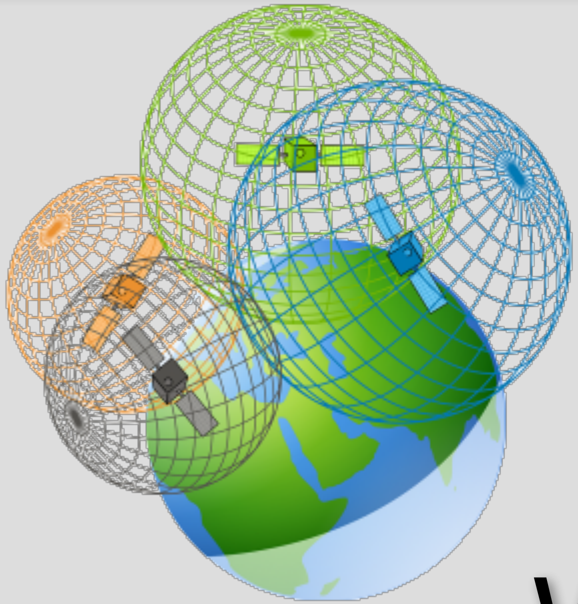
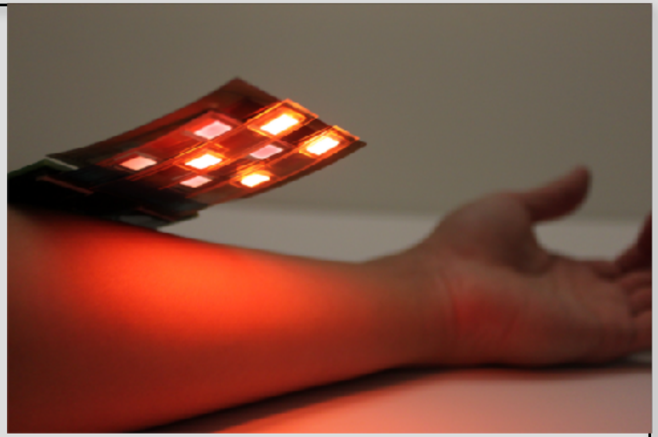




Welcome to EECS 16A!

Designing Information Devices and Systems I

Ana Arias and Miki Lustig

Fa 2022

Lecture 3A

Matrix xForms

Announcements

- Last time:
  - Proofs
  - Span
- Today:
  - Linear (in)dependance
  - Matrix Transformations

Span / Column Space / Range

- Span of the columns of A is the set of all vectors $\overrightarrow{b}$ such that $A\overrightarrow{x} = \overrightarrow{b}$ has a solution
  - i.e. the set of all vectors that can be reached by all possible linear combinations of the columns of A

- Definition:

$$\text{If } \exists \overrightarrow{x} \text{ s.t. } A\overrightarrow{x} = \overrightarrow{b} \text{ then } \overrightarrow{b} \in \text{span}\{\text{cols}(A)\}$$

Proof: Span

Theorem: $\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\} = \mathbb{R}^2$

Know:

$$\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\} \Rightarrow \left\{\vec{v} \,\middle|\, \vec{v} = \alpha\begin{bmatrix}1\\1\end{bmatrix} + \beta\begin{bmatrix}1\\-1\end{bmatrix} \quad, \alpha, \beta \in \mathbb{R}\right\} = \mathbb{S}$$

Need to show:

$$\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\} = \mathbb{R}^2$$

Concept: pick some specific $\vec{b} = \begin{bmatrix}b_1\\b_2\end{bmatrix} \in R^2$, and show that it belongs to $\mathbb{S}$

Need to solve:

Proof: Span

Theorem: $\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\} = \mathbb{R}^2$

Know:

$$\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\} \Rightarrow \left\{\vec{v} \,\middle|\, \vec{v} = \alpha\begin{bmatrix}1\\1\end{bmatrix} + \beta\begin{bmatrix}1\\-1\end{bmatrix} \quad, \alpha, \beta \in \mathbb{R}\right\} = \mathbb{S}$$

Need to show:

$$\text{span}\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix}\right\} = \mathbb{R}^2$$

Concept: pick some specific $\vec{b} = \begin{bmatrix}b_1\\b_2\end{bmatrix} \in R^2$, and show that it belongs to $\mathbb{S}$

Need to solve:

$$\underbrace{\alpha}_{\text{unknown}}\begin{bmatrix}1\\1\end{bmatrix} + \underbrace{\beta}_{\text{unknown}}\begin{bmatrix}1\\-1\end{bmatrix} = \underbrace{\begin{bmatrix}b_1\\b_2\end{bmatrix}}_{\text{Known and } \in \mathbb{R}^2} \qquad \Rightarrow \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}\alpha\\\beta\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$$

Proof: Span

Need to solve:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Gaussian Elimination:

Proof: Span

Need to solve:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\frac{b_1+b_2}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{b_1-b_2}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Constructive proof

Gaussian Elimination:

$$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 1 & -1 & b_2 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & -2 & b_2-b_1 \end{array}\right] \quad \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 1 & \frac{b_1-b_2}{2} \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & \frac{b_1+b_2}{2} \\ 0 & 1 & \frac{b_1-b_2}{2} \end{array}\right] \Rightarrow \alpha = \frac{b_1+b_2}{2}, \beta = \frac{b_1-b_2}{2},$$

Every $\overrightarrow{b} \in \mathbb{R}^2$ can be written as linear combinations!

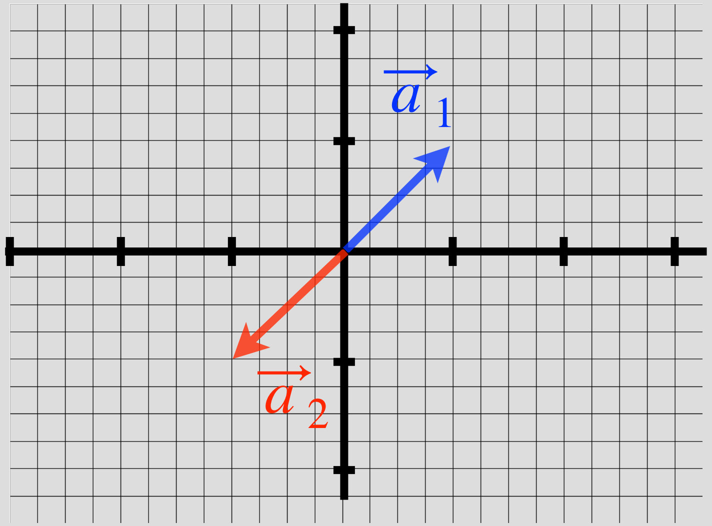
So also, $\overrightarrow{b} \in \mathbb{S}$

Linear Dependence

Recall:

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$\vec{a}_1 \quad \vec{a}_2$$



$\vec{a}_1$ and $\vec{a}_2$ are linearly dependent

$$\vec{a}_1 = -\vec{a}_2$$

Linear Dependence

- Definition 1:
  A set of vectors $\{\overrightarrow{a}_1, \overrightarrow{a}_2, \cdots, \overrightarrow{a}_N\}$ are linearly dependent if $\exists \{\alpha_1, \alpha_2, \cdots, \alpha_N\} \in \mathbb{R}$, such that:

$$\overrightarrow{a}_i = \sum_{j \neq i} \alpha_j \overrightarrow{a}_j \qquad 1 \leq i, j \leq M$$

$$\Downarrow$$

$\overrightarrow{a}_i$ in the span of all $\overrightarrow{a}_j$s

For example: if $\overrightarrow{a}_2 = 3\overrightarrow{a}_1 - 2\overrightarrow{a}_5 + 6\overrightarrow{a}_7$

Linear Dependence

Are these linearly dependent?

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$$

Need to solve:

Linear Dependence

Are these linearly dependent?

