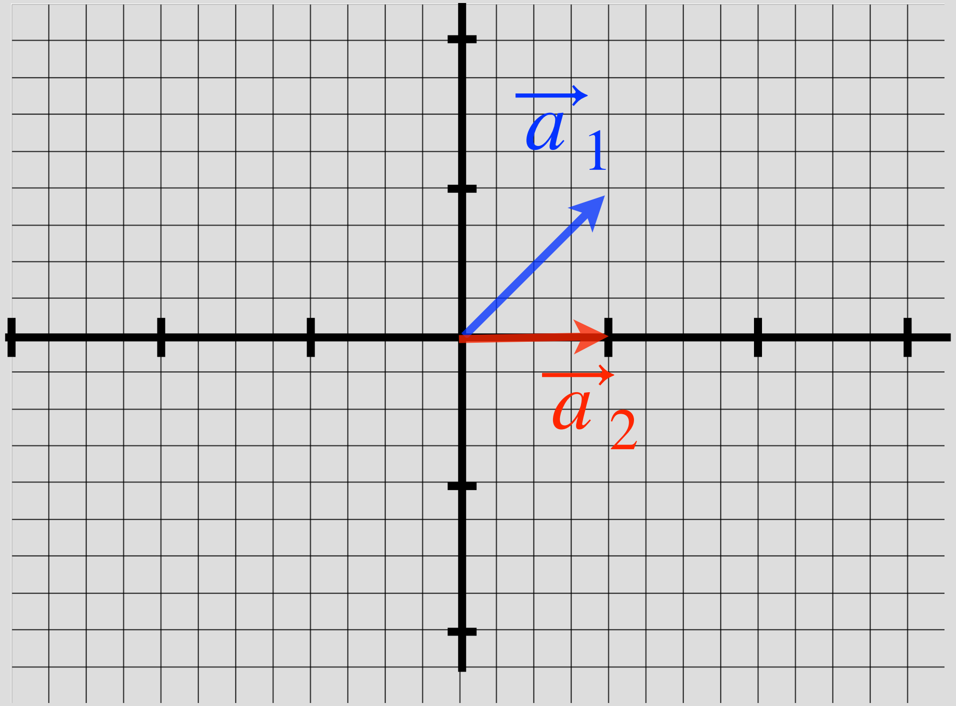


Linear Set of Equations as a Linear Combination

- Consider the problem: $A\overrightarrow{x} = \overrightarrow{b}$:

