# EECS 16A Designing Information Devices and Systems I
# Spring 2018 Final Solutions

**Solutions**

PRINT your student ID: ________________________________________

PRINT AND SIGN your name: ____________________, ____________________ ________________________
(last name) (first name) (signature)

PRINT your discussion section and GSI(s) (the one you attend): ________________________________________

Name and SID of the person to your left: ________________________________________________________

Name and SID of the person to your right: ________________________________________________________

Name and SID of the person in front of you: ________________________________________________________

Name and SID of the person behind you: ________________________________________________________

1. **Which lab was your favorite lab in EE16A? (1 Point)**

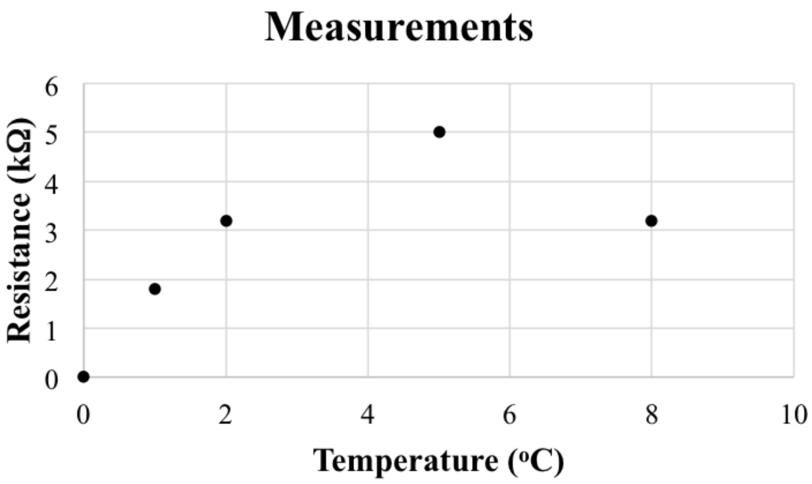
2. **What are your plans for the summer? (1 Point)**

Do not turn this page until the proctor tells you to do so. You may work on the questions above.

## 3. Temperature-Dependent Resistor (10 points)

You scored a summer research position in Professor Stojanovic's lab, where you are developing a new type of resistor with a resistance that varies as a function of its temperature $T$, $R = f(T)$. Your task is to determine an approximation for $f(T)$.

You measure the resistance by applying a test voltage and measuring the current. Then you change the temperature and repeat this procedure. Your findings can be summarized by the following scatter plot:



| Temperature (°$C$) | 0 | 1 | 2 | 5 | 8 |
|---|---|---|---|---|---|
| Resistance ($k\Omega$) | 0.0 | 1.8 | 3.2 | 5.0 | 3.2 |

You look through Professor Stojanovic's notes, and find that he believes the temperature function $f(T)$ is a polynomial that can be expressed as $R = aT^3 + bT^2 + cT + d$, where the $T$ values are temperature and $R$ values are resistance.

(a) (4 points) Set up a linear system of equations in matrix form that when solved will give the values of $a$, $b$, $c$, and $d$. *Do not solve the system of equations.*

**Solution:** To solve this with least-squares, set up the matrix equation $\mathbf{A}\vec{x} = \vec{y}$, where $\mathbf{A}$ contains the x-coordinates and $\vec{y}$ contains the y coordinates. The values $a$, $b$, $c$, and $d$ are the unknowns $\vec{x}$. We can set up the system of equations like so:

$$\begin{array}{ccccccccc}
0^3a & + & 0^2b & + & 0^1c & + & 1d & = & 0 \\
1^3a & + & 1^2b & + & 1^1c & + & 1d & = & 1.8 \\
2^3a & + & 2^2b & + & 2^1c & + & 1d & = & 3.2 \\
5^3a & + & 5^2b & + & 5^1c & + & 1d & = & 5 \\
8^3a & + & 8^2b & + & 8^1c & + & 1d & = & 3.2
\end{array}$$

Performing the exponentiations nets the following system of equations:

$$\begin{array}{ccccccccc}
0a & + & 0b & + & 0c & + & 1d & = & 0 \\
1a & + & 1b & + & 1c & + & 1d & = & 1.8 \\
8a & + & 4b & + & 2c & + & 1d & = & 3.2 \\
125a & + & 25b & + & 5c & + & 1d & = & 5 \\
512a & + & 16b & + & 8c & + & 1d & = & 3.2
\end{array}$$

The system of equations from part (a) in matrix equation form is as follows:

$$\begin{array}{ccccccccc} 0^3a & + & 0^2b & + & 0^1c & + & 1d & = & 0 \\ 1^3a & + & 1^2b & + & 1^1c & + & 1d & = & 1.8 \\ 2^3a & + & 2^2b & + & 2^1c & + & 1d & = & 3.2 \\ 5^3a & + & 5^2b & + & 5^1c & + & 1d & = & 5 \\ 8^3a & + & 8^2b & + & 8^1c & + & 1d & = & 3.2 \end{array} \rightarrow \begin{bmatrix} 0^3 & 0^2 & 0^1 & 1 \\ 1^3 & 1^2 & 1^1 & 1 \\ 2^3 & 2^2 & 2^1 & 1 \\ 5^3 & 5^2 & 5^1 & 1 \\ 8^3 & 8^2 & 8^1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1.8 \\ 3.2 \\ 5 \\ 3.2 \end{bmatrix}$$

(b) (4 points) Write the expression for the least-squares estimator for $a, b, c, d$. You can define matrices and/or vectors and use them in your solution. *You do not need to compute the values.*

**Solution:**

And from that, the Least Squares solution is:

$$\vec{x} = (\mathbf{A^T A})^{-1}\mathbf{A^T}\vec{b} = \left( \begin{bmatrix} 0^3 & 0^2 & 0^1 & 1 \\ 1^3 & 1^2 & 1^1 & 1 \\ 2^3 & 2^2 & 2^1 & 1 \\ 5^3 & 5^2 & 5^1 & 1 \\ 8^3 & 8^2 & 8^1 & 1 \end{bmatrix}^T \begin{bmatrix} 0^3 & 0^2 & 0^1 & 1 \\ 1^3 & 1^2 & 1^1 & 1 \\ 2^3 & 2^2 & 2^1 & 1 \\ 5^3 & 5^2 & 5^1 & 1 \\ 8^3 & 8^2 & 8^1 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0^3 & 0^2 & 0^1 & 1 \\ 1^3 & 1^2 & 1^1 & 1 \\ 2^3 & 2^2 & 2^1 & 1 \\ 5^3 & 5^2 & 5^1 & 1 \\ 8^3 & 8^2 & 8^1 & 1 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1.8 \\ 3.2 \\ 5 \\ 3.2 \end{bmatrix}$$

Performing the exponentiations nets the following expression:

$$\vec{x} = (\mathbf{A^T A})^{-1}\mathbf{A^T}\vec{b} = \left( \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 125 & 25 & 5 & 1 \\ 512 & 64 & 8 & 1 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 125 & 25 & 5 & 1 \\ 512 & 64 & 8 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 125 & 25 & 5 & 1 \\ 512 & 64 & 8 & 1 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1.8 \\ 3.2 \\ 5 \\ 3.2 \end{bmatrix}$$

Either of the latter two expressions are acceptable.

(c) (2 points) You solve the above expression with your handy iPython notebook and get the following values:

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ -0.2 \\ 2 \\ 0 \end{bmatrix}$$

What is the equation of the best-fit polynomial, and what kind of polynomial is it?

