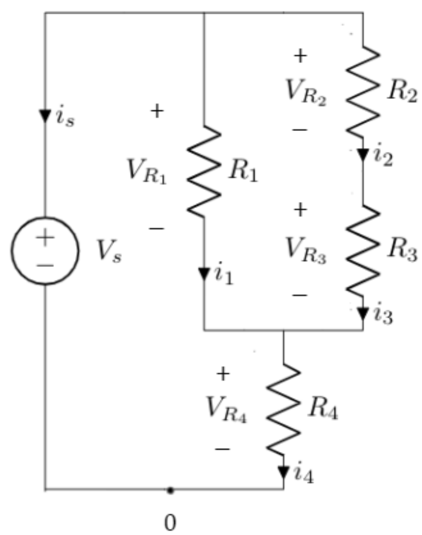
EECS 16A Designing Information Devices and Systems I
Fall 2021 Midterm 1

Midterm 1 Solution

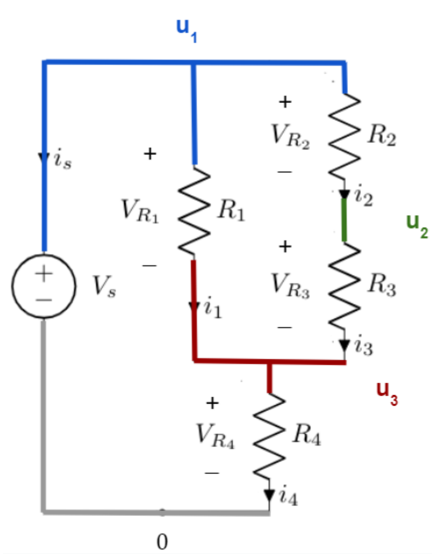
1. Circuits Part I (6 points)

You are given the following circuit for Question 1-2.



(a) What is the number of nodes in the circuit, including the ground node?

Solution: There are 4 nodes in the circuit.



(b) Complete the following KCL equations:

i_s + [value1] + [value2] = 0

[value3] - i_3 + [value4] = 0

Solution: For the first equation, the top node is the only node with three currents that includes i_s. All three currents are leaving the node, so we write $i_s + i_1 + i_2 = 0$. For the second equation, the only node with three currents that involves i_3 is the middle node, where i_1 and i_3 are entering and i_4 is leaving. To make the signs match what is given, we can write $-i_1 - i_3 + i_4 = 0$.

(c) Complete the following Ohm's law relationships:
$V_{R2} = i_2[value5]$
Solution: From Ohms law and the labeling in the diagram, $V_{R2} = i_2 R_2$

2. Circuits Part II (3 points)

(d) When solving the circuit using $\mathbf{A}\vec{x} = \vec{b}$, $\vec{x}$ consists of all element currents and node potentials, other than the zero potential node. How many elements are there in $\vec{x}$?

Solution: There are 5 unknown element currents $(i_s, i_1, i_2, i_3, i_4)$ and 3 unknown node potentials (u_1, u_2, u_3), so we have 8 unknowns in total.

(e) How many columns are there in $\mathbf{A}$?

Solution: With 8 unknowns, we need 8 columns to solve for all 8 unknowns.

3. Gaussian Elimination Part I (3 points)

(version 1) Complete the following Gaussian elimination problem by filling the blanks for the equation $\mathbf{A}\vec{x} = \vec{b}$, where $\mathbf{A} = \begin{bmatrix} 2 & 6m \\ 2 & 12 \\ 2 & 12 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 8 \\ k \end{bmatrix}$.

$$\left(\begin{array}{cc|c} 2 & 6m & 2 \\ 2 & 12 & 8 \\ 2 & 12 & k \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & [\text{value1}] & 1 \\ 1 & 6 & 4 \\ 0 & 0 & k+[\text{value2}] \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & [\text{value1}] & 1 \\ 0 & [\text{value3}] & 3 \\ 0 & 0 & k+[\text{value2}] \end{array}\right)$$

(version 2) Complete the following Gaussian elimination problem by filling the blanks for the equation $\mathbf{A}\vec{x} = \vec{b}$, where $\mathbf{A} = \begin{bmatrix} 2 & 8m \\ 2 & 12 \\ 2 & 12 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 8 \\ k \end{bmatrix}$.

$$\left(\begin{array}{cc|c} 2 & 8m & 2 \\ 2 & 12 & 8 \\ 2 & 12 & k \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & [\text{value1}] & 1 \\ 1 & 6 & 4 \\ 0 & 0 & k+[\text{value2}] \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & [\text{value1}] & 1 \\ 0 & [\text{value3}] & 3 \\ 0 & 0 & k+[\text{value2}] \end{array}\right)$$

(version 3) Complete the following Gaussian elimination problem by filling the blanks for the equation $\mathbf{A}\vec{x} = \vec{b}$, where $\mathbf{A} = \begin{bmatrix} 2 & 6m \\ 2 & 12 \\ 2 & 12 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 6 \\ k \end{bmatrix}$.

