EECS 16A Designing Information Devices and Systems I
Spring 2022 Final

Final Solution

1. **HONOR CODE**
   Please read the following statements of the honor code, and sign your name (you don't need to copy it).

   I will respect my classmates and the integrity of this exam by following this honor code. I affirm:

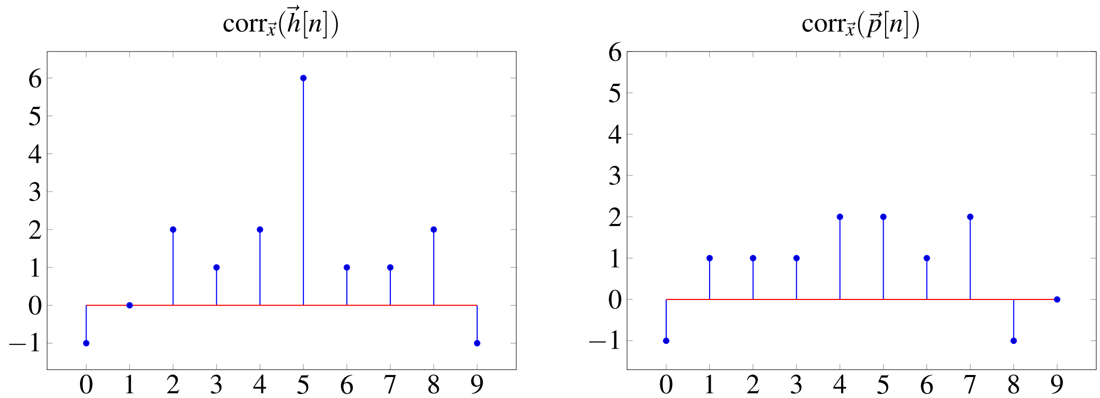
   - *I have read the instructions for this exam. I understand them and will follow them.*
   - *All of the work submitted here is my original work.*
   - *I did not reference any sources other than my unlimited printed resources.*
   - *I did not collaborate with any other human being on this exam.*

2. **Tell us about something you are looking forward to this summer. (1 point)** *All answers will be awarded full credit.*

3. Meow Madness (11 points)

Over quarantine, you adopted a new cat, and as a locked-down signals enthusiast, you spent lots of time noticing patterns in your cat's meows. For instance, the sound your cat makes when it is hungry can be represented as $\vec{h}[n]$ and the sound when your cat wants to play can be represented as $\vec{p}[n]$.

(a) (4 points) There comes a time when you have to leave your cat with a pet sitter. One day, the pet sitter calls you frantically, not knowing what your cat wants. You receive an audio signal, $\vec{x}[n]$, and determine its cross-correlation with $\vec{h}[n]$ and $\vec{p}[n]$. The two cross-correlations, $\text{corr}_{\vec{x}}(\vec{h}[n])$ and $\text{corr}_{\vec{x}}(\vec{p}[n])$ are plotted below:



i. Is your cat more likely to be hungry or want to play? Circle the answer and justify in a sentence or two.

ii. At what time-step of $\vec{x}[n]$ does the meow begin? Justify.

Solution:

i. The peak cross-correlation of the audio signal with $\vec{h}[n]$ is much higher than the peak cross-correlation with $\vec{p}[n]$. Therefore the cat appears to be hungry.

ii. The peak cross-correlation of the audio signal with $\vec{h}[n]$ occurs at time-step 5, therefore, the cat starts meowing at time-step 5.

(b) (3 points) Now, assume that there is a time delay between when your cat meows, and when the meow appears in your audio signal $\vec{x}[n]$. This delay is determined by the time it takes for the sound to reach the microphone. You have calculated that the delay is $t_{delay} = \frac{50}{3}ms$ ($1ms = 10^{-3}s$) and would like to use the delay information to determine where your cat is in the house. You know that your cat has to be in one of the litter boxes either in the bathroom, the bedroom or in the living room. The location of each litter box and of the microphone are shown below.

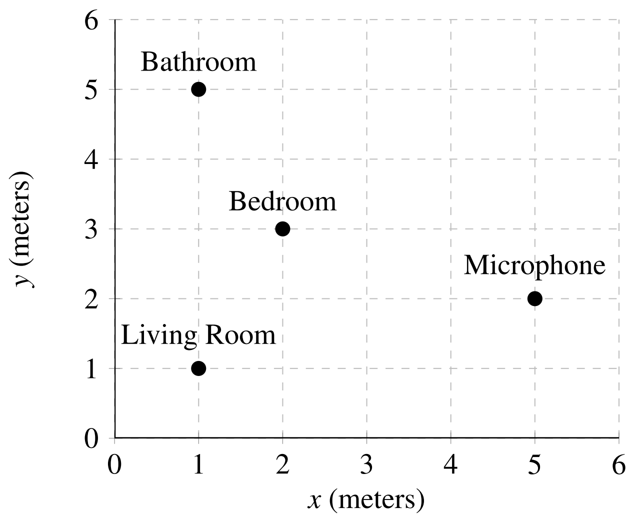


Figure 3.1: Coordinates of litter boxes in each location.