**Solution:**

This is **quadratic** with equation $R = 0T^3 - 0.2T^2 + 2T + 0 \rightarrow R = -0.2T^2 + 2T$

## 4. Completely Normal Eigenvectors (20 points)

(a) (6 points) Consider matrix $A$ that has eigenvectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

$$A = \begin{bmatrix} 2.5 & 0.5 & 1.5 \\ 0.5 & 2.5 & -0.5 \\ 0. & 0. & 4. \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Orthonormalize the eigenvectors using Gram-Schmidt to get vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$. Perform the orthonormalization in the order $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

**Solution:**

$$\vec{u_1}\prime = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$$

$$\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$$

$$\vec{u_2}\prime = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \frac{\langle \vec{u_1}\prime, \vec{v}_2 \rangle}{\|u_1\prime\|^2} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$\vec{u_3}\prime = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \frac{\langle \vec{u_1}\prime, \vec{v}_3 \rangle}{\|u_1\prime\|^2} - \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \frac{\langle \vec{u_2}\prime, \vec{v}_3 \rangle}{\|u_2\prime\|^2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{u}_3$$

(b) (5 points) Write the vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$ as a linear combination of the eigenvectors. Are any of $\vec{u}_1, \vec{u}_2, \vec{u}_3$ still eigenvectors of the matrix A? Justify your answer.

**Solution:** $\vec{u}_1$ and $\vec{u}_2$ are both still an eigenvectors, but $\vec{u}_3$ is not. This is because eigenvectors are only the same upto a constant multiplication, not linear combinations with other eigenvectors (in general). Expressing it as a linear combination is exactly what Gram-Schmidt does, so we just need to follow the same process as the previous part but leave terms in symbolic form.

$$\vec{u}_1 = \frac{1}{\sqrt{2}} \vec{v}_1$$

$$\vec{u}_2 = \frac{1}{\sqrt{2}} \vec{v}_2$$

$$\vec{u}_3 = \vec{v}_3 - \frac{1}{2} \vec{v}_2 - \frac{1}{2} \vec{v}_1$$

**Common Mistakes:**

- Assuming linear combinations of eigenvectors corresponding to different eigenvalues generate valid eigenvectors.
- Assuming that having a set of eigenvectors that spans R3 guarantees that the $u$ vectors will be eigenvectors.
- Correctly stating that $u_1$ and $u_2$ are still eigenvectors, but incorrectly stating that they have new, scaled eigenvalues.

(c) (3 points) Let $U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix}$. Calculate $U^T \cdot U$.

**Solution:**

$$U^T \cdot U = \begin{bmatrix} \vec{u}_1{}^T \\ \vec{u}_2{}^T \\ \vec{u}_3{}^T \end{bmatrix} \cdot \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix}$$

Note that the off-diagonal terms $\vec{u}_i{}^T \vec{u}_j = 0, i \neq j$ because we orthogonalized the vectors. Similarly the diagonal terms are just the norm squared of the vectors, which is 1 from the normalization. Thus

$$U^T \cdot U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(d) (6 points) Prove that if an arbitrary matrix $X$ has orthogonal eigenvectors then $X$ is symmetric i.e. $X^T = X$. You may assume that $X$ exists in $\mathbb{R}^{n \times n}$ and has $n$ linearly independent eigenvectors.

**Solution:** Since $X$ has the maximum number of eigenvectors, we can write it in terms of it's eigendecomposition. For simplicity assume that the eigenvectors chosen are orthonormal. We are allowed to do this because normalizing a vector is simply scaling it by a constant, and eigenvectors do not change on scaling.

$$X = V\Lambda V^{-1}$$

Since the eigenvectors chosen in $V$ are orthonormal, we know that $V^{-1} = V^T$. Substituting, we get

$$X = V\Lambda V^T$$

$$X^T = (V\Lambda V^T)^T = (V^T)^T \Lambda^T V^T$$

Since $\Lambda$ is a diagonal matrix, we know that $\Lambda^T = \Lambda$. Thus we have

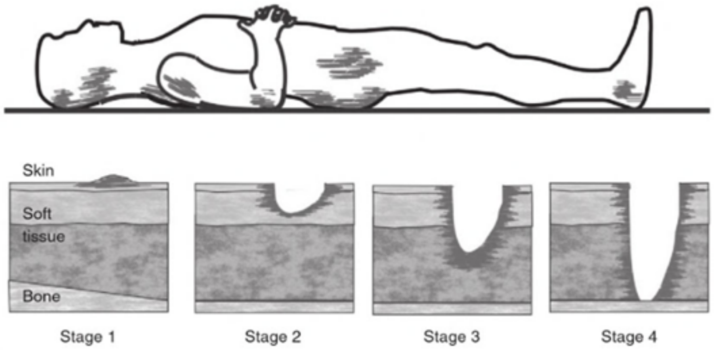
$$X^T = V\Lambda V^T = X$$

**Common Mistakes:**

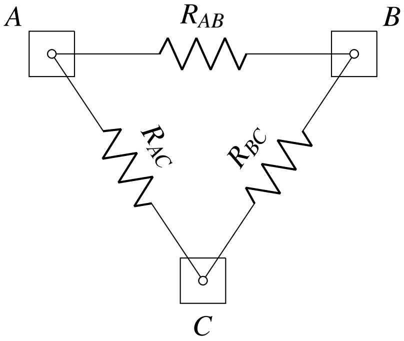
- Claiming that $X$ is an orthonormal matrix (in other words, saying $X^T X = I$). The *eigenvectors* of $X$ are orthonormal, not the columns of $X$.

5. **Smart Bandage Saves Lives (40 points)**

In the U.S. alone, there are 60,000 patient deaths per year from hospital-acquired pressure ulcers. The figure below illustrates the change in patient skin in various stages of ulcer development. Researchers in Professor Ana Arias' group in Cal's EECS department have decided to do something about it. They have developed a method to print electronics and batteries on flexible materials, and would like to get your help in designing a "smart bandage" that can detect skin deterioration and warn the hospital staff to turn the patient and treat the ulcer wounds.

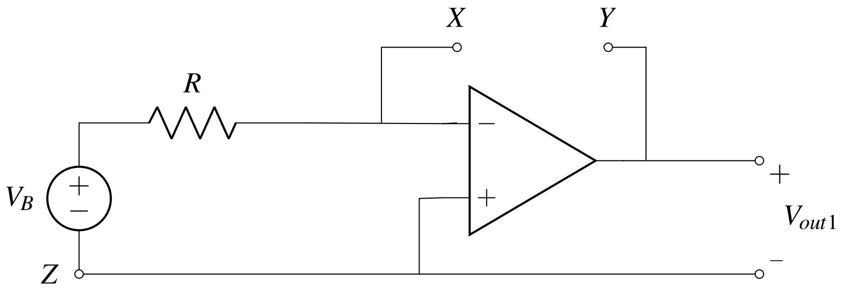


Various stages of the skin health above can be detected by measuring the skin resistance. The circuit below illustrates the skin resistance between three bandage electrodes.



The resistance between the electrodes can vary between 50Ω and 150 Ω, depending on the stage of skin health.

(a) (15 points) One of Prof. Arias' students has found the schematic of the following circuit but needs your help to figure out how to use it to turn the skin resistance measurement to voltage. $V_B$ is the voltage of battery printed on the bandage. The battery also powers up the op-amp with $\pm V_B$.

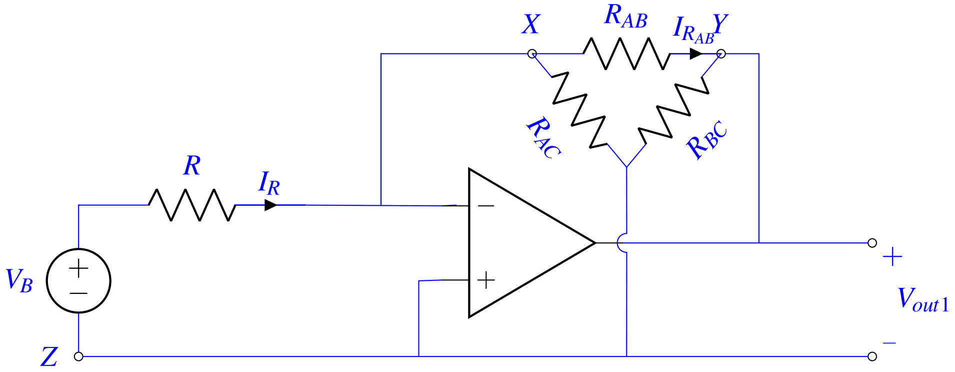


i. (3 points) You would like $V_{out1}$ to be a function of only $R$, $V_B$, and $R_{AB}$ (the resistance between electrodes $A$ and $B$).
How should you attach the bandage electrodes ($A$, $B$ or $C$) to the labeled nodes ($X$, $Y$ and $Z$) to achieve this? Each labeled node should be attached to one electrode.
**Solution:** To measure $R_{AB}$, it needs to be between nodes $X$ and $Y$. There are two possible solutions:

$X$: ______A______ $Y$: ______B______ $Z$: ______C______

$X$: ______B______ $Y$: ______A______ $Z$: ______C______

ii. (6 points) Find an expression for $V_{out1}$ as a function of circuit components above.
**Solution:**



Node $X$ is connected to the negative input to the op amp and node $Z$ is connected to the positive input. Since the op amp is in negative feedback, the voltages at nodes $X$ and $Z$ are the same. Therefore, there is no current through $R_{AC}$ in the above schematic.
This means that all of the current in $R$ must also flow through $R_{AB}$.

$$I_R = \frac{V_B}{R} = I_{R_{AB}}$$

$$V_{out1} = -R_{AB} \cdot I_{R_{AB}} = -R_{AB} I_R$$

$$V_{out1} = -\frac{R_{AB}}{R}V_B$$

iii. (6 points) Pick the value of $R$ and calculate the maximum current through the skin, such that current through the skin between any two electrodes is less than 100 $\mu$A, and $V_{out1}$ fills the range from -1.5 mV to -0.5 mV, depending on the value of skin resistance between the electrodes, as mentioned above. Assume that $V_B$ = 5 V.

**Solution:** $R_{AB}$ is in the range of 50Ω to 150Ω and we and we want $V_{out1}$ to be in the range of $-1.5mV$ to $-0.5mV$.

$$V_{out1} = -0.5mV \text{ to } -1.5mV$$
$$V_{out1} = -\frac{R_{AB}}{R}V_B \text{ to } -\frac{R_{AB}}{R}V_B$$
$$V_{out1} = -\frac{50\Omega}{R}5V \text{ to } -\frac{150\Omega}{R}5V$$
$$R = 500k\Omega$$

Now we calculate the max current through each of the skin resistors. As described above, there is no current in $R_{AB}$.

$$I_{R_{AB},max} = \frac{V_{out1}}{R_{AB}} = \frac{V_B}{R} = \frac{5V}{500k\Omega} = 10\mu A$$
$$I_{R_{BC},max} = \frac{V_{out1}}{R_{BC}} = \frac{R_{AB}}{R \cdot R_{BC}}V_B$$

$I_{R_{BC}}$ will be largest when $R_{AB}$ is largest and $R_{BC}$ is smallest.

$$I_{R_{BC},max} = \frac{150\Omega}{50\Omega \cdot 500k\Omega}5V = 30\mu A$$

In summary:

$R =$ ______ $500k\Omega$ ______ $I_{\text{skin, max}} =$ ______ $30\mu A$ ______

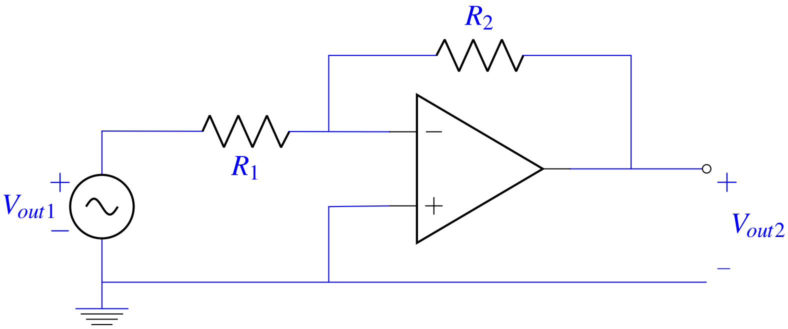
(b) (10 points) Since $V_{out1}$ is a small voltage in the range of -1.5mV to -0.5mV, we need to design an amplifying stage that will amplify $V_{out1}$ into $V_{out2}$ in the range of 500mV to 1.5V. For this you can use

- **one** op amp
- **two** resistors

You do not need to specify power supplies on the op amp.

i. (5 points) Draw your circuit below, clearly labeling the circuit components and circuit nodes. Derive an expression for $V_{out2}$ as a function of $V_{out1}$ and circuit component values.

**Solution:** We want the output to be amplified and negated, so we design an inverting amplifier.



$$V_{out2} = -\frac{R_2}{R_1} V_{out1}$$

The signal needs to be amplified by 1000, so $R_2$ must be 1000 times $R_1$.

ii. (5 points) To prevent skin damage, you need to pick the resistor values such that the two resistors each dissipate less than 1$\mu$W of power. Modify your resistor values if necessary to ensure that the patient is not burned. Show your calculations.

**Solution:** Let's first consider the power dissipated in $R_1$:

$$P = \frac{(V_{out1,max})^2}{R_1} < 1\mu W$$

$$P_{max} = \frac{(1.5mV)^2}{R_1} < 1\mu W$$

$$\frac{(1.5mV)^2}{1\mu W} < R_1$$

$$R_1 > 2.25\Omega$$

Now let's look at the power dissipated in $R_2$:

$$P = \frac{(V_{out2,max})^2}{R_2} < 1\mu W$$

$$P_{max} = \frac{(1.5V)^2}{R_2} < 1\mu W$$

$$\frac{(1.5V)^2}{1\mu W} < R_2$$

$$R_2 > 2.25M\Omega$$

The constraint on $R_2$ is more restrictive. The smallest pair of resistor values that meets the specifications is $R_1 = 2.25k\Omega$ and $R_2 = 2.25M\Omega$.
Larger pairs (where $R_2$ is 1000 times $R_1$) are also valid.

(c) (15 points) Finally, we'd like to make the bandage "smart". Students in Prof. Arias' group have figured out how to print red light emitting diodes (LEDs) on the bandage, too, and would like to turn it on when the monitored skin resistance is larger than that of reference "healthy" skin. In the figure below, $V_w$ represents the output signal from the wound sensor that we designed in Part B, and $V_h$ represents the output signal from the reference sensor placed over the healthy skin.

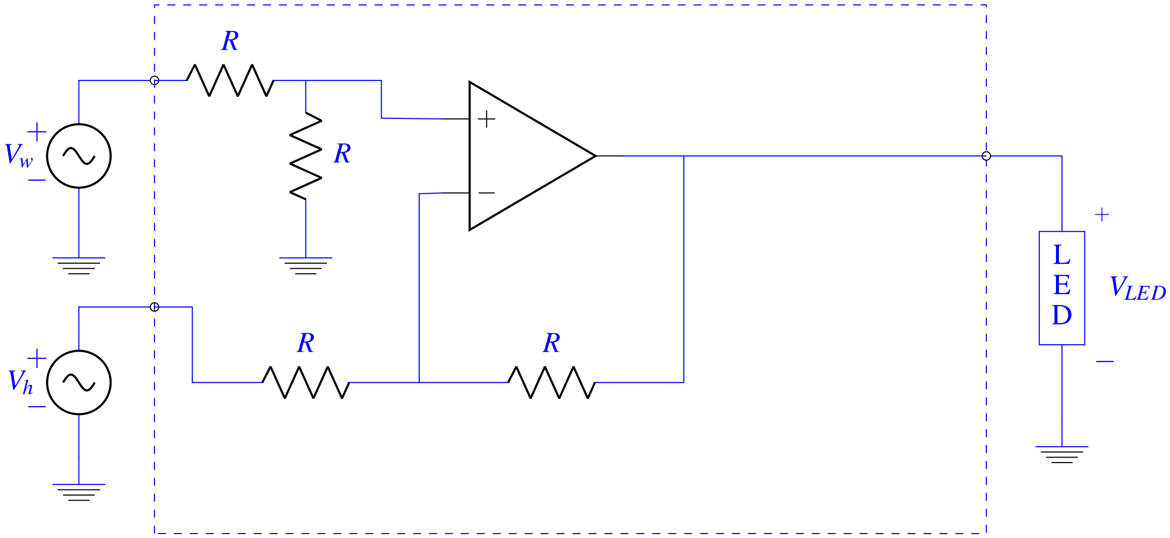
Design a circuit that provides positive voltage $V_{LED}$ to the LED diode whenever $V_w > V_h$. We'd like to make the LED shine brighter (larger $V_{LED}$) when the difference between the two voltages is larger. The LED does not shine when $V_{LED} < 0$.