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{b}$$

$\overrightarrow{a}_1$ $\overrightarrow{a}_2$

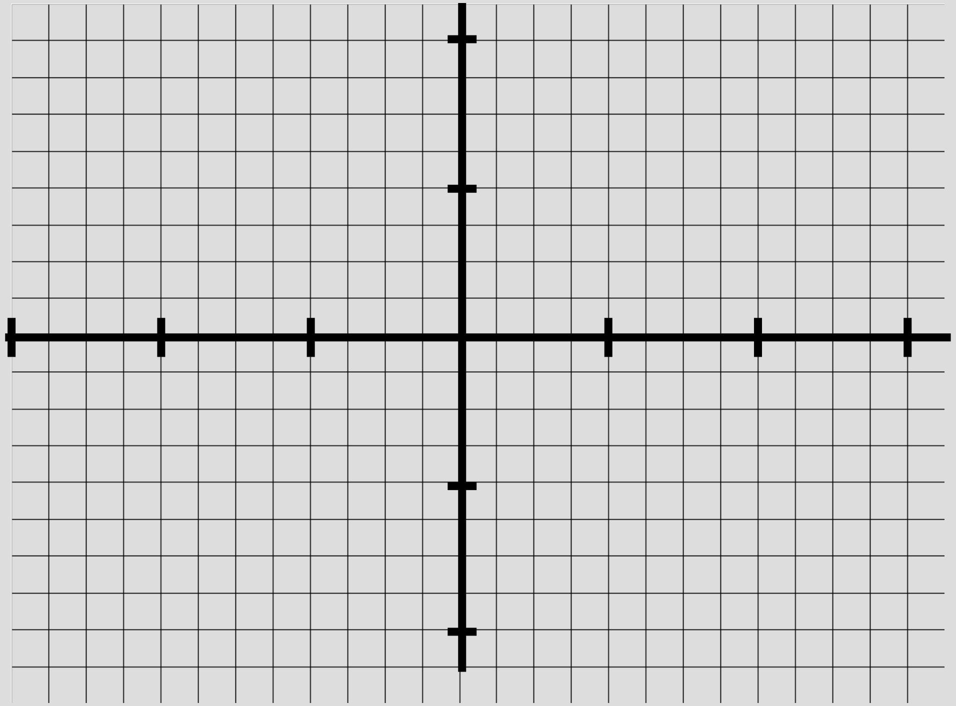


Q: Can linear combination of $\overrightarrow{a}_1, \overrightarrow{a}_2$ give any $\overrightarrow{b}$?

A: Hmmm….yes!

Span / Column Space / Range

- Span of the columns of A is the set of all vectors $\overrightarrow{b}$ such that $A\overrightarrow{x} = \overrightarrow{b}$ has a solution
  - i.e. the set of all vectors that can be reached by all possible linear combinations of the columns of A

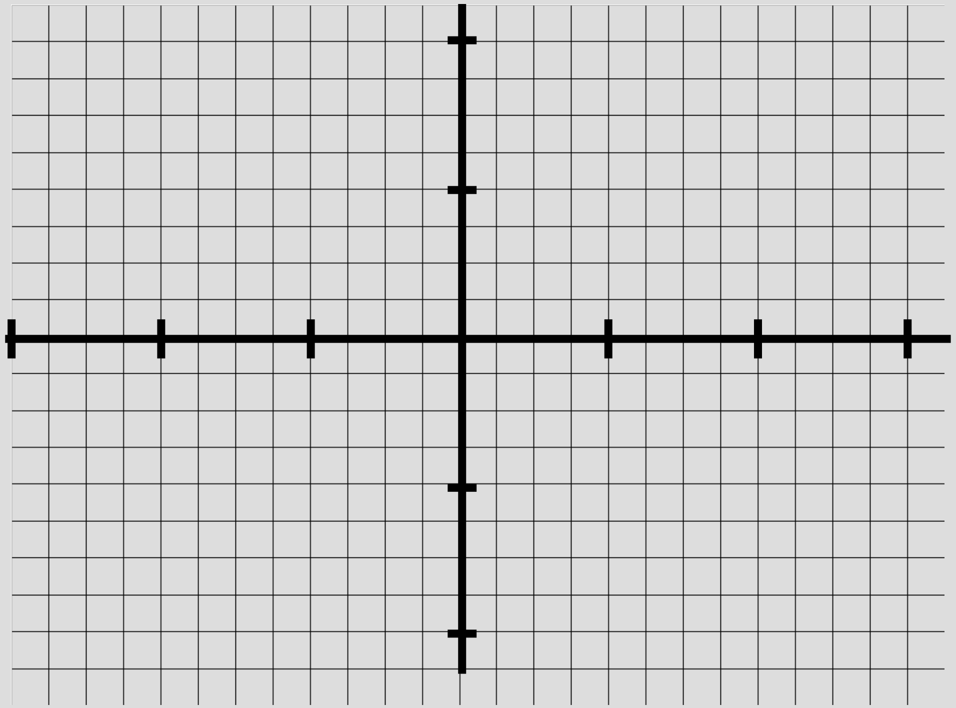


Span / Column Space / Range

- Span of the columns of A is the set of all vectors $\overrightarrow{b}$ such that $A\overrightarrow{x} = \overrightarrow{b}$ has a solution
  - i.e. the set of all vectors that can be reached by all possible linear combinations of the columns of A

Example: What is the span of the cols of A?