$$\left(\begin{array}{cc|c} 2 & 6m & 2 \\ 2 & 12 & 6 \\ 2 & 12 & k \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & [\text{value1}] & 1 \\ 1 & 6 & 3 \\ 0 & 0 & k+[\text{value2}] \end{array}\right) \Longrightarrow \left(\begin{array}{cc|c} 1 & [\text{value1}] & 1 \\ 0 & [\text{value3}] & 2 \\ 0 & 0 & k+[\text{value2}] \end{array}\right)$$

Note that multiple row reduction steps were taken between the arrows.
Express your answers in terms of m and/or k.

value1 = [ ]

value2 = [ ]

value3 = [ ]

Solution: The row reduction operations performed to get to the second matrix from the first matrix is subtracting row 2 from row 3 and then dividing row 1 and row 2 by 2. The row reduction operations performed to get to the third matrix from the second matrix is subtracting row 1 from row 2. This means that
for version 1: value1 $= 3m$, value2 $= -8$, value3 $= 6-3m$;
for version 2: value1 $= 4m$, value2 $= -8$, value3 $= 6-4m$;
for version 3: value1 $= 3m$, value2 $= -6$, value3 $= 6-3m$.

4. Gaussian Elimination Part II (6 points)

You performed some row reduction steps to solve the equation $\mathbf{A}\vec{x} = \vec{b}$ and you arrived at the following result:

(version 1) $\left(\begin{array}{cc|c} 1 & 5m & 1 \\ 0 & 2-m & 3 \\ 0 & 0 & 3k \end{array}\right)$

(version 2) $\left(\begin{array}{cc|c} 1 & 5m & 1 \\ 0 & 3-m & 3 \\ 0 & 0 & k-3 \end{array}\right)$

(version 3) $\left(\begin{array}{cc|c} 1 & 5m & 1 \\ 0 & 2-2m & 3 \\ 0 & 0 & 5k \end{array}\right)$

(a) For what value of k could the solution exist?

(b) For what value of m does the system always have no solutions, independent of k?

(c) For what value of m does the matrix **A** have a non-trivial null space?

Solution:

(a) The solution will only exist if the third row is all zeros, because otherwise there will be a contradiction. This means that for version 1: $k = 0$; for version 2: $k = 3$; for version 3: $k = 0$.

(b) The solution does not exist if both elements of the second row in the row reduced matrix is zero. This means that for version 1: $m = 2$; for version 2: $m = 3$; for version 3: $m = 1$.

(c) The matrix A has a non-trivial nullspace if the columns are linearly dependent. This means that for version 1: $m = 2$; for version 2: $m = 3$; and for version 3: $m = 1$.

5. Berkeley boba stores Part I (6 points) *The following description applies to Question 5-6.*
Your EECS 16A TA Francis is a bobaholic. She is hosting a boba party with her lab ASEs to try out a secret recipe with a unique combination of toppings. Instead of buying each topping separately, Francis orders from Berkeley boba stores (U-cha, Yi Fang, Hay Tea) to get the combination of toppings.

Number of drinks	Boba store	Toppings		
		Grass jelly serving	Red bean serving	Mango pudding serving
x_u	U-cha	2	2	2
x_y	Yi Fang	1	3	4
x_h	Hay Tea	4	3	1

The secret recipe	
Toppings	Servings
Grass jelly	10
Red bean	9
Mango pudding	14

$\vec{x} = \begin{bmatrix} x_u \\ x_y \\ x_h \end{bmatrix}$ represents the number of drinks Francis is ordering from each store. $\vec{y}$ represents the servings of each topping Francis will get.

To try out the secret recipe, Francis sets up a system of linear equations $\mathbf{A}\vec{x} = \vec{y}$ to figure out how many drinks she needs from each store.

Fill in the blanks. You do not need to solve for $\vec{x}$

$$A = \begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{bmatrix}, \vec{y} = \begin{bmatrix} y1 \\ y2 \\ y3 \end{bmatrix}$$

Solution: Each row of the A matrix corresponds to the number of servings of each ingredient from the three different boba stores. For example, the first row reflects the servings of grass jelly from U-cha (2), Yi Fang (1), and Hay Tea (4). The vector $\vec{y}$ is simply the secret recipe listed above because that is the combination we are trying to get.

$$A = \begin{bmatrix} 2 & 1 & 4 \\ 2 & 3 & 3 \\ 2 & 4 & 1 \end{bmatrix}, \vec{y} = \begin{bmatrix} 10 \\ 9 \\ 14 \end{bmatrix}$$

6. Berkeley boba stores Part II (7 points)

From another recipe, Francis sets up another system of linear equations: $\mathbf{A}\vec{x} = \vec{y}$ where $\vec{y} \neq \vec{0}$. The solution has this form $\vec{x} = \begin{bmatrix} x_u \\ x_y \\ x_h \end{bmatrix} = \begin{bmatrix} \alpha+2 \\ -3\alpha+4 \\ \alpha \end{bmatrix}$, where α is a free variable. Given that we can only add drinks, the number of drinks cannot have a negative value ($x_u \geq 0, x_y \geq 0, x_h \geq 0$). Which of the following statements are implied? (Select all that apply)