For this part, you can use

- **one** op amp
- **four identical** resistors

i. (7 points) Draw your circuit in the box in the schematic provided below. You do not need to specify power supplies on the op amp.

**Solution:**



ii. (8 points) Find the expression for $V_{LED}$ as a function of $V_w$ and $V_h$.

**Solution:** We can find $V_{LED}$ using superposition. When only $V_h$ is on, this is a inverting amplifier:

$$V_{LED,h} = -\frac{R}{R}V_h = -V_h$$

When only $V_w$ is on, we use the voltage divider equation to calculate $V_+$.

$$V_+ = \frac{R}{R+R}V_w$$

The rest of the circuit is a non-inverting op amp.

$$V_{LED,w} = (1+\frac{R}{R})V_+$$

$$V_{LED,w} = (1+\frac{R}{R})\frac{R}{R+R}V_w = V_w$$

Finally we add $V_{LED,h}$ and $V_{LED,w}$:

$$V_{LED} = V_w - V_h$$

**Common Mistakes:**

- Changing the rails of the opamps to use $V_h$ or $V_w$ was not acceptable, the problem explicitly mentions not drawing the rails.
- If you want partial credit for calculation you HAVE to show work. An incorrect answer with no explanation can get no additional partial credit, if any has been given. Please do not submit regrade requests for this.
- Comparator: Just saying the voltage is positive or negative gets no points, you need to indicate either that the op-amp will rail, or write an expression for the output voltage of an op-amp to get partial credit.
- Incorrect polarity: This could mean either incorrect based on diagram (wrote $V_w - V_h$ when your diagram creates $V_h - V_w$) or correct diagram but wrote $V_h - V_w$.
- Inverting $V_h$ and then adding it to $V_w$ using a voltage summer (without an op-amp) doesn't work because the LED current affects this voltage.
- A lot of errors were off by a factor of 2 or $1/2$, please check your work before submitting a regrade request.

## 6. Track Timer (40 points)

Jewanna Befast is an up and coming track star that has asked you to design a smart stopwatch to tell her exactly how fast she runs around the track. As an enthusiastic 16A student, you immediately agree.

Your idea is to have Jewanna start the stopwatch by stepping on a pressure sensor, time her as she runs along the track, and stop the timer when she steps on the pressure sensor again.

(a) (10 points) In order to start and stop your timer, you decide to use a resistive pressure sensor and edge-triggered switch. An edge-triggered switch has the following properties:

- If the switch is initially open, it will close when it sees a voltage change from 0.5 V to 2 V.
- If the switch is initially closed, it will open when it sees a voltage change from 0.5 V to 2 V.

You want to design a circuit to control the switch. When Jewanna steps off the sensor, you want to output a change in voltage from 0.5 V to 2 V to open/close the switch, which will correspondingly start/stop the stopwatch.

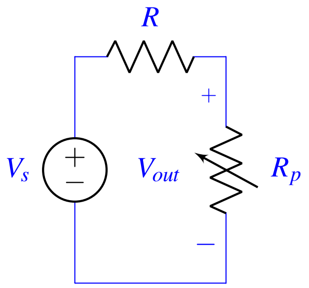
Suppose you have read the specs of your pressure sensor and calculated that any time Jewanna steps on the sensor, the resistance is 1 kΩ, and any time she is not on it, the resistance is 10 kΩ. We model the pressure sensor as a variable resistor with resistance $R_p$, shown below.

Design a circuit that:

- outputs 0.5 V when Jewanna is on the pressure sensor
- outputs 2 V when Jewanna is not on the sensor

**You are only allowed to use resistors and voltage sources.** Clearly label where your output voltage, $V_{out}$, is, and label the values of all resistors and voltage sources.

**Solution:** Use a voltage divider to solve this circuit. Specifically:



We then formulate two equations, corresponding to when Jewanna is stepping and not stepping on the sensor:

$$V_{0.5} = \frac{R_{p1}}{(R_{p1}+R)}V_s$$

$$V_2 = \frac{R_{p2}}{(R_{p2}+R)}V_s$$

Which then becomes:

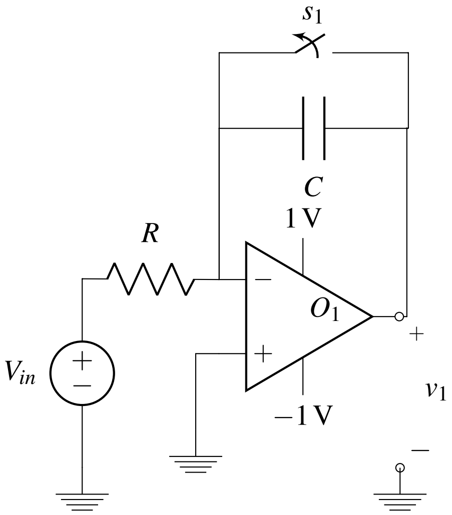
$$0.5 = \frac{1}{(1+R)}V_s$$

$$2 = \frac{10}{(10+R)}V_s$$

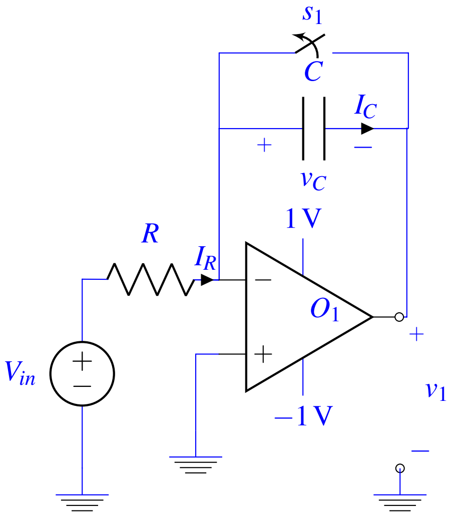
Solving leads to $R = 5\,\text{k}\Omega$ and $V_s = 3\,\text{V}$

(b) (10 points) You would like to measure how much time it takes for Jewanna to run the track. You start with the RC circuit below where $s_1$ is the edge-triggered switch from part A. Assume the switch is closed until time $t = 0$, after which it opens because Jewanna starts running.

Find an expression of $v_1$ with respect to $V_{in}$, $R$, $C$ and time $t$. Assume that the capacitor is discharged at time $t = 0$ and that $V_{in}$ is a constant voltage. You must show all work to receive credit.



**Solution:**



First find the currents through the resistor and capacitor, which are the same:

$$i_R = i_C = \frac{V_{in} - 0}{R} = \frac{V_{in}}{R}$$

Derive the expression for voltage across the capacitor:

$$i_c = C\frac{dv_C}{dt}$$

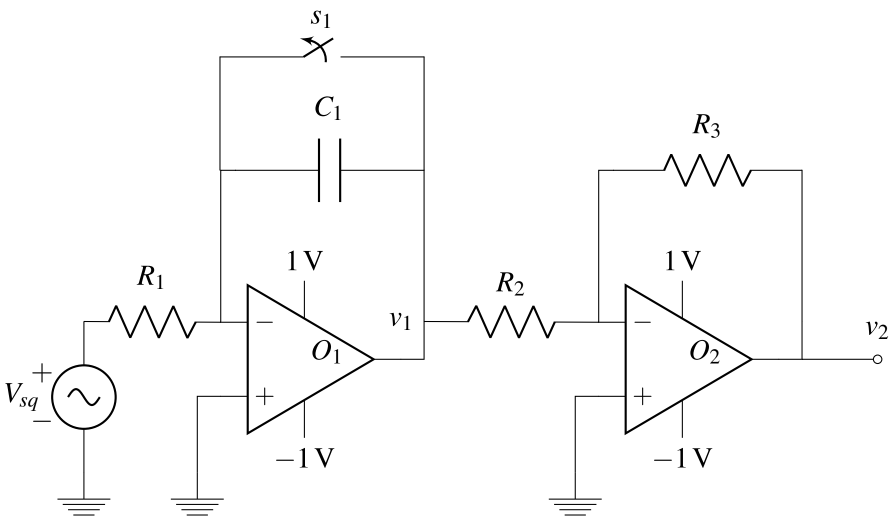
Integrating, and taking into account the capacitor being initially discharged leads to:

$$v_C(t) = v_c(0) + \frac{V_{in}}{RC}t = \frac{V_{in}}{RC}t$$

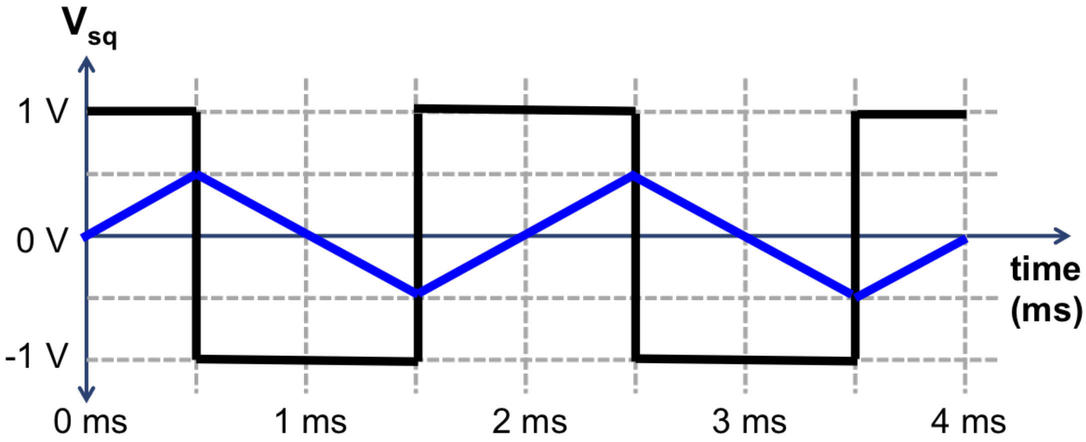
We know that the the voltage across $C$ is the same as the negative of the voltage across the node $v_1$. The voltage at the output node $v_1$ is then

$$v_1 = -v_C = -\frac{V_{in}}{RC}t$$

(c) (8 points) Now consider the following circuit. The voltage source $V_{sq}$ outputs a square wave, which is plotted below. On the same plot, draw the output voltage $v_2$ as a function of time. Assume that $C_1 = 1\mu F$ and is initially discharged, and all resistors have resistance of $1k\Omega$. Show your work in the box below.



**Solution:**



First we notice that the output $v_2$ is connected to $v_1$ by an inverting amplifier, so $v_2 = -v_1$. From Part B, we know that the output will increase linearly over time when a constant voltage is applied:

$$v_2 = \frac{V_{in}}{RC}t$$

$$RC = 1k\Omega \cdot 1\mu F = 1ms$$

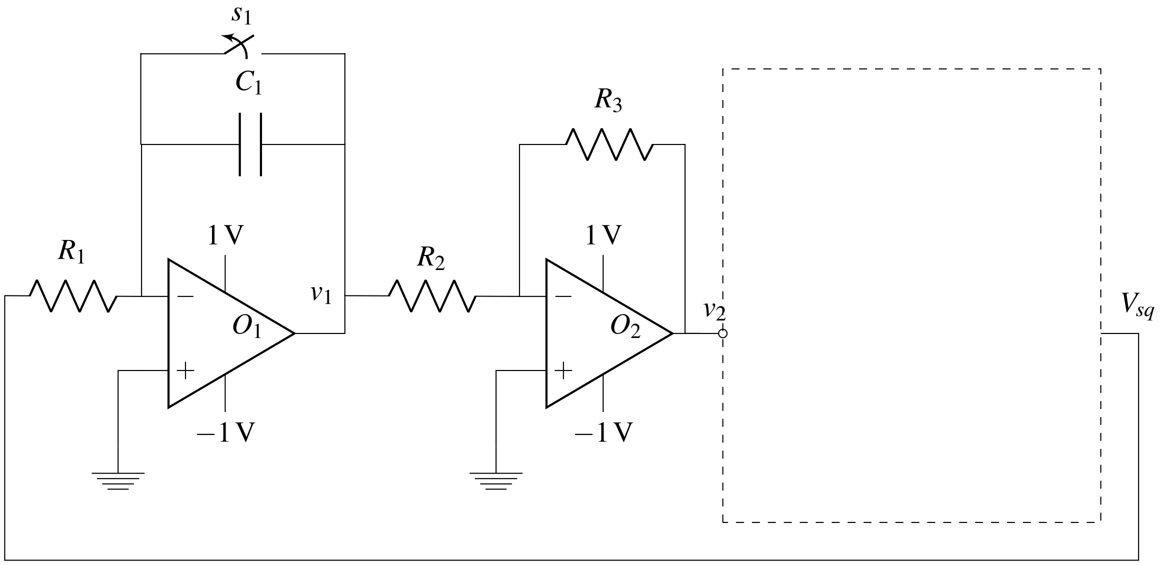
If 1V is applied for $0.5ms$, as is the case with the first part of the square wave, then $v_2$ will linearly rise to $0.5V$ by the end of $0.5ms$.

When the square wave switches from $1V$ to $-1V$, the voltage starts to linearly decrease at the same rate. Note that the capacitor stays charged when this happens, so there are no discontinuities in $v_2$.

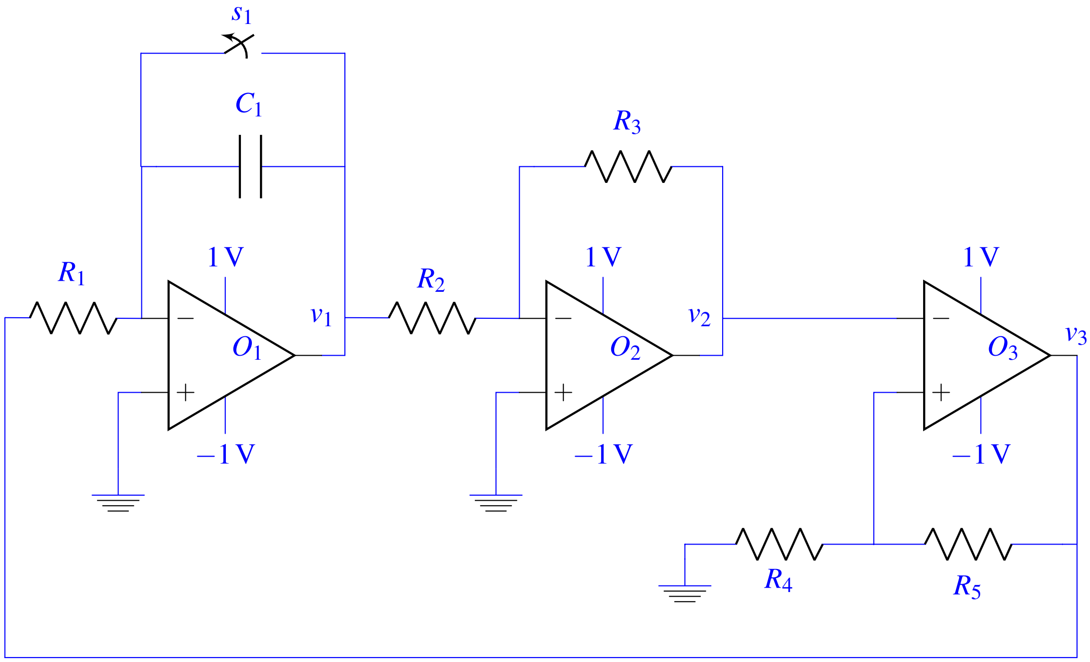
(d) (10 points) Design a circuit that will convert $v_2$ into the square wave, $V_{sq}$ shown in the previous part. You can use the following components only:

- **one** op amp (assume $\pm 1V$ power supplies)
- **two identical** resistors

Draw your circuit in the dotted box and show any work below.



**Solution:** This is very similar to the timer circuit from discussion.



We would like to convert a changing signal ($v_2$) into a constant signal (the square wave), so we use a comparator. When our input square wave is high, we want the output to be high, and we want it to switch from high to low when $v_2$ crosses $0.5V$.