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$$

Are linearly dependent

Need to solve:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

but we showed that....

$$\frac{b_1+b_2}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{b_1-b_2}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

So....

$$\frac{3+1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{3-1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Linear dependence / independence

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\} \quad \Rightarrow 2\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 0$$

- Definition 2:
  A set of vectors $\left\{\overrightarrow{a}_1, \overrightarrow{a}_2, \cdots, \overrightarrow{a}_N\right\}$ are linearly dependent if $\exists \left\{\alpha_1, \alpha_2, \cdots, \alpha_N\right\} \in \mathbb{R}$, such that:

$$\sum_{i=1}^{N} \alpha_i \overrightarrow{a}_i = 0$$

As long as not all $a_i = 0$

- Definition:
  A set of vectors $\left\{\overrightarrow{a}_1, \overrightarrow{a}_2, \cdots, \overrightarrow{a}_N\right\}$ are linearly independent if they are not dependent

Linear dependence / independence

Are these linearly dependent?

$$\underbrace{\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\text{span} = \mathbb{R}^2}, \begin{bmatrix} \pi \\ \sqrt{2} \end{bmatrix} \right\}$$

linearly dependent!

$\in \mathbb{R}^2$

Solutions for linear equations

- Theorem: if the columns of the matrix A are linearly dependent then, $A\overrightarrow{x} = \overrightarrow{b}$ does <u>not</u> have a unique solution

PROOF Consider the counter-example $\mathbb{S} \triangleq \{\circ, \bullet\}$, $\tau \triangleq \{\langle\bullet,\bullet\rangle, \langle\bullet,\circ\rangle, \langle\circ,\circ\rangle\}$ so that $\mathcal{M}_\tau = \{\langle i, \lambda\ell\cdot\bullet\rangle, \langle j, \lambda\ell\cdot\circ\rangle,$ $\langle k, \lambda\ell\cdot(\ell < m \mathrel{?} \bullet \mathrel{¿} \circ)\rangle\}$. We let $\mathcal{X} \triangleq \{\langle i,\sigma\rangle \mid \forall j < i :$ $\sigma_j = \bullet\}$ so that $\neg FD(\mathcal{X})$. We have $\mathcal{M}_{\tau\downarrow\bullet} = \{\langle i,\lambda\ell\cdot\bullet\rangle,$ $\langle k, \lambda\ell\cdot(\ell < m \mathrel{?} \bullet \mathrel{¿} \circ)\rangle \mid k < m\}$, $\mathcal{M}_{\tau\downarrow\circ} = \{\langle j,\lambda\ell\cdot\circ\rangle,$ $\langle k, \lambda\ell\cdot(\ell < m \mathrel{?} \bullet \mathrel{¿} \circ)\rangle \mid k \geq m\}$ and $\oplus\llbracket\mathcal{X}\rrbracket = \{\langle i,\sigma\rangle \mid \forall j \leq$ $i : \sigma_j = \bullet\}$. We have $\alpha^\vee_{\mathcal{M}_\tau}(\oplus\llbracket\mathcal{X}\rrbracket) = \{s \mid \mathcal{M}_{\tau\downarrow s} \subseteq \oplus\llbracket\mathcal{X}\rrbracket\} = \{\bullet\}$ whereas $\widetilde{pre}[\tau](\alpha^\vee_{\mathcal{M}_\tau}(\mathcal{X})) = \widetilde{pre}[\tau](\{s \mid \mathcal{M}_{\tau\downarrow s} \subseteq \mathcal{X}\}) = \widetilde{pre}[\tau](\{\bullet\})$ $= \{s \mid \forall s' : t(s, s') \Rightarrow s' = \bullet\} = \emptyset$ since $t(s, \bullet)$ implies $s = \bullet$ and $t(\bullet, \circ)$ holds. ■

Solutions for linear equations

- Theorem: if the columns of the matrix A are linearly dependent then, $A\vec{x} = \vec{b}$ does <u>not</u> have a unique solution

Proof for $A \in \mathbb{R}^{3\times 3}$

know: columns are linearly dependent

show: more than 1 solution

Concept: pick some specific solution $\vec{x}^*$, and show that there's another one

Let: $A\vec{x}^* = \vec{b}$ and $A = [\vec{a}_1 \ \ \vec{a}_2 \ \ \vec{a}_3]$

From linear dependence Def 2:

$$\alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \alpha_3\vec{a}_3 = \vec{0}$$

Solutions for linear equations

- Theorem: if the columns of the matrix A are linearly dependent then, $A\vec{x} = \vec{b}$ does <u>not</u> have a unique solution

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From linear dependence Def 2:

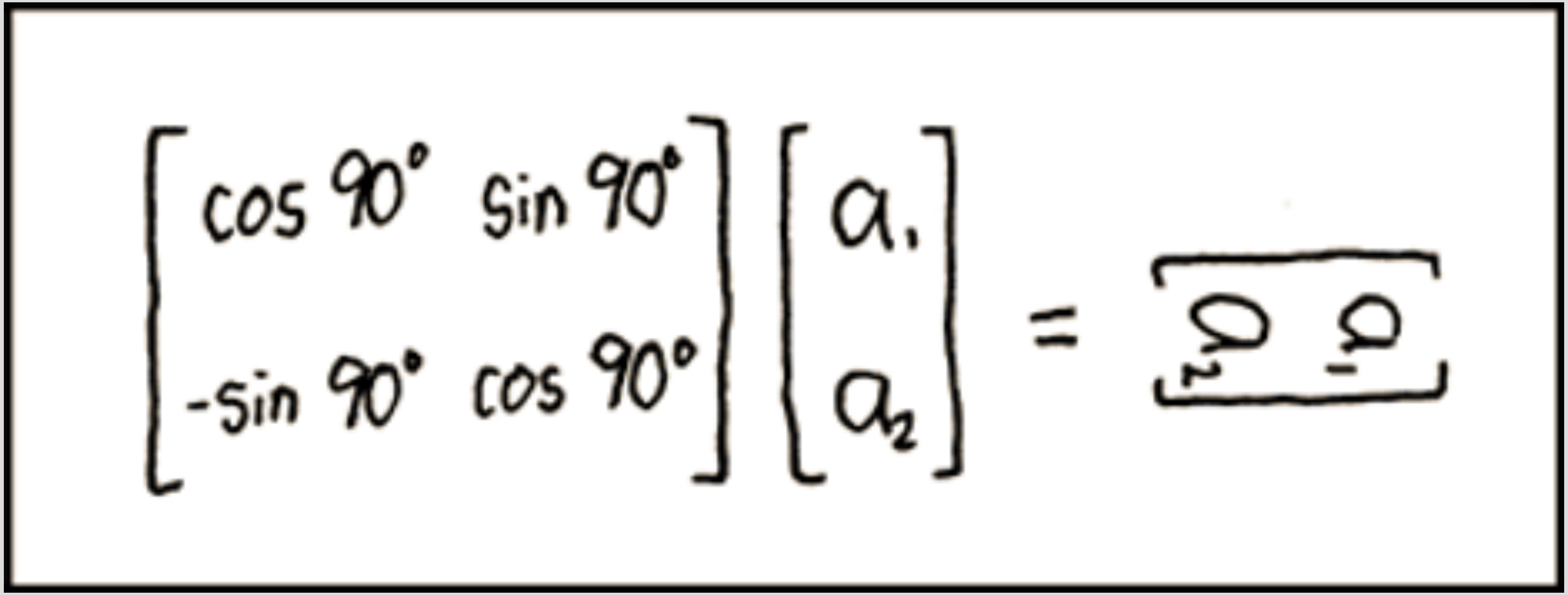
$$\alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \alpha_3\vec{a}_3 = \vec{0} \longrightarrow \begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \vec{a}_3\end{bmatrix}\underbrace{\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix}}_{\vec{\alpha}} = 0 \qquad \Rightarrow A\vec{\alpha} = 0$$

Set $\vec{x}^\dagger = \vec{x}^* + \vec{\alpha}$

$$\Rightarrow A\vec{x}^\dagger = A(\vec{x}^* + \vec{\alpha}) = A\vec{x}^* + A\vec{\alpha} = \vec{b} + 0$$

So $\vec{x}^\dagger$ is another solution!