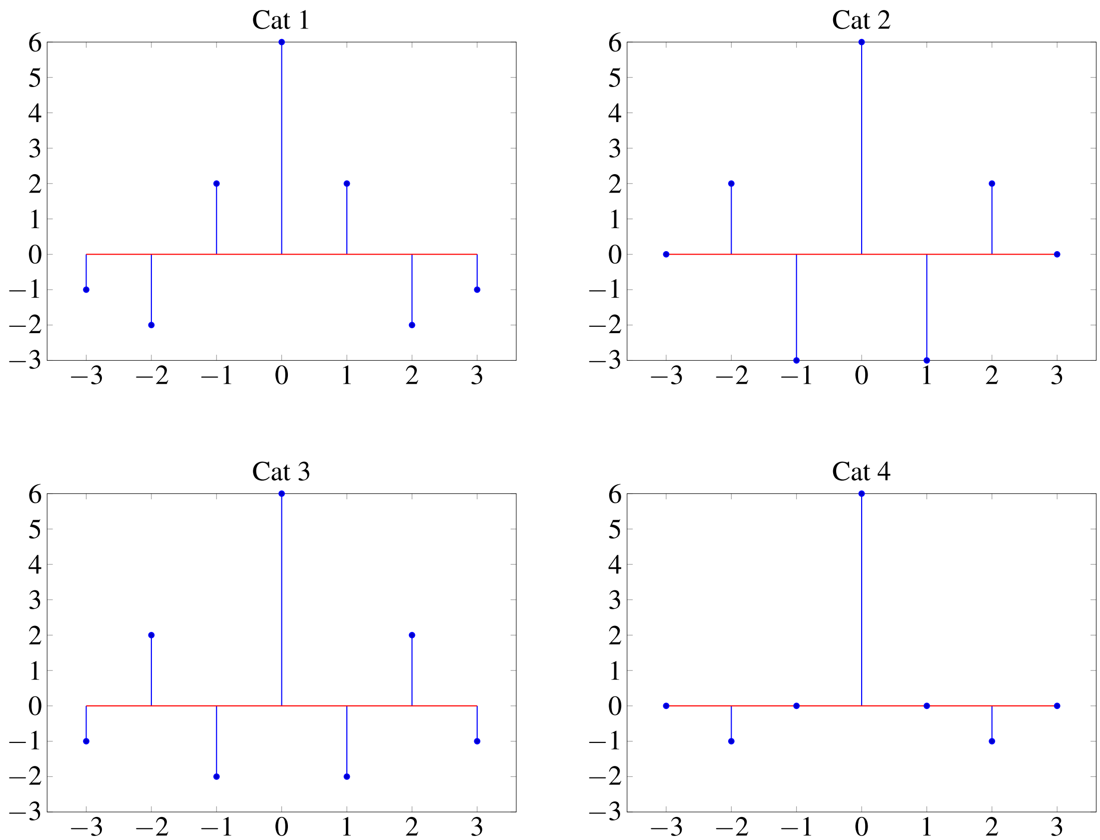
Which location is your cat in? Justify your reasoning. *You can use that the speed of sound is* $v_s = 300m/s$.

Solution: Knowing the time delay and the speed of sound you can solve for the distance between the microphone and the cat as:

$$d = v_s \cdot t_{delay} = 300 \cdot \frac{50}{3} 10^{-3} = 5$$

The cat is 5 meters away from the microphone. The bedroom is $\sqrt{10}$ meters away, the living room is $\sqrt{10}$ meters away and the bathroom is $\sqrt{25} = 5$ meters away. Therefore the cat must be in the litter box in the bathroom.

(c) (4 points) When you go to pick up your cat, you are met with a sea of cats that all look the same! Even your pet sitter is confused by how this came to be. Luckily, it is mealtime, and all of the cats are meowing hungrily. You compute the autocorrelation functions of the meow signals from each of the four cats. The autocorrelation function is defined as $\text{corr}_{\vec{h}}(\vec{h}[n])$ where $h[n]$ is the meow signal. Below are the plots of the resulting autocorrelation functions:



Knowing that your cat's "hungry" meow is

$$\vec{h}[n] = \begin{bmatrix} 1 & -1 & 2 & 0 \end{bmatrix}$$

which of the cats is yours? Justify your reasoning.