When our input square wave is low, we want our output to be low, and we want it to switch from low to high when $v_2$ crosses $-0.5V$.

We can do this by connecting the reference voltage of our op amp to the output $V_{sq}$. We set the reference voltage to be half of $V_{sq}$ by designing a voltage divider that halves the voltage using our identical resistors. This means that the absolute value of the threshold for flipping the comparator is at 0.5 V.

For more details on how this timer circuit works, see Discussion 10B.

(e) (2 points) Suppose you have a device that counts the number of pulses of your square wave (ie. you get one count per period of the square wave). How would you use this information to get the total time it takes for Jewanna to run around the track?

**Solution:** Multiply the number of pulses with the period of the square wave (2 ms).

**7. Cactus Care (30 points)**

On Midterm 2 you designed a light sensor to check that there is sufficient light in your room for your cactus to be happy and healthy. But you want to monitor the light levels over the course of the day, when you aren't around. You design a transmitter that sends the following periodic code of length $N = 5$:

$$\vec{c} = \begin{bmatrix} 1 & -3 & 2 & 1 & 2 \end{bmatrix}^T$$

You encode information about the light by multiplying the code with the light intensity ($y$). With your cell phone, you receive a shifted version of the code (since it had to travel an unknown distance), multiplied by the light intensity.

(a) (4 points) Write a matrix $A$ such that

$$A\vec{y} = \vec{r}$$

where $\vec{r}$ is the received signal (length 5) and $\vec{y}$ is a vector of all zeros except one entry which contains the light intensity $y$. (*Hint: The position of $y$ in the vector $\vec{y}$ will depend on the unknown shift in the signal.*)

**Solution:** $A$ is a circulant matrix containing all of the possible shifts of $\vec{c}$ in it's columns:

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 & -3 \\ -3 & 1 & 2 & 1 & 2 \\ 2 & -3 & 1 & 2 & 1 \\ 1 & 2 & -3 & 1 & 2 \\ 2 & 1 & 2 & -3 & 1 \end{bmatrix}$$

You could also write this with the shift notation we used in class, where $\vec{c}^{(k)}$ is $\vec{c}$ circularly shifted by $k$.

$$A = \begin{bmatrix} \vec{c}^{(0)} & \vec{c}^{(1)} & \vec{c}^{(2)} & \vec{c}^{(3)} & \vec{c}^{(4)} \end{bmatrix}$$

(b) (6 points) This semester your learned several techniques for solving linear systems of equations. For each of the following techniques, could you use it to solve the matrix equation from Part A? Justify your answer in 1-2 sentences. Assume there is no noise.

**Gaussian Elimination** ○ yes ○ no

**Solution:** **Yes.**

$A$ is a square matrix with linearly independent rows, so we can use Gaussian elimination to solve for $\vec{y}$.

**Least Squares** ○ yes ○ no

**Solution:** **Yes.**

Least squares can be used to solve when there are at least as many rows as columns. In this case, where $A$ is square and invertible, the least squares solution is the same as the one you would get with Gaussian elimination.

**Orthogonal Matching Pursuit** ○ yes ○ no

**Solution: Yes.**

Orthogonal Matching Pursuit can be used for solving for vectors that are mostly zero (sparse vectors). Since $\vec{y}$ contains all zeros except one element, it is sparse and we can use OMP.

Alternative justification: OMP can be used to extract the messages from a small number of beacons sending periodic signals, which is what is happening in this problem.

You're not sure that your room is really the right place for your cactus, so you set up another light detector in the lab to see if it's better. Each of the two light detectors has a transmitter with a different periodic code $c_1, c_2$.

$$\vec{c_1} = \begin{bmatrix} 1 & -3 & 2 & 1 & 2 \end{bmatrix}^T$$

$$\vec{c_2} = \begin{bmatrix} 3 & 1 & 2 & -2 & -1 \end{bmatrix}^T$$

As before, the codes are multiplied by the light intensities at each location, $y_1$ and $y_2$, and your cell phone receives the sum of shifted codes, each weighted by the light at that location.

(c) (5 points) Write a new matrix $A$ such that

$$A\vec{y} = \vec{r}$$

where $\vec{r}$ is the received signal (length 5) and $\vec{y}$ is a vector of all zeros except two entries which contain $y_1$ and $y_2$.

*Hint: The positions of $y_1$ and $y_2$ in the vector $\vec{y}$ will depend on the unknown shifts in $c_1$ and $c_2$, respectively.*

**Solution:** $A$ contains all of the possible shifts of each code in it's columns.

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 & -3 & 3 & -1 & -2 & 2 & 1 \\ -3 & 1 & 2 & 1 & 2 & 1 & 3 & -1 & -2 & 2 \\ 2 & -3 & 1 & 2 & 1 & 2 & 1 & 3 & -1 & -2 \\ 1 & 2 & -3 & 1 & 2 & -2 & 2 & 1 & 3 & -1 \\ 2 & 1 & 2 & -3 & 1 & -1 & -2 & 2 & 1 & 3 \end{bmatrix}$$

You could also write this with the shift notation we used in class, where $\vec{c}^{(k)}$ is $\vec{c}$ circularly shifted by $k$.

$$A = \begin{bmatrix} \vec{c}_1^{(0)} & \vec{c}_1^{(1)} & \vec{c}_1^{(2)} & \vec{c}_1^{(3)} & \vec{c}_1^{(4)} & \vec{c}_2^{(0)} & \vec{c}_2^{(1)} & \vec{c}_2^{(2)} & \vec{c}_2^{(3)} & \vec{c}_2^{(4)} \end{bmatrix}$$

(d) (6 points) For each of the following techniques, could you use it to solve the matrix equation from Part D, with two different light sensors? Justify your answer in 1-2 sentences. Assume there is no noise.

**Gaussian Elimination** ○ yes ○ no

**Solution: No.**

There are more columns than rows in $A$ so if we attempt Gaussian Elimination, there will not be a pivot in every column. This means there are infinitely many possible solutions.

**Least Squares** ○ yes ○ no

**Solution: No.**

Least squares can only be used to solve a system of equations when there are at least as many rows as columns. In this case there are fewer rows than columns, so we cannot use least squares.

To see this, let's look at the least squares equation:

$$\hat{\vec{y}} = (A^T A)^{-1} A^T \vec{r}$$

If $A$ has more columns than rows, $A^T A$ cannot be full rank, so it is not invertible.

**Common Mistakes:**

- Many students said that least squares only works for OVER-determined systems, however that's not true because it works for perfectly determined systems as well.

**Orthogonal Matching Pursuit** ○ yes ○ no

**Solution: Yes.**

Orthogonal Matching Pursuit can be used for solving for vectors that are mostly zeros, even when the system of equations is underdetermined. Since $\vec{y}$ contains all zeros except two elements, it is sparse and we can use OMP. Alternative justification: OMP can be used to extract the messages from a small number of beacons sending periodic signals, which is what is happening in this problem.

(e) (3 points) In order to judge if your codes are "good", you want to calculate the autocorrelations and cross-correlation of your codes. Professor Waller helps you calculate the following:

$$\text{autocorrelation of } \vec{c}_1 = \begin{bmatrix}19 & -3 & \textbf{??} & -2 & -3\end{bmatrix}^T$$

$$\text{autocorrelation of } \vec{c}_2 = \begin{bmatrix}19 & 0 & -5 & -5 & 0\end{bmatrix}^T$$

$$\text{cross-correlaton of } \vec{c}_1 \text{ with } \vec{c}_2 = \begin{bmatrix}0 & -10 & 12 & 11 & -4\end{bmatrix}^T$$

Finish the set by calculating the unknown term in the autocorrelation of $c_1$.