(a) rank($\mathbf{A}$) = 1

(b) rank($\mathbf{A}$) = 2

(c) the set of possible solutions forms a subspace

(d) the span of vector $\vec{x}$ forms the null space of $\mathbf{A}$

(e) the set of possible solutions is a subset of a vector space

(f) the set of possible solutions forms an eigenspace

(g) matrix $\mathbf{A}$ has a non-trivial null space

Solution:

Knowing that we can express x_u and x_y with x_h, we can conclude that x_h will correspond to the one and only linearly dependent column from matrix A. Because A is a 3x3 matrix, the rank of A will be 2. Because matrix A has linearly dependent columns, it also has a non-trivial null space. $\rightarrow$ **(a) is wrong. (b) is correct. (g) is correct**

The set of possible solutions is a set of vectors and any set of vectors would be a subset of a vector space. $\rightarrow$ **(e) is correct.**

However, a subset of a vector space is different from a subspace. A subspace needs to satisfy additional conditions: containing the zero vector, closure under multiplication, and closure under addition. Because the set of possible solutions will NOT contain vectors with negative elements, it does not form a subspace. Because nullspaces and eigenspaces are types of vector subspaces, the set of possible solutions cannot form either of these spaces. $\rightarrow$ **(c)(d)(f) are wrong.**

7. Linear Dependence (3 points)

(version 1) Given that $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4} \in \mathbb{R}^3$ and $\vec{v_1} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 4 \\ 7 \\ \alpha \end{bmatrix}, \vec{v_4} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

Find α such that $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}$ are linearly dependent. Choose the best answer.

(a) 0

(b) 10

(c) Any scalar multiple of 10

(d) Any real number

(e) There is no value such that the vectors are linearly dependent

(version 2) Given that $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4} \in \mathbb{R}^3$ and $\vec{v_1} = \begin{bmatrix} 2 \\ 3 \\ 10 \end{bmatrix}, \vec{v_2} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}, \vec{v_3} = \begin{bmatrix} 4 \\ 7 \\ \alpha \end{bmatrix}, \vec{v_4} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$

Find α such that $\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}$ are linearly dependent. Choose the best answer.

(a) 0

(b) 20

(c) Any scalar multiple of 20

(d) Any real number

(e) There is no value such that the vectors are linearly dependent

Solution: Given that we have vectors in $\mathbb{R}^3$, at most we can have only 3 linearly independent vectors. Thus, if vectors v_1, v_2, v_4 are linearly independent, v_3 must be some linear combination of the others regardless of the value of α. If v_1, v_2, and v_4 are linearly dependent, then once again, the value of α does not change the fact that the four vectors will be linearly dependent on each other. Therefore, the correct answer is any real number. (Note: if you selected the "any scalar multiple of..." option, your answer was also marked as correct, because any scalar multiple of a non-zero number is equivalent to any real number.)

8. Eigenspace and Nullspace. (4 points)

Given $\alpha \in \mathbb{R}$, $\mathbf{A} = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$, and $\mathbf{B} = \begin{bmatrix} 2-\alpha & 1 & 2 \\ 0 & 1-\alpha & 4 \\ 0 & 0 & 2-\alpha \end{bmatrix}$, if there exists a vector $\vec{x}$ such that $\mathbf{B}\vec{x} = \vec{0}$ and $\vec{x} \neq \vec{0}$, which of the following are true?

(a) $\text{rank}(\mathbf{A}) = 3$

(b) $\vec{x}$ is in an eigenspace of $\mathbf{B}$.

(c) $\vec{x}$ is in the null space of $\mathbf{B}$.

(d) $\vec{x}$ is in an eigenspace of $\mathbf{A}$.

Solution:

(a) Notice that matrix $\mathbf{A}$ has three pivot columns, so that the rank of $\mathbf{A}$ is 3. $\rightarrow$ **a) is correct**

(b) Given the fact that there exists a vector $\vec{x}$ such that $\mathbf{B}\vec{x} = \vec{0}$ and $\vec{x} \neq \vec{0}$, we have $\mathbf{B}\vec{x} = \vec{0} = 0\vec{x}$ and $\vec{x} \neq \vec{0}$. Therefore, $\vec{x}$ is in the eigenspace of $\mathbf{B}$ that associated with eigenvalue $\lambda = 0$.$\rightarrow$ **b) is correct**

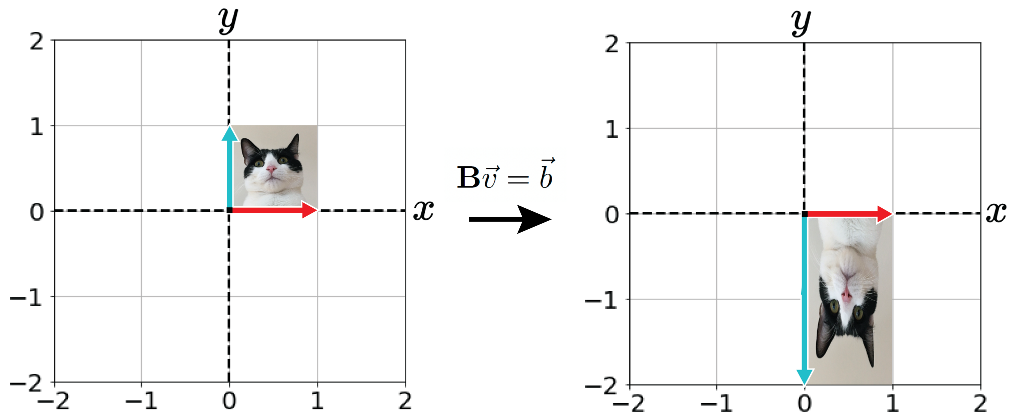
(c) $\mathbf{B}\vec{x} = \vec{0}$ follows the definition of the null space. $\rightarrow$ **c) is correct**