More space is provided on the next page if needed.

Solution: By computing the autocorrelation function of your signal, $\vec{h}[n]$ you obtain $\begin{bmatrix} 0 & 2 & -3 & 6 & -3 & 2 & 0 \end{bmatrix}$ which corresponds to Cat 2.

4. Eurovision Song Contest (12 points)

The Eurovision Song Contest is a music contest between countries. A participant from each country gives a performance and the remaining countries score this performance. The performance obtaining the highest total score wins. In this example, there are n countries voting for three performances from Country A, Country B and Country C. You can assume that the performing countries do not vote.

Each of the n voting countries gives 3 points to the country with the best performance, 2 to the second best and 1 for the last. The points received from each country are summed together to obtain the total score.

(a) (3 points) Miki and Ana are very enthusiastic about the contest and make predictions of the total score. Their predictions (p_{Ana} and p_{Miki}) and the true total scores (p_{true}) are:

$$p_{\text{Ana}} = \begin{bmatrix} 14 \\ 34 \\ 13 \end{bmatrix} \quad p_{\text{Miki}} = \begin{bmatrix} 12 \\ 37 \\ 10 \end{bmatrix} \quad p_{\text{true}} = \begin{bmatrix} 13 \\ 35 \\ \alpha \end{bmatrix}$$

where α is unknown. If $||p_{\text{Ana}} - p_{\text{true}}||^2 < ||p_{\text{Miki}} - p_{\text{true}}||^2$, what are the possible values for α?

Solution:

$$
\begin{gathered}
||p_{\text{Ana}} - p_{\text{true}}||^2 < ||p_{\text{Miki}} - p_{\text{true}}||^2 \\
(14-13)^2 + (34-35)^2 + (13-\alpha)^2 < (12-13)^2 + (37-35)^2 + (10-\alpha)^2 \\
1 + 1 + 169 - 26\alpha + \alpha^2 < 1 + 4 + 100 - 20\alpha + \alpha^2 \\
171 - 26\alpha < 105 - 20\alpha \\
66 < 6\alpha \\
\alpha > 11
\end{gathered}
$$

(b) (6 points) The total points can be expressed as the result of a matrix-vector multiplication:

$$
\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} p_A \\ p_B \\ p_C \end{bmatrix}
$$

where A is an unknown matrix which stores information on scoring and $\vec{p} = [p_A \quad p_B \quad p_C]^T$ is the total points received by countries A, B and C.

You are given that the eigenvalues of $\mathbf{A}$ are $\lambda_1 = 6$, $\lambda_2 = 0.5$ and $\lambda_3 = 0$. The corresponding eigenvectors are:

$$
\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}
$$

What would be the resulting total points, $\vec{p}$, that would be obtained with this matrix $\mathbf{A}$?

Solution: We would like to decompose our input vector $\vec{x} = [3 \quad 2 \quad 1]^T$ in terms of our eigenvectors, i.e. we would like to find α_1, α_2 and α_3 such that

$$
\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}
$$

From here we can see that

$$
\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \vec{v}_1 + 2\vec{v}_2
$$

Then we can write

$$
\mathbf{A}\vec{x} = \mathbf{A}(\vec{v}_1 + 2\vec{v}_2) = \mathbf{A}\vec{v}_1 + 2\mathbf{A}\vec{v}_2 = \lambda_1\vec{v}_1 + 2\lambda_2\vec{v}_2 = 6\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} 7 \\ 1 \\ 6 \end{bmatrix}
$$

(c) (3 points) Somehow your voting system has gotten corrupted. The corrupted total points you obtain is $\vec{p}_{\text{corrupt}} = [20 \quad 15 \quad 5]^T$. You have been told that the uncorrupted (true) points satisfy $p_A = 2p_B = 2p_C$. You know that the corruption is small, so the true points must be close to what you received after the corruption. Use projection to find a vector, $\vec{p}*$, that satisfies the constraint and is closest to $\vec{p}_{\text{corrupt}}$.