**Solution:**

The missing term of the autocorrelation is the inner product of the code with a version of itself, circularly shifted by 2.

$$\vec{c}_1 = \begin{bmatrix}1 & -3 & 2 & 1 & 2\end{bmatrix}^T$$

$$\text{autocorr. at lag 2 } = (1)(2) + (-3)(1) + (2)(2) + (1)(1) + (2)(-3) = -2$$

(f) (6 points) Consider the following set of codes ($c_3$ and $c_4$).

$$\vec{c}_3 = \begin{bmatrix}1 & -2 & -3 & 2 & 1\end{bmatrix}^T \qquad \vec{c}_4 = \begin{bmatrix}1 & 1 & 2 & -2 & -3\end{bmatrix}^T$$

$$\text{autocorr. of } \vec{c}_3 = \begin{bmatrix}19 & 1 & -10 & -10 & 1\end{bmatrix}^T$$

$$\text{autocorr. of } \vec{c}_4 = \begin{bmatrix}19 & 2 & -11 & -11 & 2\end{bmatrix}^T$$

$$\text{cross-correlation of } \vec{c}_3 \text{ with } \vec{c}_4 = \begin{bmatrix}-14 & -16 & 5 & 18 & -2\end{bmatrix}^T$$

If you use OMP to solve for the light intensities, which set of codes ($c_1$,$c_2$ OR $c_3$,$c_4$) is more robust to noise in the received signal? Justify your answer. For the set of codes that is worse, what mistake will is most likely to happen during the OMP algorithm in the presence of noise?

**Solution:** $\mathbf{c_1}$ **and** $\mathbf{c_2}$ are more robust to noise. This is because they are "more orthogonal" than $c_3$ and $c_4$ for all possible shifts, ie. the autocorrelation (at non-zero shift) and cross-correlations of $c_1, c_2$ are generally closer to zero.

Specifically, the cross-correlation of $c_3, c_4$ has a peak of magnitude 18, which is almost as high as the autocorrelation at zero shift. This means that if we try to use OMP with $c_3, c_4$ we are likely to accidentally mistake $c_3$ for $c_4$ at a different shift (or vice versa).

**Common Mistakes:**

- Saying that $c_1$,$c_2$ are more robust because they are orthogonal at time shift zero (first term of the cross-correlation is 0). We need to check that the codes are nearly orthogonal for all time shifts, not just 0. This did not get credit because you could have a pair of codes that are orthogonal at lag 0 but have a very high cross correlation at a different lag. Additional explanation is needed to get credit for your justification.
- Saying that $c_1$,$c_2$ are more robust because they have autocorrelations that are more different from each other than those of $c_3$,$c_4$. In OMP we know which code we cross-correlated with the received signal, so we don't need the autocorrelations to be different. We just want a high peak at 0 lag and low every where else.
- Thinking that the cross-correlation given is the cross-correlation which is calculated during OMP, in other words, the cross-correlation that we are trying to find peaks in. This is not the case, it is the cross-correlation between the two codes, which does not include information from the received signal.

## 8. APS Lab by Hand (38 points)

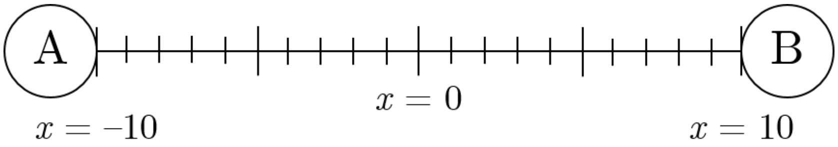
In the APS labs, you wrote code for cross-correlation functions that separated raw recorded signals from known beacon signals. These beacon signals were very large–upwards of 10,000 samples long. In this problem, you will walk through a similar process as the APS lab except your beacon signals are much smaller and much much slower.

For the subsequent parts, the beacon signals are comprised of binary numbers. Each element can be treated as a unit of time, $t$, in seconds with the signals traveling at a velocity $v = 1$ meter per second (m/s) away from beacons located some distance $d$ away from our microphone in meters. Friendly reminder: $v = d/t$.

First, we will look at a 1D system where we are only trying to find our $x$-coordinate based on signals from two beacons: Beacon $A$ and Beacon $B$. The signals coming from each beacon, which are **periodic** and have a length of $N = 7$, are represented below:

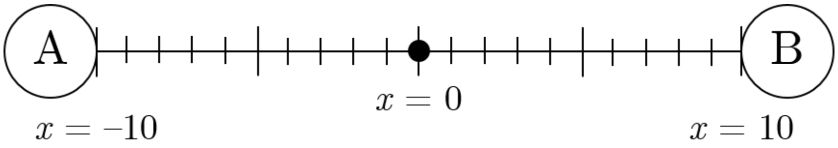
$$\vec{A} = \begin{bmatrix}1 & 0 & 1 & 0 & 0 & 0 & 0\end{bmatrix}$$
$$\vec{B} = \begin{bmatrix}1 & 0 & 0 & 1 & 0 & 0 & 0\end{bmatrix}$$

Our 1D system is only the $x$-axis, with Beacon $A$ and Beacon $B$ at opposite ends, as shown below:



For all parts of this problem, assume that Beacon $A$ and Beacon $B$ both start sending their signals at the same time, but we don't know exactly what time they started transmitting. Therefore, we can only know *relative* information about our location compared to $A$ and $B$.

(a) (4 points) What is a **raw signal** ($\vec{raw}$) the microphone could record at the position $x = 0$? The 1D system with the microphone position at $x = 0$ ($d_A = 10$ m, $d_B = 10$ m) is shown below:



**Solution:** Since both beacons start transmitting at the same time and the microphone is equal distance between them, the raw signal is a sum of the two beacon signals:

$$\vec{raw} = \vec{A} + \vec{B} = \begin{bmatrix}2 & 0 & 1 & 1 & 0 & 0 & 0\end{bmatrix}$$

Circularly shifted versions of the above signal are also correct.

(b) (8 points) Suppose you have a raw signal recorded by the microphone as follows:

$$\overrightarrow{raw} = \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

What is the smallest time delay between when the signals arrive? That is, what is their smallest time difference of arrival (TDOA)? Reminder, the signals are traveling at 1 m/s over a distance in meters, and each element in a signal array corresponds to some time $t$ related by $v = d/t$.

**Solution:** Cross-correlating the raw signal with the two beacon signals (keeping raw signal stationary and rolling reference beacons to the right) gives the following:

$$\overrightarrow{raw}CCwith\vec{A} = \begin{bmatrix} 2 & 1 & 0 & 2 & 0 & 3 & 0 \end{bmatrix}$$
$$\overrightarrow{raw}CCwith\vec{B} = \begin{bmatrix} 3 & 0 & 1 & 2 & 1 & 1 & 0 \end{bmatrix}$$

Looking at where the cross-correlations are the maximum, $\vec{A}$ peaks at $n = 5$ or $n = -2$ and $\vec{B}$ peaks at $n = 0$, meaning that signal A is either lagging by five time steps $(5-0)$ or leading by two time steps $(-2-0)$. Since we want the smallest time delay, we should go with signal A leading by two time steps.

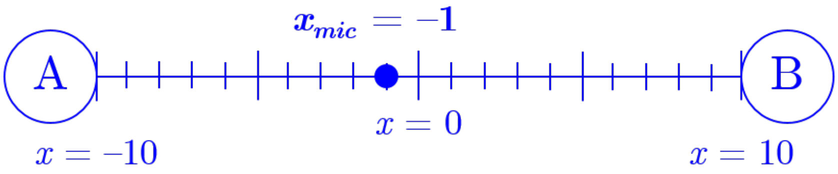
**Common Mistakes:**

- When calculating correlation, a common mistake was putting the vectors in the circulant matrix as columns instead of rows.

(c) (2 points) Based on your answer from Part B, where is the microphone located in our 1D system? Please mark the location on the axis below.

*Hint: Don't forget, if you move closer to Beacon A, you're moving farther away from Beacon B.*

**Solution:** Our smallest time delay was $t = 2$ when the cross-correlation of $\vec{A}$ with $\overrightarrow{raw}$ peaks at $n = -2$, meaning Beacon $A$ arrived first 2 time steps ahead of Beacon $B$. This means the difference between the distance from each beacon is $d_B - d_A = (1m/s)(2sec) = 2m = 11m - 9m$, which corresponds to $x = -1$ on our axis.