Matrix Transformations

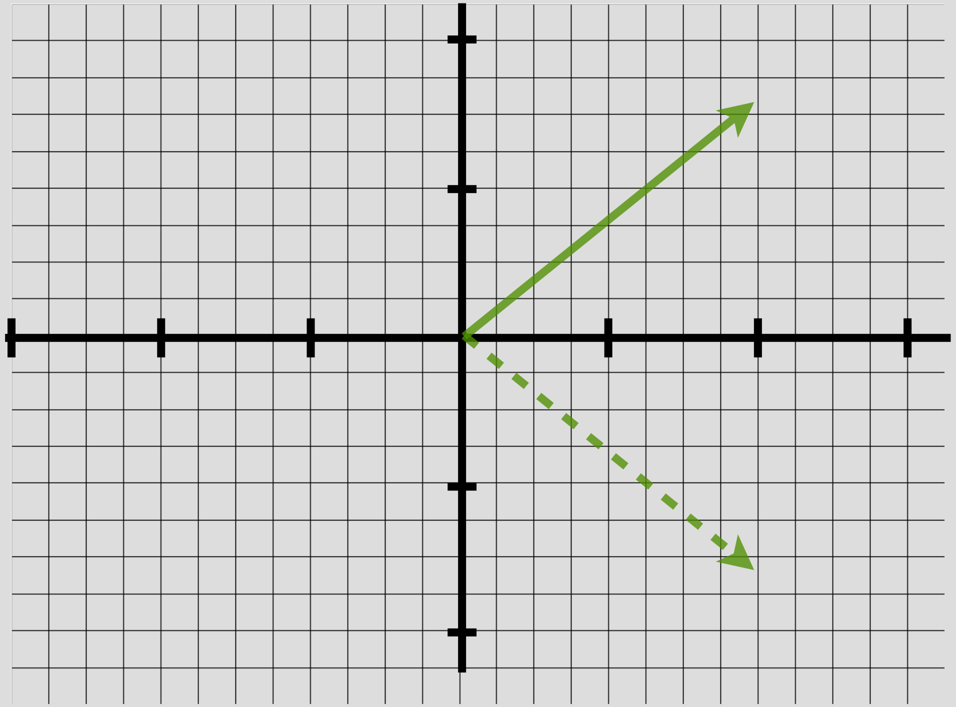


Matrices are operators that transform vectors

$$A\overrightarrow{x} = \overrightarrow{b}$$

Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$



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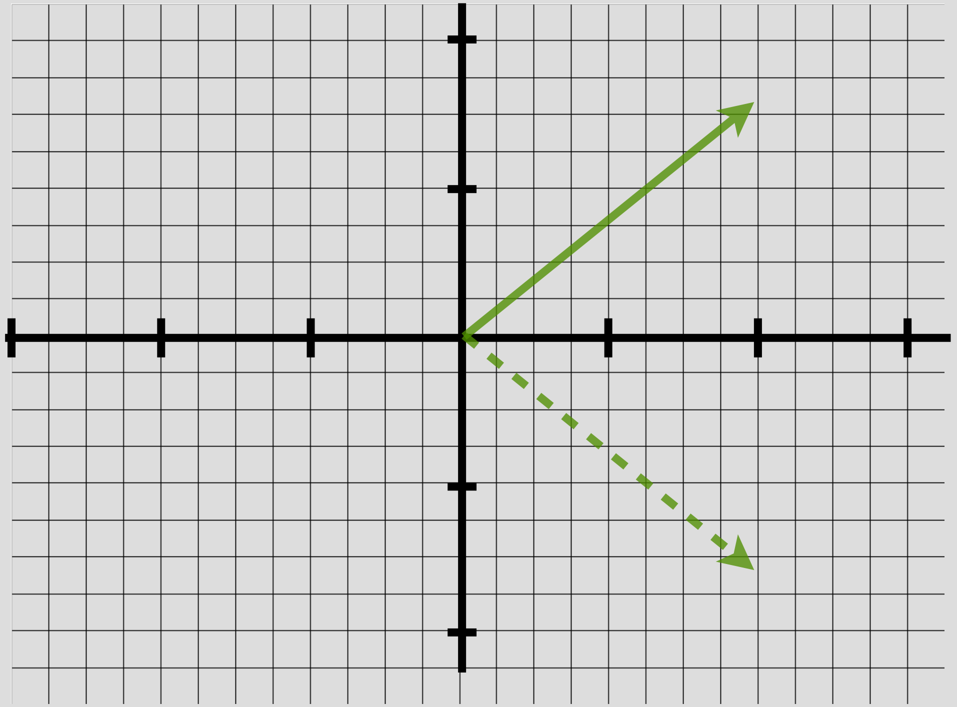

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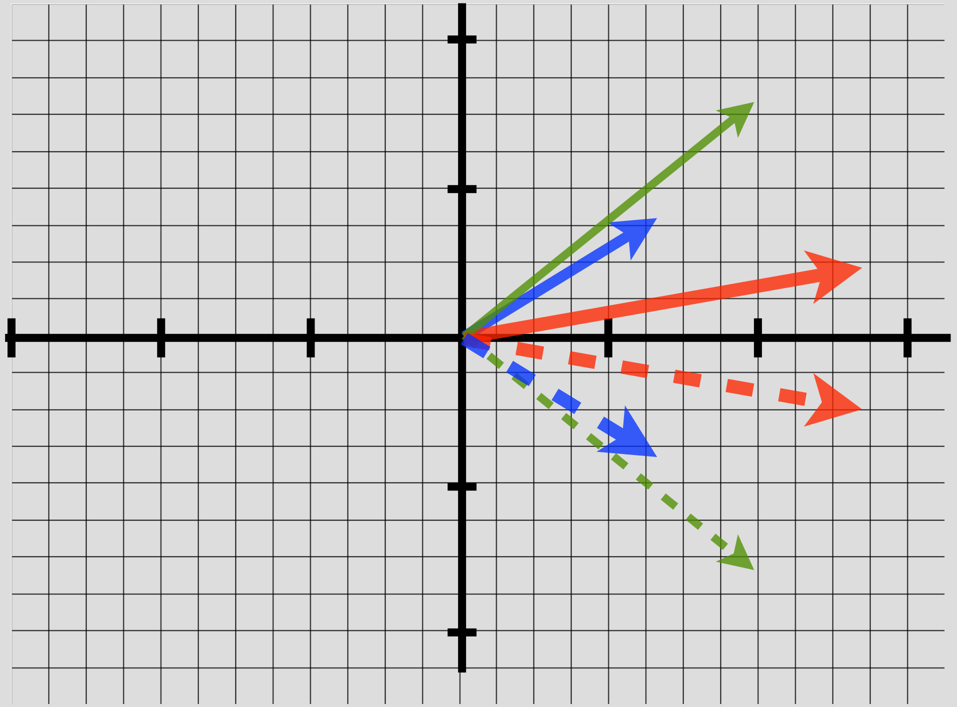
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

Reflection Matrix!



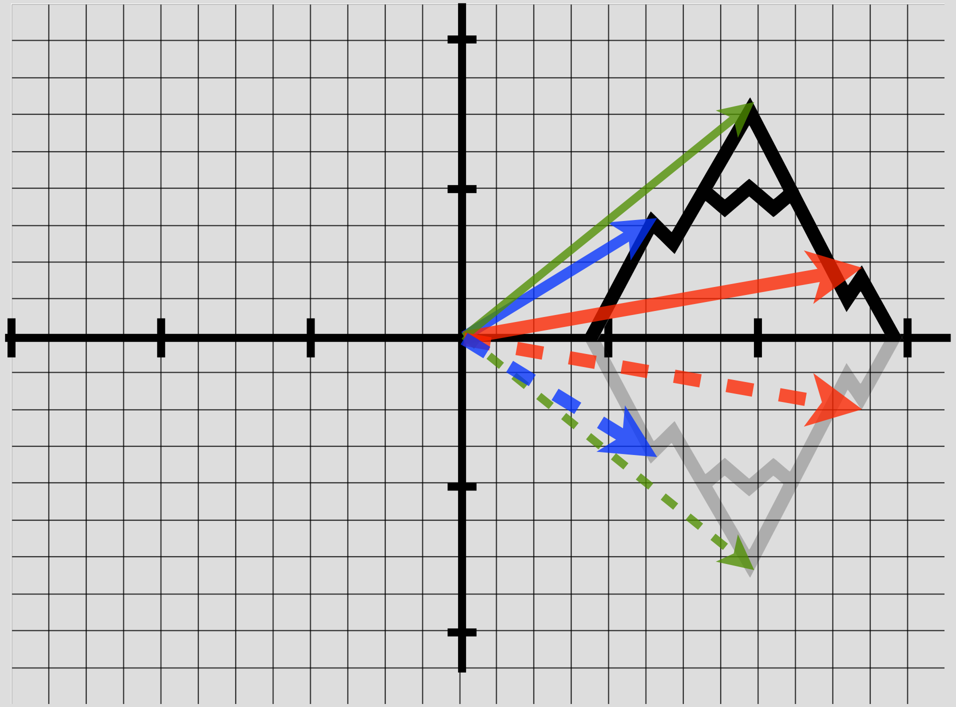
Matrices are operators that transform vectors

$$A\overrightarrow{x} = \overrightarrow{b}$$

Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

Reflection Matrix!