(d) Given $\mathbf{A}$ and $\mathbf{B}$, We have $\mathbf{B} = \begin{bmatrix} 2-\alpha & 1 & 2 \\ 0 & 1-\alpha & 4 \\ 0 & 0 & 2-\alpha \end{bmatrix} = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} = \mathbf{A} - \alpha\mathbf{I}$. Since $\mathbf{B}\vec{x} = (\mathbf{A} - \alpha\mathbf{I})\vec{x} = \vec{0}$, we have $\mathbf{A}\vec{x} = \alpha\vec{x}$. Therefore, $\vec{x}$ is in the eigenspace of $\mathbf{A}$ that associated with eigenvalue $\lambda = \alpha$. $\rightarrow$ **d) is correct**

According to the solutions, the correct answers are a), b), c), and d).

9. Transition matrix Part I (2 points) You want to practice some of the matrix transformation skills you've learned in EECS 16A, and decide to try them out on some cat photos.

Choose the correct matrices to transform the photo on the left to the photo on the right. Assume $\vec{v}$ represents a vector pointing to any pixel in the image, and $\vec{b}$ is the resulting vector after applying the transformation matrix to $\vec{v}$.



$\mathbf{B}\vec{v} = \vec{b}$, where $\vec{v} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \vec{b} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$. Which matrix $\mathbf{B}$ will result in the transformation seen above?

(a) $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$

(d) $\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}$

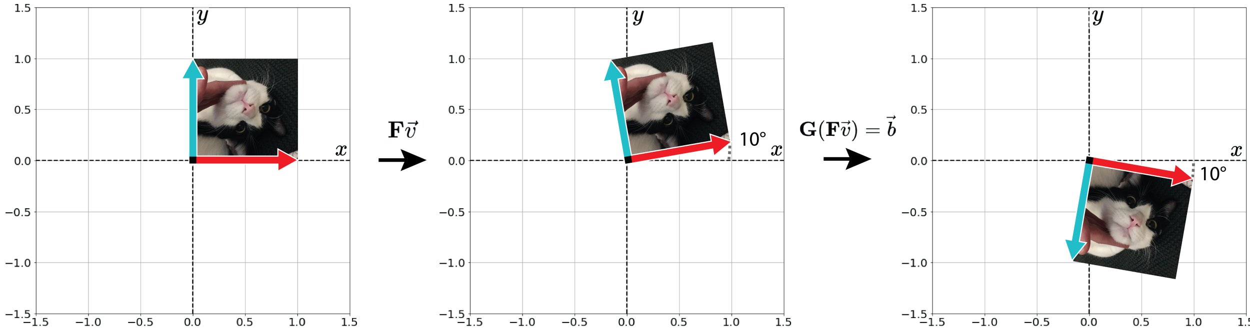
Solution: We can see that this cat picture is being both flipped over the x-axis, and scaled along the y-axis by a factor of two. Reflection over the x-axis is achieved by negating the y dimension, while a vertical stretch is also achieved by scaling the y dimension. The matrix that achieves this transformation is:

$$\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

For example, the transformation that this matrix would do to the blue arrow would be:

$$\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

10. Transition matrix Part II (3 points)



$\mathbf{G}(\mathbf{F}\vec{v}) = \vec{b}$, where $\vec{v} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$.

Which matrices **G** and **F** will result in the transformation seen above?

(a) Choose matrix **G** and **F** from the following matrices.

$\mathbf{M}1 = \begin{bmatrix} \cos(-10°) & -\sin(-10°) \\ \sin(-10°) & \cos(-10°) \end{bmatrix}$,

$\mathbf{M}2 = \begin{bmatrix} \cos(10°) & \sin(10°) \\ -\sin(10°) & \cos(10°) \end{bmatrix}$,

$\mathbf{M}3 = \begin{bmatrix} \cos(10°) & -\sin(10°) \\ \sin(10°) & \cos(10°) \end{bmatrix}$,

$\mathbf{M}4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{M}5 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\mathbf{M}6 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Solution: This cat picture is first being rotated **counterclockwise** by 10°, and then reflected over the x-axis. Recall the rotation matrix, which rotates a vector counterclockwise by the angle θ:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

We want to rotate our picture by 10°counterclockwise, thus our first transformation is:

$$\mathbf{F} = \begin{bmatrix} \cos(10°) & -\sin(10°) \\ \sin(10°) & \cos(10°) \end{bmatrix}$$

Next, in order to flip our picture over the x-axis, we need to negate the y coordinate, which can be achieved using the following transformation matrix:

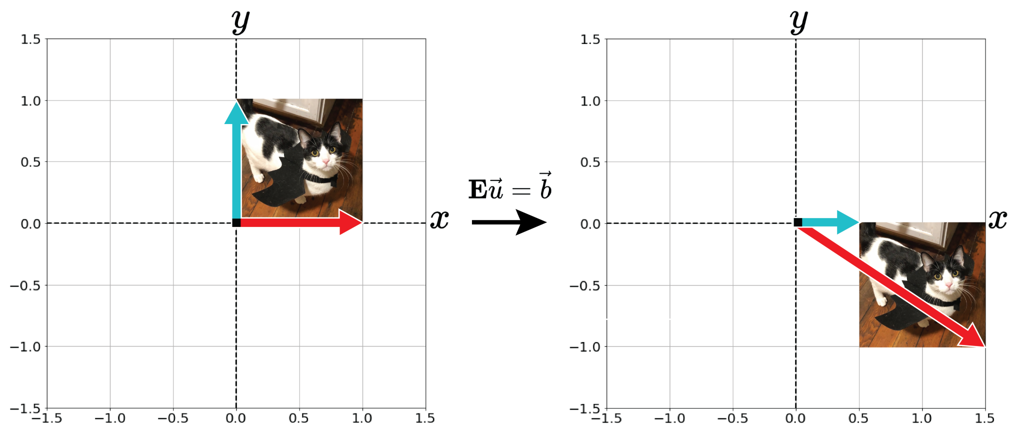
$$\mathbf{G} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(b) Would $\mathbf{F}(\mathbf{G}\vec{v})$ result in the same vector $\vec{b}$?

Solution: In general, the order of transformations matter when doing rotations and reflections (the exception being rotations by 45°or 90°). Because this rotation is by 10°, $\mathbf{F}(\mathbf{G}\vec{v}) \neq \mathbf{G}(\mathbf{F}\vec{v})$.

11. Transition matrix Part III (3 points)

Expressing translation in matrix multiplication



$\vec{b} = \vec{v} + \begin{bmatrix} 0.5 \\ -1 \end{bmatrix}$, where $\vec{v} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$.

Consider $\vec{u} = \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$, we can express the translation in matrix multiplication form: $\mathbf{E}\vec{u} = \vec{b}$.

Which matrix $\mathbf{E}$ will result in the transformation seen above?

(a) $\begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & -1 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0.5 \\ 0 & -1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}$

(e) such matrix $\mathbf{E}$ does not exist

Solution: In this case, the cat picture is being translated from the origin. It is moving 0.5 units right along the x-axis, and 1 unit down along the y-axis. We are told that this transformation results from adding the vector $\begin{bmatrix} 0.5 \\ -1 \end{bmatrix}$ to the original vector, $\vec{v}$. The original vector is preserved by the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Thus, the transformation matrix

$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & -1 \end{bmatrix}$$

will translate the original vector 0.5 units right and 1 unit down without reflecting or stretching. For example, the transformation that this matrix would do to the blue arrow would be:

$$\begin{bmatrix} 1 & 0 & 0.5 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$$

12. Null space (4 points) Which of the following vectors are in the null space of matrix $\mathbf{A}A$? (Select all that apply)

(version 1) $\mathbf{A} = \begin{bmatrix} 1 & 5 & 8 & 9 \\ 2 & 4 & 10 & 12 \end{bmatrix}$

(a) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} -4 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$

(version 2) $\mathbf{A} = \begin{bmatrix} 8 & 9 & 1 & 5 \\ 10 & 12 & 2 & 4 \end{bmatrix}$

(a) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} 8 \\ 10 \end{bmatrix}$ (d) $\begin{bmatrix} 0 \\ 1 \\ -4 \\ -1 \end{bmatrix}$

Solution: (version 1) In order to find the free variables, we must first perform Gaussian elimination:

$$\begin{bmatrix} 1 & 5 & 8 & 9 \\ 2 & 4 & 10 & 12 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 8 & 9 \\ 0 & -6 & -6 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 8 & 9 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 8 & 9 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{aligned} x_1 + 5x_2 + 8x_3 + 9x_4 &= 0 \\ x_2 + x_3 + x_4 &= 0 \end{aligned}$$

We then assign free variables $x_3 = a$, and $x_4 = b$ and substitute in:

$$\begin{aligned} x_1 &= -5x_2 - 8a - 9b = -5(-a-b) - 8a - 9b = -3a - 4b \\ x_2 &= -a - b \\ x_3 &= a \\ x_4 &= b \end{aligned}$$

We then write these equations in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -4 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the nullspace of $\mathbf{A}$ is spanned by the vectors:

$$\begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

(version 2) In order to find the free variables, we must first perform Gaussian elimination:

$$\begin{bmatrix} 8 & 9 & 1 & 5 \\ 10 & 12 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{9}{8} & \frac{1}{8} & \frac{5}{8} \\ 10 & 12 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{9}{8} & \frac{1}{8} & \frac{5}{8} \\ 0 & \frac{3}{4} & \frac{3}{4} & -\frac{9}{4} \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{9}{8} & \frac{1}{8} & \frac{5}{8} \\ 0 & 1 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 4 \\ 0 & 1 & 1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{aligned} x_1 - x_3 + 4x_4 &= 0 \\ x_2 + x_3 - 3x_4 &= 0 \end{aligned}$$