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$



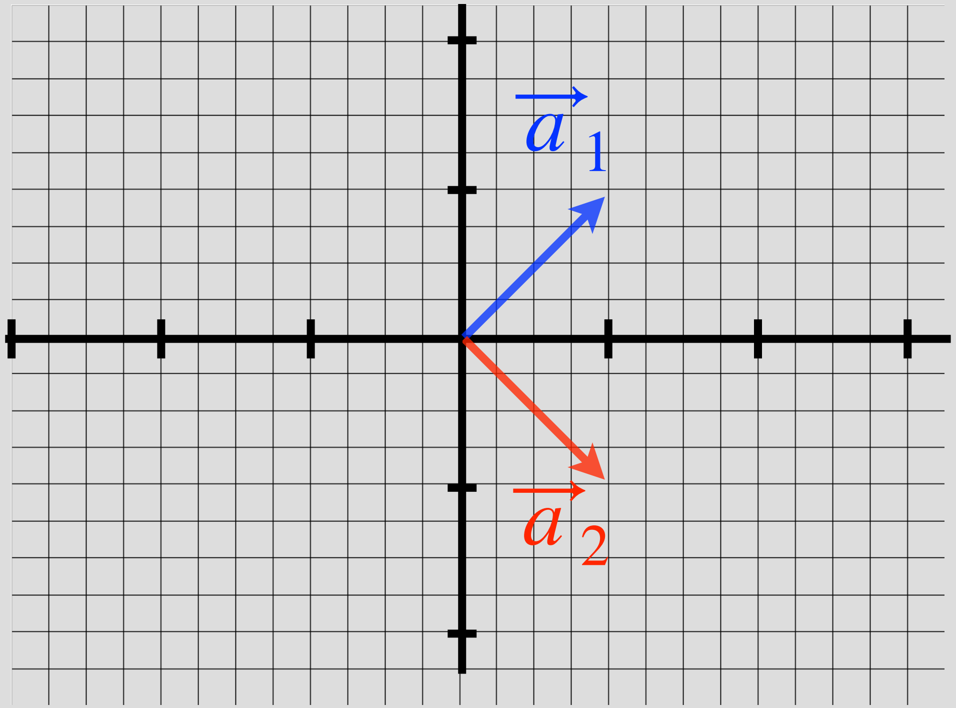
Span / Column Space / Range

- Span of the columns of A is the set of all vectors $\vec{b}$ such that $A\vec{x} = \vec{b}$ has a solution
  - i.e. the set of all vectors that can be reached by all possible linear combinations of the columns of A

Example: What is the span of the cols of A?

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

A: $\mathbb{R}^2$!



Span / Column Space / Range

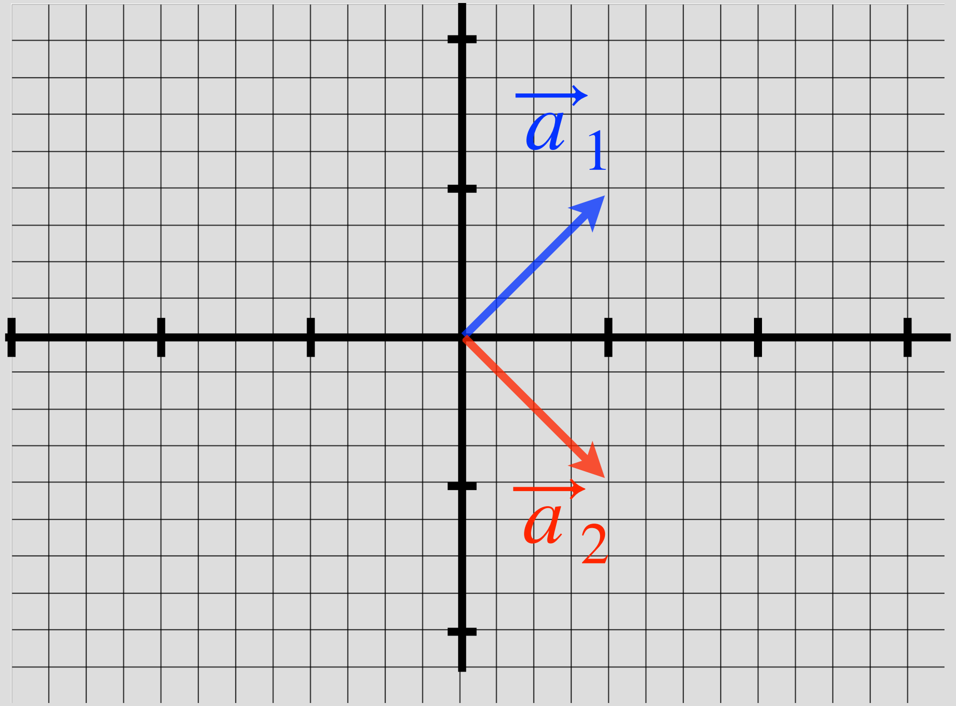
- Span of the columns of A is the set of all vectors $\overrightarrow{b}$ such that $A\overrightarrow{x} = \overrightarrow{b}$ has a solution
  - i.e. the set of all vectors that can be reached by all possible linear combinations of the columns of A