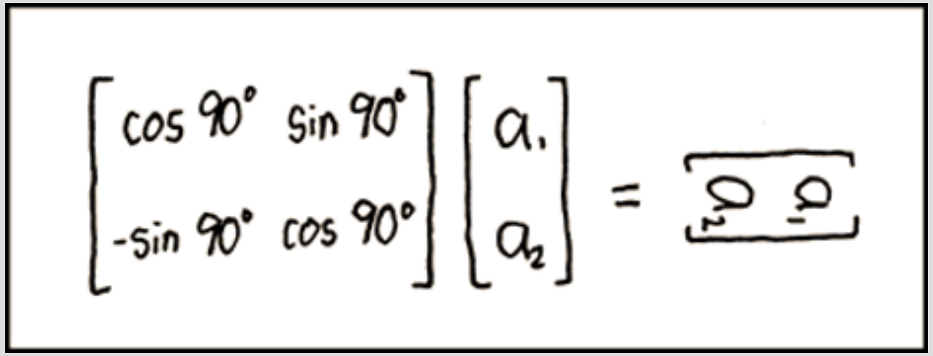


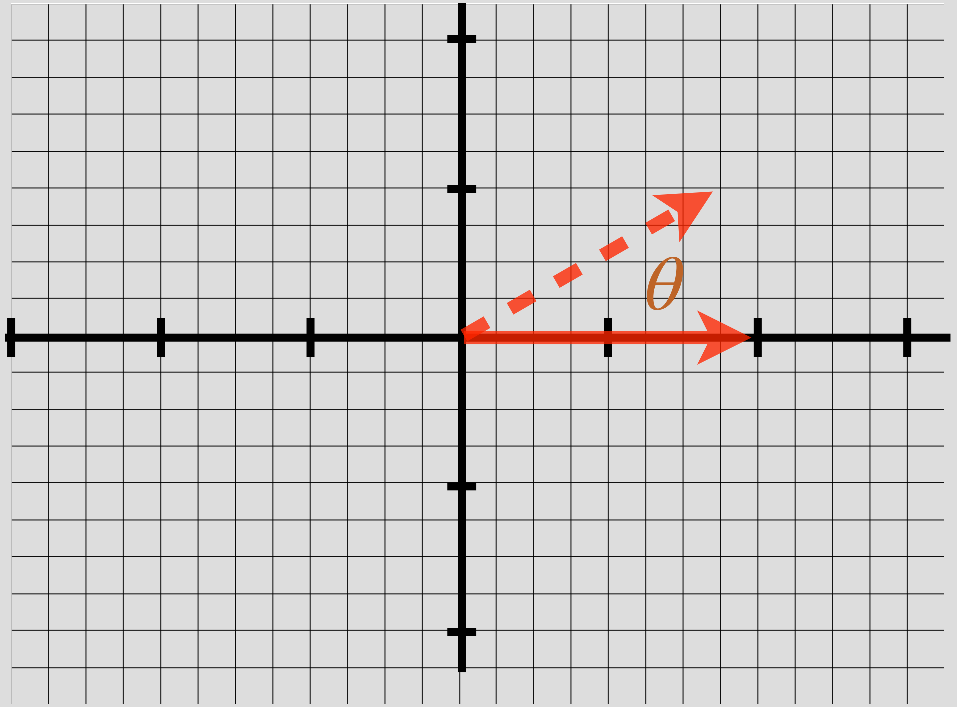
Matrices are operators that transform vectors

Example 2:
$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta)x_1 - \sin(\theta)x2 \\ \sin(\theta)x_1 + \cos(\theta)x_2 \end{bmatrix}$$

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

Rotation Matrix!





Linear Transformation of vectors

f: is a linear transformation if:

$$f(\alpha \overrightarrow{x}) = \alpha f(\overrightarrow{x}) \qquad \alpha \in \mathbb{R}$$

$$f(\overrightarrow{x} + \overrightarrow{y}) = f(\overrightarrow{x}) + f(\overrightarrow{y})$$

Claim: Matrix-vector multiplications satisfy linear transformation

$$A \cdot (\alpha \overrightarrow{x}) = \alpha A \overrightarrow{x}$$

Proof via explicitly writing the elements

$$A \cdot (\overrightarrow{x} + \overrightarrow{y}) = A \overrightarrow{x} + A \overrightarrow{y}$$

Vectors as states, Matrices as state transition

Vectors can represent states of a system

Example: The state of a car at time = t

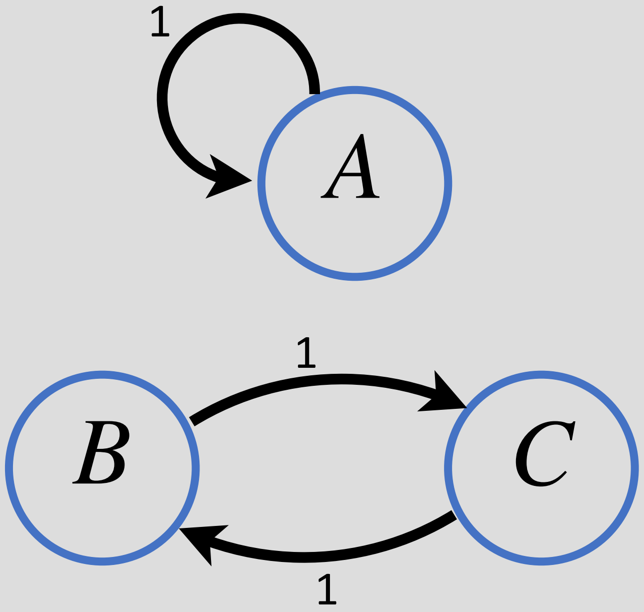
$$\vec{s}(t) = \begin{bmatrix} x(t) \\ y(t) \\ v(t) \\ \theta(t) \end{bmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{position} \\ \} \text{ velocity} \\ \\ \end{array}$$

Q: Is that enough?

A: need orientation or $v_x(t), v_y(t)$

Graph Transition Matrices

Example: Reservoirs and Pumps



Q: What is the state?

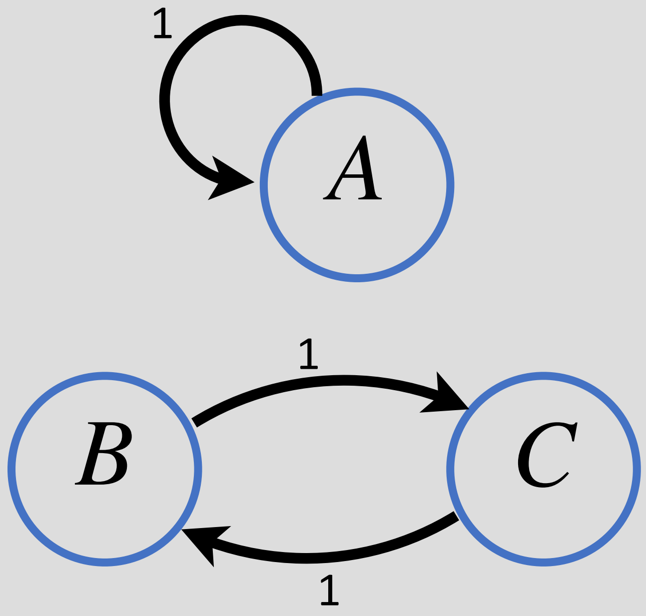
A: Water in each reservoir

$$\vec{x}(t) = \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

Pumps move water…

What would the state be tomorrow?