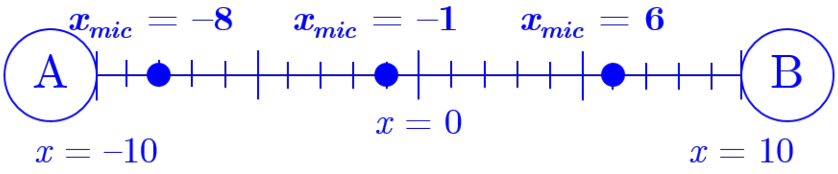
(d) (10 points) In part (b), we asked for the smallest time delay, which is implied to be less than the length of our signal, $N = 7$, since our signals are periodic. (If you got a delay > 7 time steps in part (b), you may want to double check your work!) Now, suppose we consider all time delays in which the microphone is located at an **integer number** on the $x$-axis. Given the same raw signal below:

$$\overrightarrow{raw} = \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

please identify all **integer** solutions of where the microphone is in the 1D system, $x_{mic}$. Show any relevant work, and **please mark any found solutions on the axis**. Reminder, the signals are traveling at 1 m/s over a distance in meters, and each element in a signal array corresponds to some time $t$ related by $v = d/t$.

*Hint: Again, don't forget, if you move closer to Beacon A, you're moving farther away from Beacon B.*

**Solution:** We know that the period of the signal is $N = 7$, so moving $7/2 = 3.5$ m in either direction (adding 3.5 to $d_A$ and subtracting 3.5 to $d_B$, and vice-versa) would result in all of the possible positions of the given raw signal. However, since we only want integer solutions, we simply move in increments of 7 m in each direction to find our possible integer solutions:



More explicitly, if we define the number of periods a signal could be shifted as $k_{period}$:

$$\begin{aligned}
\text{Beacon A leading: } -2 - 7k_{period} \longrightarrow k_{period} = 0, \text{delay} &= -2, x_{mic} = -1 \\
k_{period} = 1, \text{delay} &= -9, x_{mic} = -4.5 \\
k_{period} = 2, \text{delay} &= -16, x_{mic} = -8 \\
\text{Beacon A lagging: } -2 + 7k_{period} \longrightarrow k_{period} = 1, \text{delay} &= 5, x_{mic} = 2.5 \\
k_{period} = 2, \text{delay} &= 12, x_{mic} = 6 \\
k_{period} = 3, \text{delay} &= 19, x_{mic} = 9.5
\end{aligned}$$

(e) (10 points) Uh-oh! It turns out we forgot to look up and are actually in 2D. My bad. Given the same raw signal from part (b):

$$\vec{raw} = \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

assuming the signals are offset by the smallest time difference (as in part (b)), please write an expression for **all possible solutions** of where the microphone is in the **2D system** in terms of $x_{mic}, y_{mic}$. Show any relevant work. Some useful terms might be the coordinates of the beacons ($x_A = -10$, $y_A = 0$) and ($x_B = 10$, $y_B = 0$) and the distance of the microphone from the beacons $d_A$ and $d_B$.

(*Hint 1: the same raw signal means the same delay in signal arrival.*)
(*Hint 2: think about APS 2, specifically.*)
(*Hint 3: as a final sanity check, check if your solution matches the 1D solution when* $y_{mic} = 0$.)

**Solution:** Since we are now in 2D, our distance from each beacon should be represented as a circle:

$$\text{Beacon A: } \sqrt{(x_{mic} - x_A)^2 + (y_{mic} - y_A)^2} = d_A$$

$$\text{Beacon B: } \sqrt{(x_{mic} - x_B)^2 + (y_{mic} - y_B)^2} = d_B$$

Assuming the same 2 time step interval is the smallest, that means $d_B = d_A + 2$, making our equations:

$$\text{Beacon A: } \sqrt{(x_{mic} - x_A)^2 + (y_{mic} - y_A)^2} = d_A$$

$$\text{Beacon B: } \sqrt{(x_{mic}-x_B)^2+(y_{mic}-y_B)^2} = d_A+2$$

Subtracting our equation for Beacon $A$ from Beacon $B$ gives us:

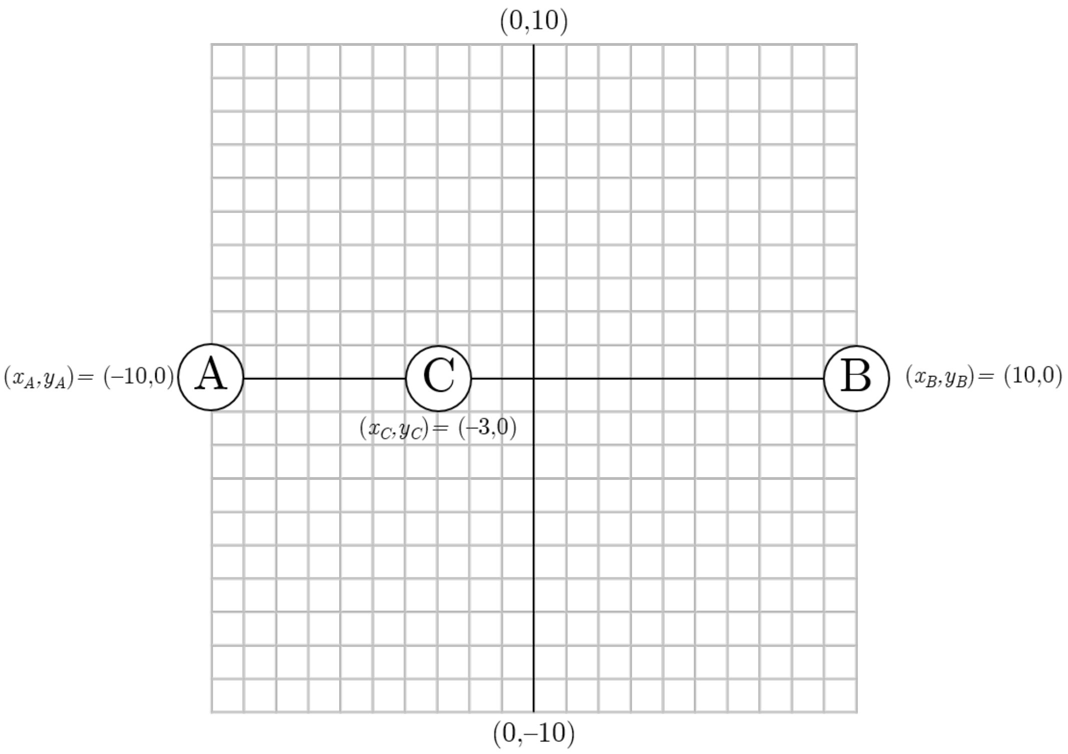
$$\sqrt{(x_{mic}-x_B)^2+(y_{mic}-y_B)^2} - \sqrt{(x_{mic}-x_A)^2+(y_{mic}-y_A)^2} = 2$$

Which you may recall from APS 2 is the equation for a hyperbola. As a final sanity check, plugging in $(x_{mic}=-1, y_{mic}=0)$ and our beacon position values gives us:

$$\begin{aligned} \sqrt{(-1-10)^2+(0-0)^2} - \sqrt{(-1+10)^2+(0-0)^2} &= 2 \\ \sqrt{(-11)^2} - \sqrt{(9)^2} &= 2 \\ 11-9 &\overset{\checkmark}{=} 2 \end{aligned}$$

(f) (4 points) To narrow down where we are in the 2D space, Thomas the Lab TA decides to add a third beacon, Beacon $C$, with its own beacon signal. All of the system's beacon signals and their respective locations are shown below:

$$\begin{aligned} \vec{A} &= \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \vec{B} &= \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \\ \vec{C} &= \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{aligned}$$



Do you see any issues with the new setup? If so, please note them and explain why they are issues.

**Solution:**

Issue 1: Beacon $C$ is co-linear with beacons $A$ and $B$, which will still give us multiple intersecting points when we try to do trilateration. Putting beacons in the same line in 2D will not give a unique solution.

Issue 2: The signal for Beacon $C$ is not orthogonal to the other beacons. More specifically, it is merely a shifted version of the signal for Beacon $B$, shifted by 3 units. This will cause issues if we try to do cross-correlation, because won't be able to distinguish between receiving signals from beacons $B$ and $C$.