We then assign free variables $x_3 = a$, and $x_4 = b$ and substitute in:

$$\begin{aligned} x_1 &= a - 4b \\ x_2 &= -a + 3b \\ x_3 &= a \\ x_4 &= b \end{aligned}$$

We then write these equations in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -4 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the nullspace of $\mathbf{A}$ is spanned by the vectors:

$$\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

Note that

$$\begin{bmatrix} 0 \\ 1 \\ -4 \\ -1 \end{bmatrix}$$

can be found by the linear combination:

$$-4 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} -4 \\ 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -4 \\ -1 \end{bmatrix}$$

Another way to solve either version of this problem would be to try to multiply each given vector by $\mathbf{A}$ to determine whether it satisfies $\mathbf{A}\vec{x} = \vec{0}$.

13. Null space dimension (3 points)

How many basis vectors are needed to span the null space of matrix **B**?

$$\text{(version 1) } \mathbf{B} = \begin{bmatrix} 2 & 1 & 3 & 0 & 8 \\ 4 & 2 & 6 & 0 & 16 \\ 6 & 3 & 9 & 0 & 24 \\ 8 & 4 & 12 & 0 & 32 \end{bmatrix}$$

$$\text{(version 2) } \mathbf{B} = \begin{bmatrix} 2 & 1 & 4 & 0 & 8 \\ 4 & 2 & 8 & 0 & 16 \\ 6 & 3 & 12 & 0 & 24 \\ 8 & 4 & 16 & 0 & 32 \end{bmatrix}$$

Solution: We can see by inspection that all of the columns of our matrix are linearly dependent. Thus, the column space of this matrix is one dimensional. Based on the rank-nullity theorem, the dimension of the nullspace is equal to the number of columns of the matrix minus the dimension of the column space: 5 - 1 = 4. Thus, the dimension of the null space is 4. Alternatively, we can perform Gaussian elimination, which will result in the following matrix:

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix has 4 non-pivot columns, and thus a 4 dimensional null space.

14. Null space, rank and linear independence (4 points)

Suppose $\vec{u}$ and $\vec{v}$ are unique vectors ($\vec{u} \neq \vec{v}$) in the null space of a square matrix **D**, and vector $\vec{w} = \vec{u} - \vec{v}$.

For each of the following statements, choose either "True", "False", or "Not enough information".

(a) Matrix **D** is not full-rank.

(b) Vector $\vec{w}$ is in the null space of matrix **D**.

(c) Vectors =$\vec{u}$ and $\vec{v}$ are linearly independent.

Solution:

(a) Matrix **D** is not full-rank: **True**. If the nullspace of **D** has at least 2 dinstinct vectors in it, it cannot be trivial. This means that the matrix has some linearly dependent column, and thus is not full rank.

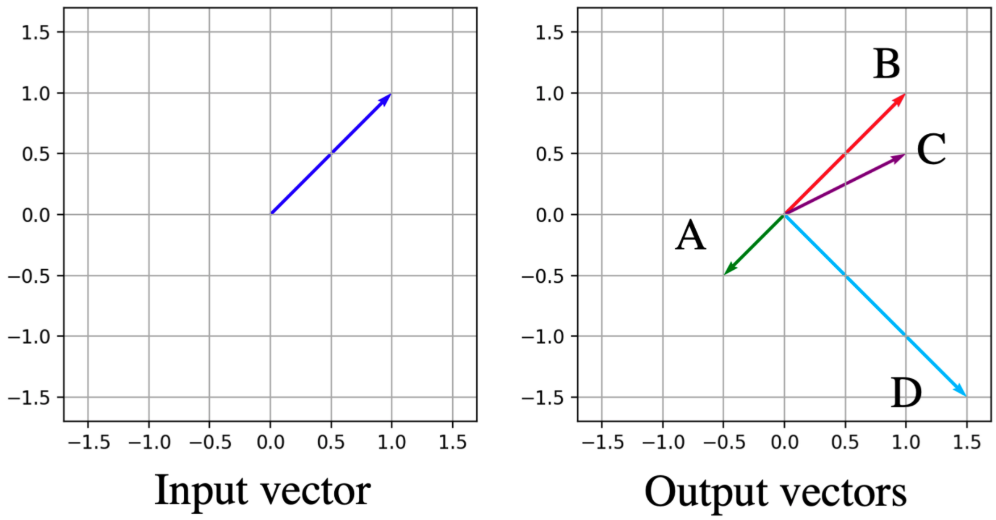
(b) Vector $\vec{w}$ is in the null space of matrix **D**: **True**. $\vec{w}$ is in the span of $\vec{u}$ and $\vec{v}$ because it is a linear combination of these two vectors. Thus, it exists in the null space of matrix **D**.

(c) Vectors $\vec{u}$ and $\vec{v}$ are linearly independent: **Not enough information**. Although $\vec{u}$ and $\vec{v}$ are distinct, one could be a scaled multiple of the other. We cannot tell based on the provided information.