State Transition Matrices

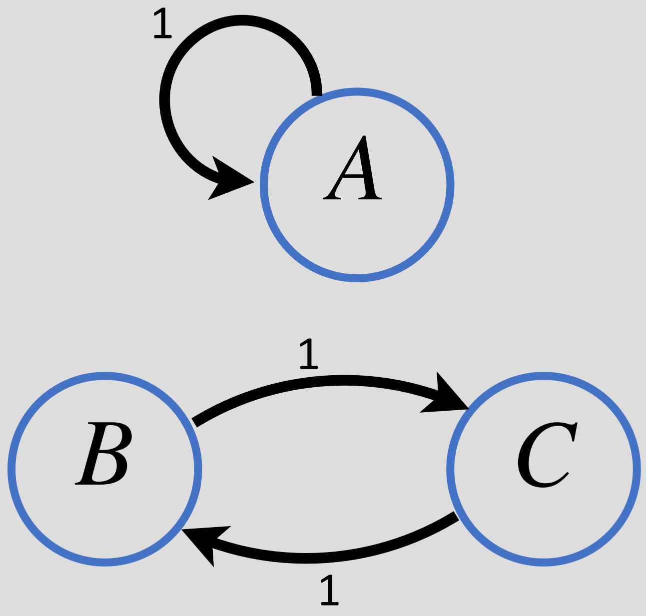


State Transition Matrices

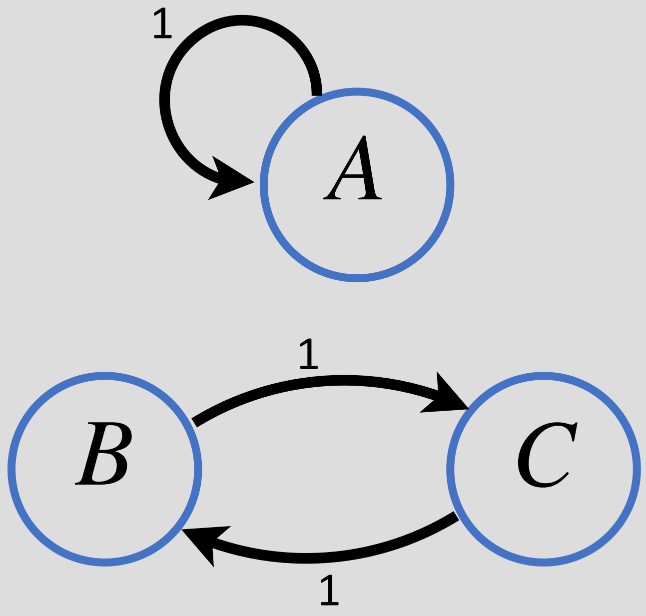
$$x_A(t+1) = x_A(t)$$
$$x_B(t+1) = x_C(t)$$
$$x_C(t+1) = x_B(t)$$

Write as a matrix-vector multiplication:

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



State Transition Matrices



$$x_A(t+1) = x_A(t)$$
$$x_B(t+1) = x_C(t)$$
$$x_C(t+1) = x_B(t)$$

Write as a matrix-vector multiplication:

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

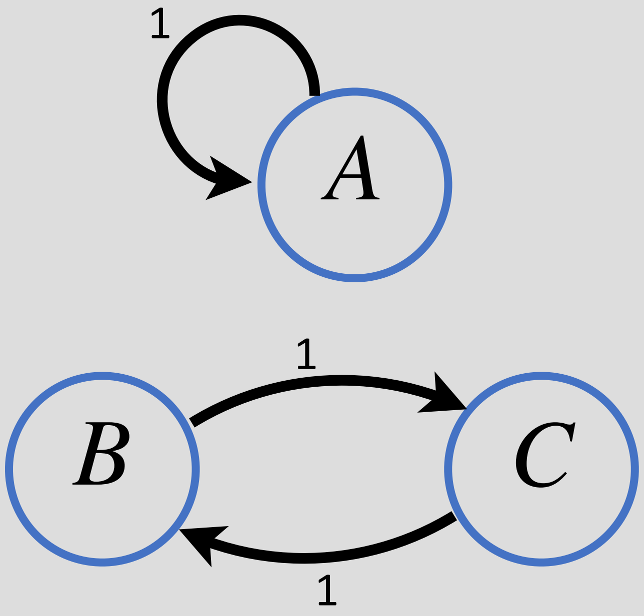
or $\overrightarrow{x}(t+1) = Q\overrightarrow{x}(t)$

What is the state after 2 times?

$$\overrightarrow{x}(t+2) = Q\overrightarrow{x}(t+1) = QQ\overrightarrow{x}(t) = Q^2\overrightarrow{x}(t)$$

State Transition Matrices

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

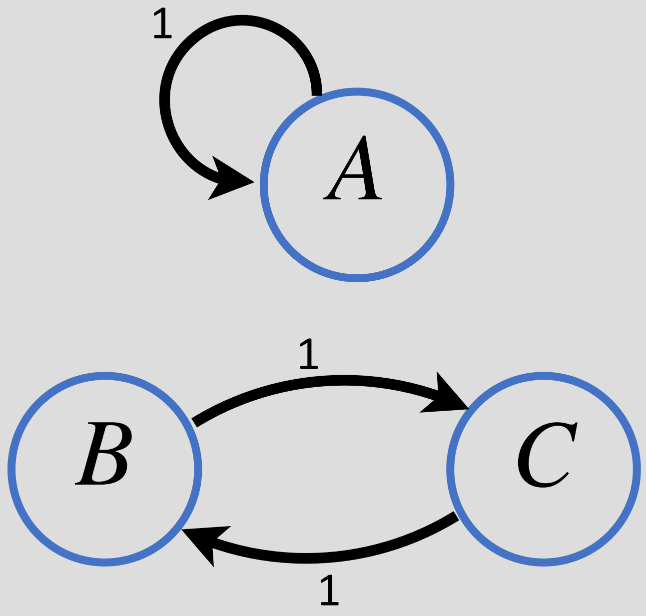


$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

What is the state after at t=1, 2?

State Transition Matrices

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

What is the state after at t=1, 2?

(1)

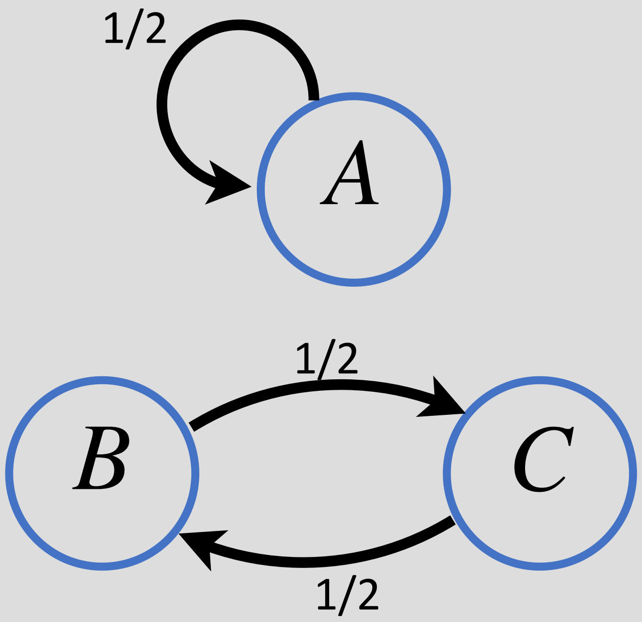
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

(2)

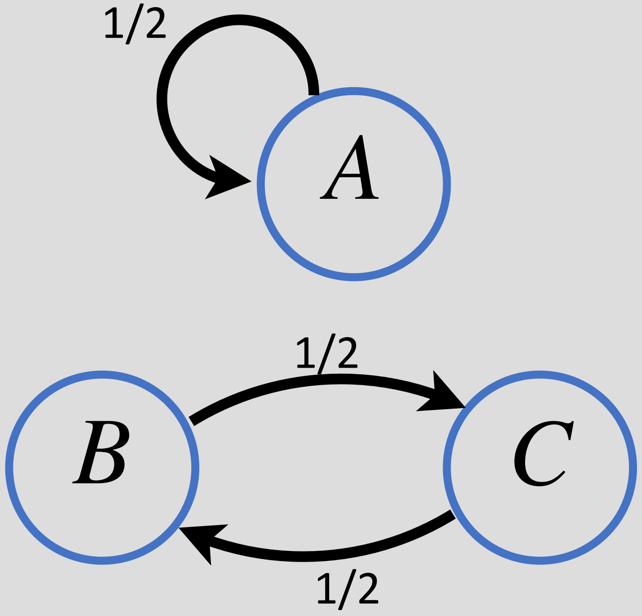
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\varphi \cdot \varphi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