Example: What is the span of the cols of A?

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

A: $\mathbb{R}^2$!

$$\text{span}(\text{cols of } A) = \left\{ \vec{v} \middle| \; \vec{v} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \alpha, \beta \in \mathbb{R} \right\}$$



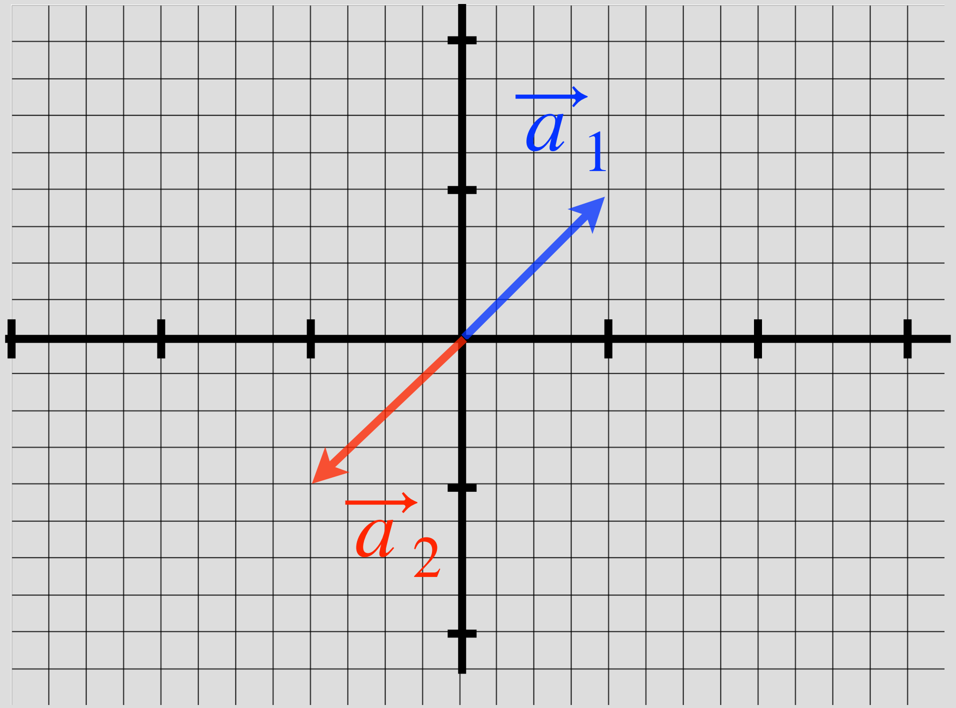
Span / Column Space / Range

Example 2: What is the span of the cols of A?

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

A: The line $x_1 = x_2$

$$\text{span(cols of } A) = \left\{ \vec{v} \middle| \ \vec{v} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \alpha \in \mathbb{R} \right\}$$



Span / Column Space / Range

- Definition:

  If $\exists \overrightarrow{x}$ s.t. $A\overrightarrow{x} = \overrightarrow{b}$ then $\overrightarrow{b} \in \text{span}\{A\}$

Converse: $\overrightarrow{b} \in \text{span}\{\text{cols}(A)\}$, there is a solution for $A\overrightarrow{x} = \overrightarrow{b}$

Q: What if $\overrightarrow{b} \notin \text{span}\{\text{cols}(A)\}$?

A: There is no solution for $A\overrightarrow{x} = \overrightarrow{b}$