15. Eigenvector. Bcourses Question 15

A 2×2 matrix $\mathbf{A}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 0.5$.

In the left figure, we have a vector $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Given $\vec{v}$ is an eigenvector of $\mathbf{A}$ and $\vec{u} = \mathbf{A}\vec{v}$, which vector(s) in the right figure is/are a possible $\vec{u}$ (Select all that apply)

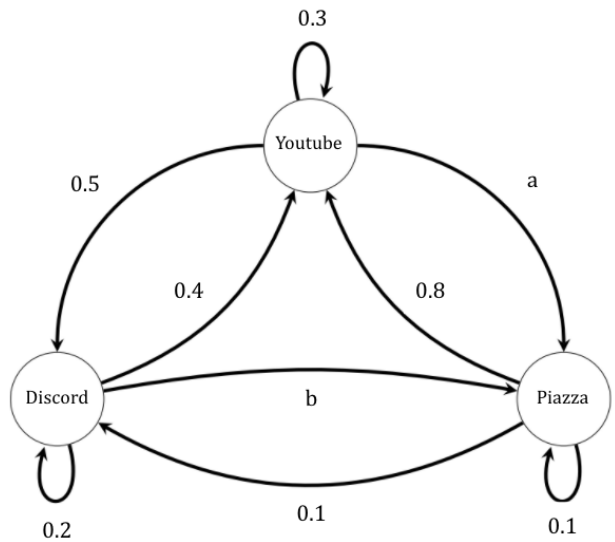


Solution: Given $\vec{v}$ is an eigenvector of $\mathbf{A}$, it can be associated with either eigenvalue $\lambda_1 = 1$ or $\lambda_2 = 0.5$. Then, we have either $\vec{u} = \mathbf{A}\vec{v} = \lambda_1\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or $\vec{u} = \mathbf{A}\vec{v} = \lambda_2\vec{v} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$. Therefore, only vector B is a possible $\vec{u}$.

16. Transition matrix Part I (7 points)

Prof. Arias decides to study the internet behavior of EECS 16A students on a typical weekday night. The number of students on the top three websites (Youtube, Piazza, and Discord) at time LaTeX: tt can be expressed as follows: $x_y[t], x_p[t], x_d$, respectively.

She finds that the flow of students across the three websites can be shown as follows:



(a) Let $\vec{x}[t] = \begin{pmatrix} x_y[t] \\ x_p[t] \\ x_d[t] \end{pmatrix}$ where $x_y[t], x_p[t], x_d[t]$ represent the number of students on Youtube, Piazza, and Discord at time t. Determine $\mathbf{A}$ such that $\vec{x}[t+1] = \mathbf{A}\vec{x}[t]$.

Solution: Reading out the diagram,

$$\mathbf{A} = \begin{bmatrix} 0.3 & 0.8 & 0.4 \\ a & 0.1 & b \\ 0.5 & 0.1 & 0.2 \end{bmatrix}$$

(b) Determine a and b such that the system is conservative.

Solution: All columns must sum to 1.

$$0.3 + a + 0.5 = 1 \implies a = 0.2$$

$$0.4 + b + 0.2 = 1 \implies b = 0.4$$

17. Transition matrix Part II (12 points)

You are given the following transition matrix $\mathbf{B}$, from another study. There are in total 1200 students.

$$\text{(version 1)}\mathbf{B} = \begin{pmatrix} 0.5 & 0 & 1 \\ 0 & 0.8 & 0 \\ 0.5 & 0.2 & 0 \end{pmatrix}$$

$$\text{(version 2)}\mathbf{B} = \begin{pmatrix} 0.5 & 0 & 1 \\ 0 & 0.4 & 0 \\ 0.5 & 0.6 & 0 \end{pmatrix}$$

(c) Assume that $\vec{x}[t+1] = \mathbf{B}\vec{x}[t]$. If $\vec{x}[2]$ is measured to be $\begin{pmatrix} 600 \\ 160 \\ 440 \end{pmatrix}$, determine the state vector at the previous timestep $\vec{x}[1]$.

Solution: To go back one timestep, we find the inverse of the matrix, and multiply by our state vector. For version 1, the inverse is equal to $\begin{bmatrix} 0 & -0.5 & 2 \\ 0 & 1.25 & 0 \\ 1 & 0.25 & -1 \end{bmatrix}$, and the final answer is $\begin{bmatrix} 800 \\ 200 \\ 200 \end{bmatrix}$.

For version 2, the inverse is equal to $\begin{bmatrix} 0 & -3 & 2 \\ 0 & 2.5 & 0 \\ 1 & 1.5 & -1 \end{bmatrix}$ and the final answer is $\begin{bmatrix} 400 \\ 400 \\ 400 \end{bmatrix}$

(d) You are given that one of the eigenvalues of $\mathbf{B}$ is $\lambda = 1$. Determine its corresponding eigenvector. Scale your solution such that the last element in the vector is 1.

Solution: Given the eigenvalue of 1, we want to solve for the vector $\vec{v}$ such that $(\mathbf{B}-\mathbf{I})\vec{v} = \vec{0}$.