State Transition Matrices



State Transition Matrices



$$x[t+1] = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} x(t)$$

Non-conservative!

$$Q^2 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

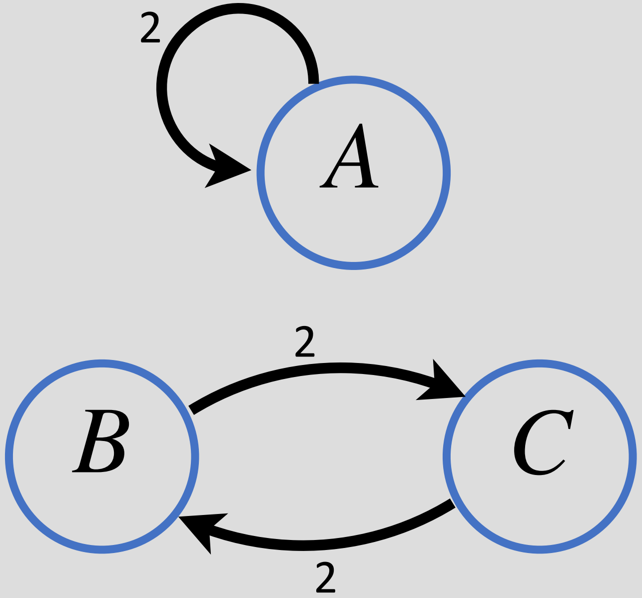
Q) What will happen if we keep going?

A) Numbers will diminish to zero


Google
DEAD SEA SOUTH
1984


פני הים
Niveau De La Mer, Sea Level
TRESPASS

State Transition Matrices



$$x(t+1) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} x(t)$$

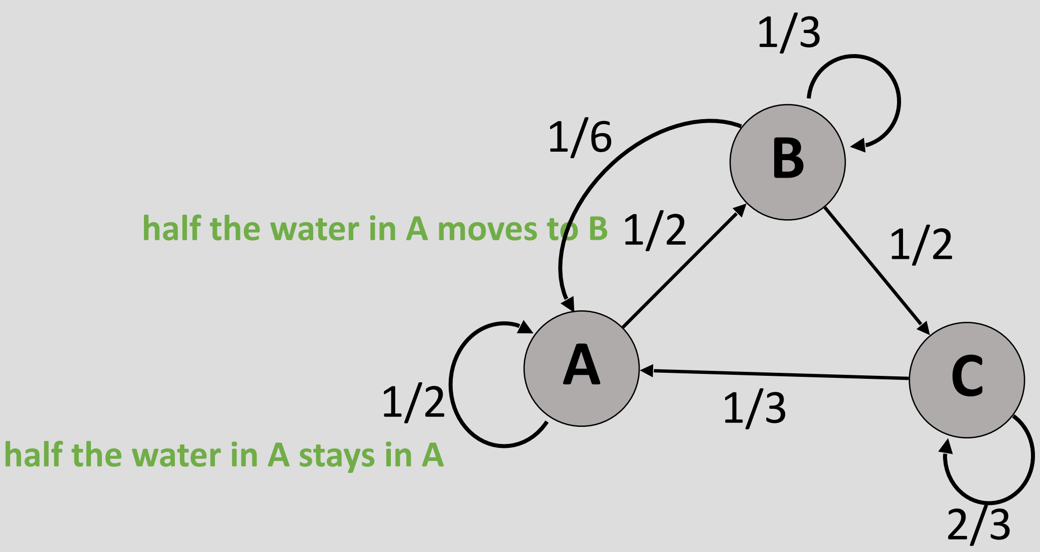
$$\Phi^2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Q) What will happen if we keep going?

A) Numbers will explode to infinity

Graph Representation

Ex: Reservoirs and Pumps



Nodes

I have 3 reservoirs: A,B,C
and I want to keep track of how much water is in each

When I turn on some pumps, water moves between the reservoirs.

Where the water moves and what fraction is represented by arrows.

Edge weights **Edges**

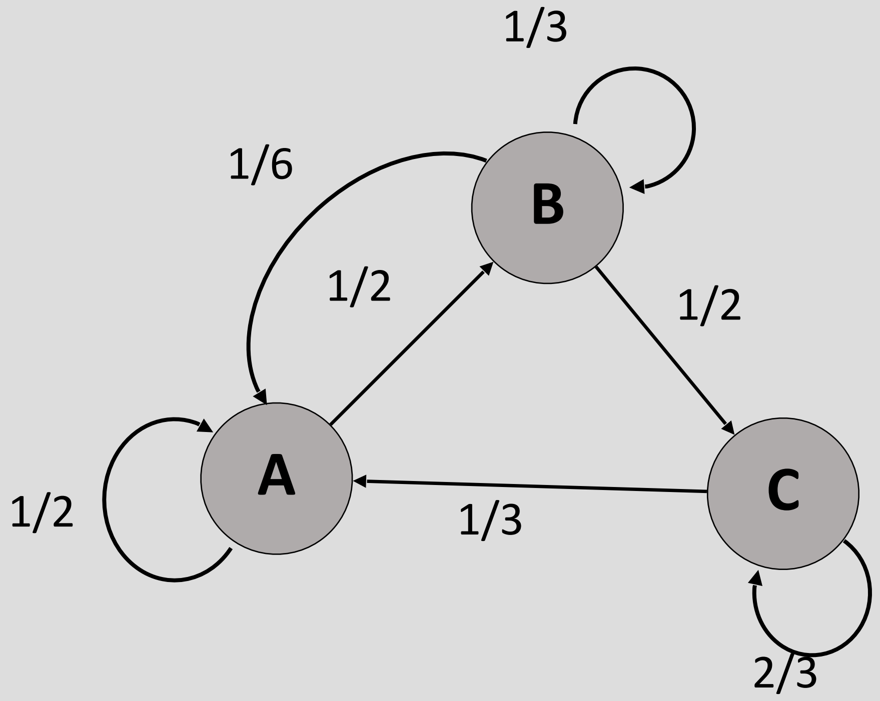
"directed" graph because arrows have a direction

Where does the rest of the water in A go? Need to label that too…

Can you tell me how much water in each after pumps start? Need to know initial amounts

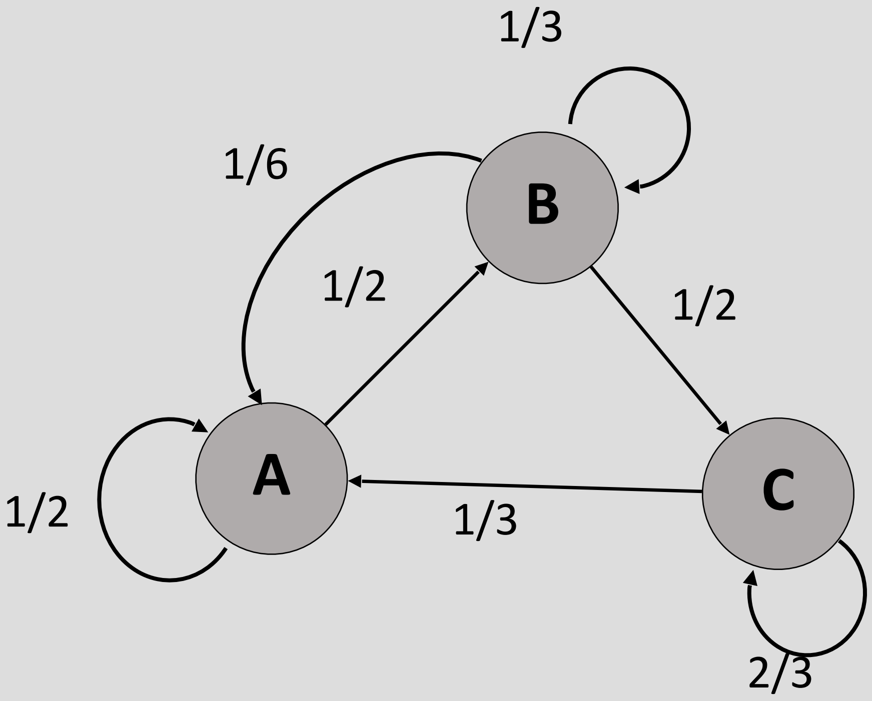
Exercise:

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} A \to A & B \to A & C \to A \\ A \to B & B \to B & C \to B \\ A \to C & B \to C & C \to C \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



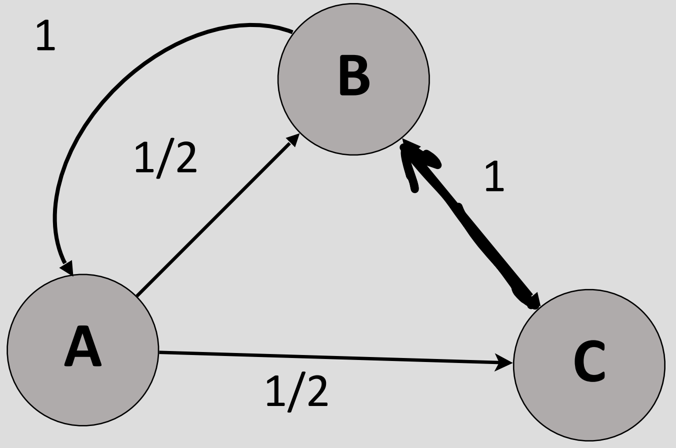
Exercise:

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} \underset{A \to A}{\frac{1}{2}} & \underset{B \to A}{\frac{1}{6}} & \underset{C \to A}{\frac{1}{3}} \\ \underset{A \to B}{\frac{1}{2}} & \underset{B \to B}{\frac{1}{3}} & \underset{C \to B}{0} \\ \underset{A \to C}{0} & \underset{B \to C}{\frac{1}{2}} & \underset{C \to C}{\frac{2}{3}} \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



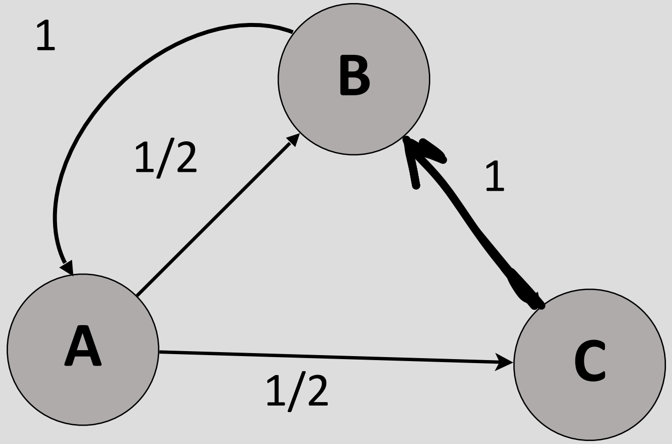
Example 2:

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} A \to A & B \to A & C \to A \\ A \to B & B \to B & C \to B \\ A \to C & B \to C & C \to C \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



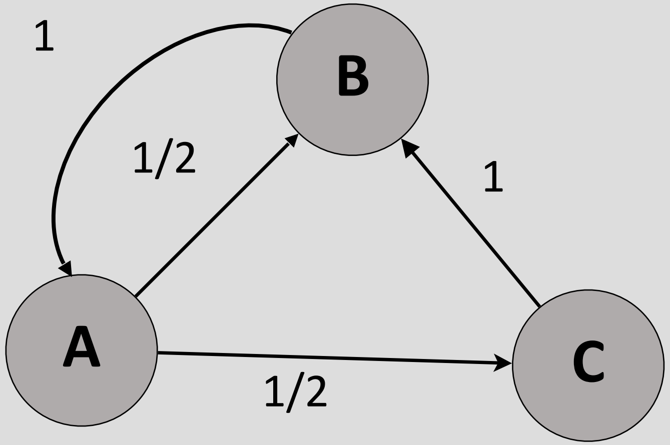
Example 2:

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} \underset{A \to A}{0} & \underset{B \to A}{1} & \underset{C \to A}{0} \\ \underset{A \to B}{\frac{1}{2}} & \underset{B \to B}{0} & \underset{C \to B}{1} \\ \underset{A \to C}{\frac{1}{2}} & \underset{B \to C}{0} & \underset{C \to C}{0} \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



What about the reverse?

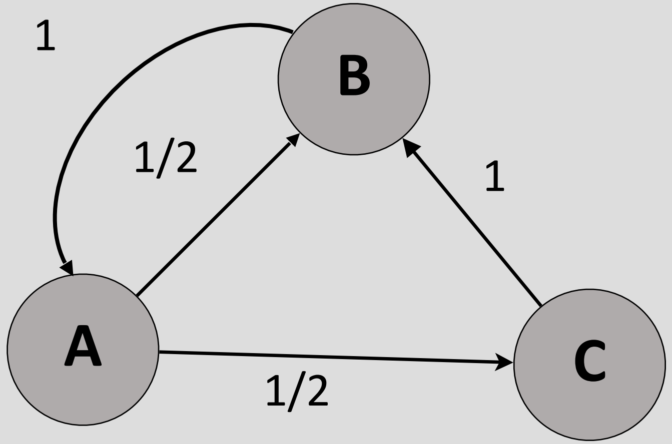
$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



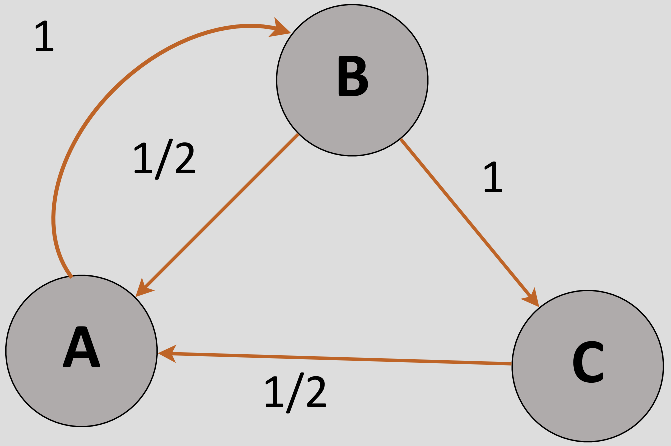
Q) Will flipping the arrows make us go back in time?

What about the reverse?

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} \underset{A\to A}{0} & \underset{B\to A}{1} & \underset{C\to A}{0} \\ \underset{A\to B}{\tfrac{1}{2}} & \underset{B\to B}{0} & \underset{C\to B}{1} \\ \underset{A\to C}{\tfrac{1}{2}} & \underset{B\to C}{0} & \underset{C\to C}{0} \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

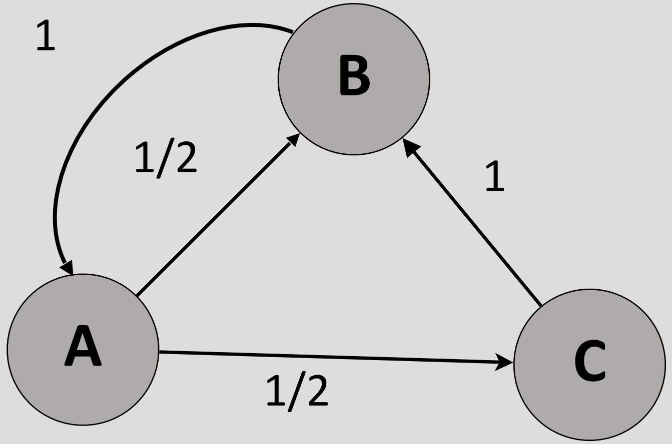


Q) Will flipping the arrows make us go back in time?