For version 1, we can solve this equation with the following augmented matrix: $\left[\begin{array}{ccc|c} -0.5 & 0 & 1 & 0 \\ 0 & -0.2 & 0 & 0 \\ 0.5 & 0.2 & -1 & 0 \end{array}\right]$

Solving for the eigenvector, we get $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

For version 2, we solve the equation with the augmented matrix $\left[\begin{array}{ccc|c} -0.5 & 0 & 1 & 0 \\ 0 & -0.6 & 0 & 0 \\ 0.5 & 0.6 & -1 & 0 \end{array}\right]$. Solving for the eigenvector, we get $\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$

(e) From part (c), we know that $\vec{x}[2] = \begin{pmatrix} 600 \\ 160 \\ 440 \end{pmatrix}$. After infinite time points, what is the number of students on each website? That is, find $\vec{x}[\infty]$.

Solution: Because this is a conservative system, the number of students must be the same at every timestep. This indicates the number of students cannot explode, and all of $\mathbf{B}$'s eigenvalues must be ≤ 1. So the system will converge to the steady state. We therefore need to scale our eigenvector according to the total number of students, which is 1200. Scaled, we get

$$\begin{bmatrix} 800 \\ 0 \\ 400 \end{bmatrix}$$

18. A proof in 2 steps Part I (5 points) You are given two matrices, $A, B \in \mathbb{R}^{N \times N}$. In Question 18-19, we try to prove the following theorem in 2 steps. Please read through both questions before answering Question 18.

Theorem: If the columns of A are linearly dependent, then the columns of (AB) are also linearly dependent.

Case: 1

(1) Consider the case where [choose from below]

(a) A is not invertible

(b) A is invertible

(c) B is non-invertible

(d) B is invertible

(2) Based on [choose from below] there exists $\vec{u} \neq \vec{0}$ such that $B\vec{u} = 0$

(a) the theorem if-statement

(b) (1)

(3) $(AB)\vec{u} = A(B\vec{u}) = 0$ because of [choose from below]

(a) distributivity property

(b) associativity property

(c) 'A' having a trivial null-space

(d) 'A' having a non trivial null-space

Hence, the columns of (AB) are linearly dependent in this case.

Solution: In this proof, we know a critical fact about matrix A: the columns of A are linearly dependent. Thus, we also know that A is not invertible. With this fact, if we wish to separate the proof into 2 cases and prove each case, we know that matrix B could be either invertible or non-invertible.

To figure out if (1) should be the invertible or non-invertible case, we look at the wording of (2). The second line of the proof states that there is a non-zero $\vec{u}$ such that $B\vec{u} = 0$. This means that B has been assumed to be a linearly dependent matrix. Thus, (1) should read "B is non-invertible". (2) itself only makes sense if we assume B is non-invertible. If it read "the columns of A are linearly dependent", we wouldn't be able to make any assumptions about $B\vec{u}$.

To be able to move a pair of parenthesis around an equation without changing the result is the definition of the associativity property (of matrices in this case). Thus, (3) should be "associativity property".

19. A proof in 2 steps Part II (6 points)

Case: 2

(1) Consider the case where [choose from below]

(a) A is invertible

(b) A is non invertible

(c) B is invertible

(d) B is non invertible

(2) We [choose from below] there exists $\vec{v} \neq \vec{0}$ such that $A\vec{v} = \vec{0}$

(a) assume

(b) know

(3) Because of (1), [choose from statements a-d]

(4) Therefore, [choose from statements a-d]

Hence, the columns of (AB) are linearly dependent in this case.

Bank of statements you can use in (3) and (4):

(a) There exists $\vec{v} \neq \vec{0}$ such that $B\vec{u} = \vec{v}$

(b) There exists $\vec{u} \neq \vec{0}$ such that $B\vec{u} = \vec{v}$

(c) $(AB)\vec{u} = A(B\vec{u}) = A\vec{v} = 0$

(d) $(AB)\vec{v} = (BA)\vec{v} = B(A\vec{v}) = 0$

Solution: Because in the previous question, we used the case where B is non-invertible, it follows that in the second case, we should use (1) "B is invertible". Because A is linearly dependent, there is no need to assume there exists a non-zero $\vec{v}$ such that $A\vec{v} = \vec{0}$. We (2) "know" this to be true based on the "if" statement of the theorem.

Looking at the bank of statements we can use for (3) and (4), we should immediately be able to eliminate (d) because it is not always true. We cannot assume $(AB) = (BA)$ in general for matrices (commutativity property is not generalizable for matrices).

For (3), the only conclusion we can draw from (1) "B is invertible", is (b) "there exists $\vec{u} \neq \vec{0}$ such that $B\vec{u} = \vec{v}$". We know this because B is invertible and the only vector such that $B\vec{u} = \vec{0}$ is the zero vector itself. So as long as $\vec{u}$ is non-zero, the product $B\vec{u}$ will be non-zero.

Putting everything together (A is linearly dependent, (3), and the associativity principle), we can see that (4) should be (c) $(AB)\vec{u} = A(B\vec{u}) = A\vec{v} = 0$.