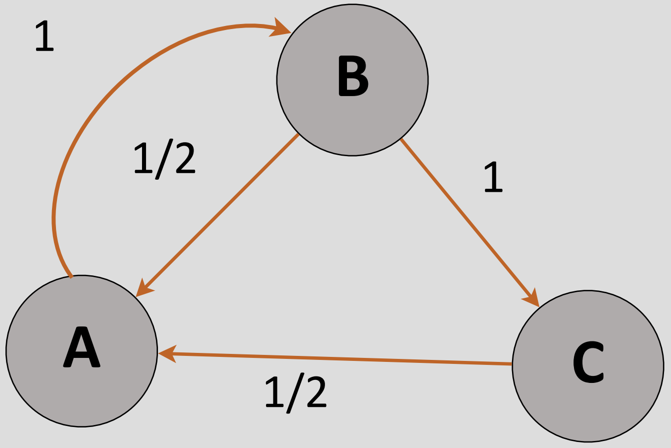
What about the reverse?

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



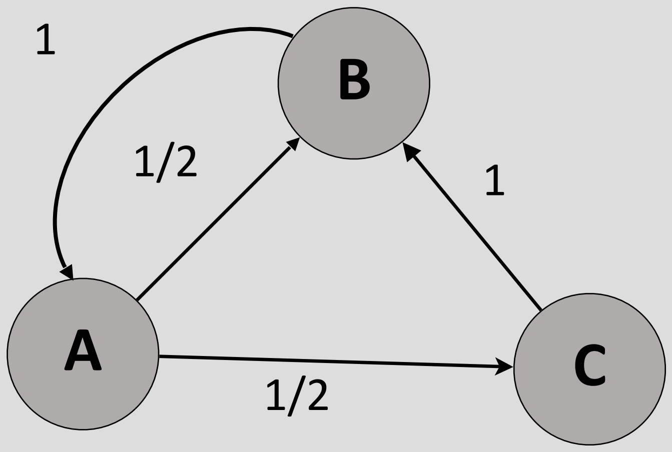
Q) Will flipping the arrows make us go back in time?

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$



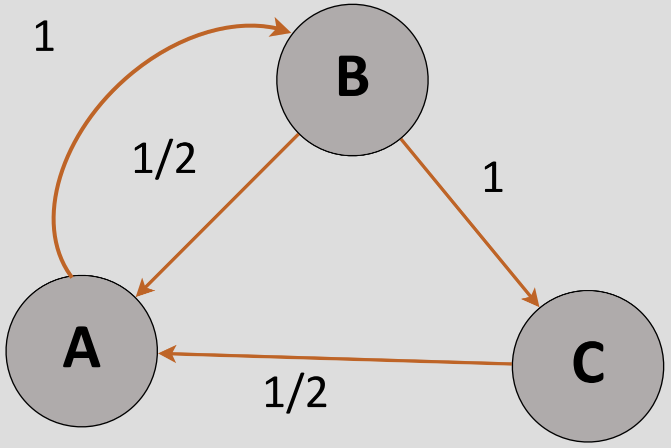
What about the reverse?

$$\begin{array}{c} 6 \\ 10 \\ 2 \end{array} \begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} \underset{A \to A}{0} & \underset{B \to A}{1} & \underset{C \to A}{0} \\ \underset{A \to B}{\frac{1}{2}} & \underset{B \to B}{0} & \underset{C \to B}{1} \\ \underset{A \to C}{\frac{1}{2}} & \underset{B \to C}{0} & \underset{C \to C}{0} \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix} \begin{array}{c} 4 \\ 6 \\ 8 \end{array}$$



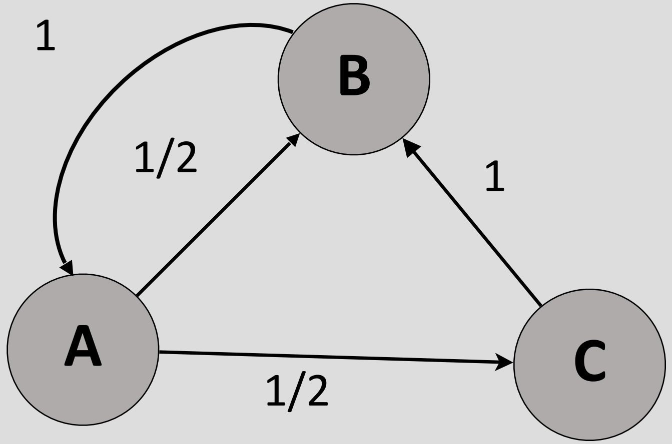
Q) Will flipping the arrows make us go back in time?

$$\begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$



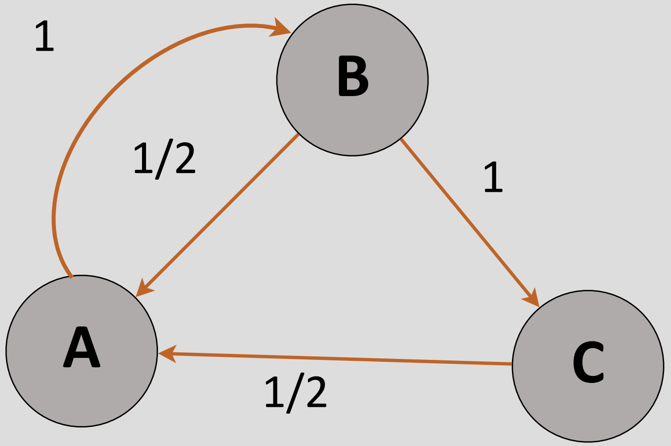
What about the reverse?

$$
\begin{array}{c}
6\\ 10\\ 2
\end{array}
\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix}
=
\begin{bmatrix}
\underset{A\to A}{0} & \underset{B\to A}{1} & \underset{C\to A}{0} \\
\underset{A\to B}{\tfrac{1}{2}} & \underset{B\to B}{0} & \underset{C\to B}{1} \\
\underset{A\to C}{\tfrac{1}{2}} & \underset{B\to C}{0} & \underset{C\to C}{0}
\end{bmatrix}
\begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}
\begin{array}{c}
4\\ 6\\ 8
\end{array}
$$



Q) Will flipping the arrows make us go back in time?

$$
\begin{bmatrix} 7 \\ 6 \\ 10 \end{bmatrix}
=
\begin{bmatrix}
0 & \tfrac{1}{2} & \tfrac{1}{2} \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix} 6 \\ 10 \\ 2 \end{bmatrix}
$$



A) In general, no!

Matrix Transpose

If the elements of the matrix $A \in \mathbb{R}^{N \times M}$ are a_{ij}

The elements of $A^T \in \mathbb{R}^{M \times N}$ are a_{ji}

Matrix transpose is not (generally) an inverse!

$$A \in \mathbb{R}^{N \times M} \quad \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_m \end{bmatrix} \qquad \begin{bmatrix} \vec{a}_1^{\,T} \\ \vec{a}_2^{\,T} \\ \vdots \\ a_m^T \end{bmatrix} \quad A^T \in \mathbb{R}^{M \times N}$$

Matrix Transpose

If the elements of the matrix $A \in \mathbb{R}^{N \times M}$ are a_{ij}

The elements of $A^T \in \mathbb{R}^{M \times N}$ are a_{ji}

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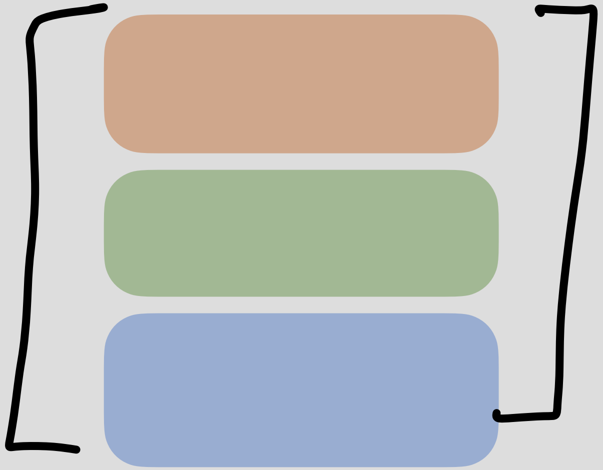
$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_m \end{bmatrix} \qquad \begin{bmatrix} \vec{a}_1^{\,T} \\ \vec{a}_2^{\,T} \\ \vdots \\ a_m^T \end{bmatrix}$$

$A \in \mathbb{R}^{N \times M}$ $\qquad$ $A^T \in \mathbb{R}^{M \times N}$

$$\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$$



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$A \in \mathbb{R}^{N\times M}$ $\qquad$ $A^T \in \mathbb{R}^{M\times N}$

$$\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Matrix Inversion