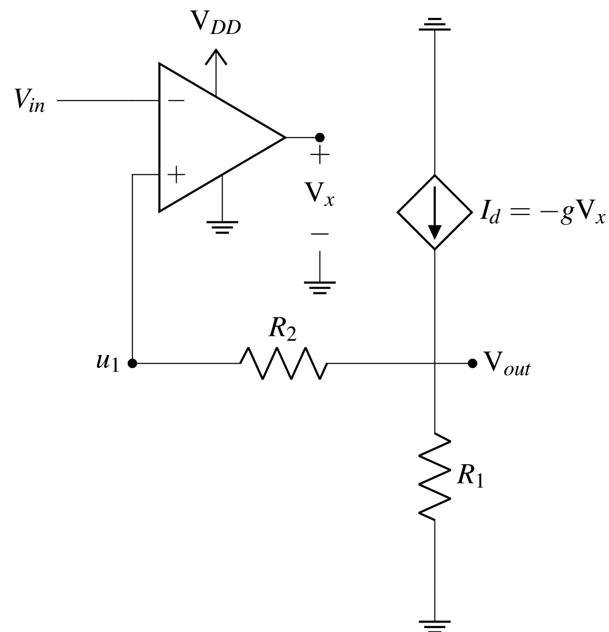
Solution: We would like to project $\vec{p}_{\text{corrupt}}$ onto the vector $\vec{a} = [2 \quad 1 \quad 1]^T$.

$$\vec{p}* = \frac{<\vec{a}, \vec{p}_{\text{corrupt}}>}{||\vec{a}||^2}\vec{a} = \frac{60}{6}\vec{a} = 10\vec{a}$$

$$\vec{p}* = \begin{bmatrix} 20 \\ 10 \\ 10 \end{bmatrix}$$

5. Can I Give You Some Feedback? (4 points)

The following circuit is a linear voltage regulator.



V_{DD} and V_{in} are both connected to ideal voltage sources. g is the gain factor of the dependent current source. The opamp has finite gain A.

Using the method for negative feedback analysis, if V_{out} **increases**, determine what happens to the following values. Circle one of the two options for each line below. *Note that if a quantity is getting more negative, that means it is decreasing.*

Voltage at u_1 will: Increase Decrease

V_x will: Increase Decrease

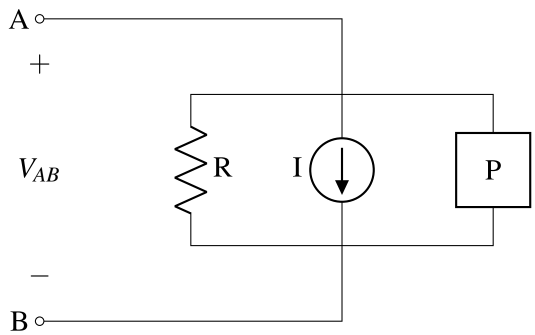
Dependent current I_d will: Increase Decrease

The circuit is in: Negative Feedback Positive Feedback

Solution: Increase, increase, decrease, and negative feedback.
When V_{out} increases, because no current can go into the positive terminal of the op-amp, there cannot be voltage drop across R_2. As a result,voltage at u_1 must follow V_{out} and increase.
By definition of op-amp output voltage, $V_x = A(u_1 - V_{in})$, as u_1 increases, V_x increases.
Thus, because V_x increases, $I_d = -gV_x$ must decrease. Finally, because I_d decreases, the current through R_1 to decreases, therefore V_{out} ends decreases. This is the expected result of negative feedback.

6. R-I-P Loads (11 points)

One model for representing the behavior of a household appliance under different conditions is with the R-I-P model, which models the device as the parallel combination of a resistor, a current source, and a component that consumes a constant amount of power (shown as a box labeled P):



(a) (3 points) Suppose we apply a test voltage V_{AB} to this device. How much power will the device use? Express your answer in terms of V_{AB}, I, R, and/or P.

Solution: To find the total power consumed, we can add together the power consumed by each element of the device. Because the elements are in parallel, the voltage across each of them will be equal to V_{AB}. Using Ohm's law, the current through the resistor will be equal to V_{AB}/R, so from the power equation $P = IV$, $P_R = V_{AB}^2/R$. The current through the current source is equal to I, so power consumed by the current source will be $P_I = IV_{AB}$. Finally, we know the power from the last component is always equal to P. Adding together each contribution, we get that $P_{tot} = V_{AB}^2/R + IV_{AB} + P$.

(b) (2 points) Usually the values of R, I and P are not known ahead of time, so we need to estimate them by measuring the power dissipated by the device at different values of V_{AB}. What is the minimum number of values of V_{AB} we could measure at in order to generate a least squares estimate for the values of R, I and P?

Solution: We are trying to estimate for three unknown parameters, so our matrix A will have three columns. For least squares to work, we need A to have linearly independent columns, so that $A^T A$ is invertible. In order for a set of three vectors to be linearly independent, they must all have at least three elements, so we must take at least three measurements to generate three rows in A.

(c) (6 points) Now assume that we have a new model where the I-V relationship can be modeled as

$$I = k_0 + \frac{k_1}{V},$$

where k_0 and k_1 are device-specific constants. You want to estimate k_0 and k_1 for a given device, so you apply several test voltages and measure the corresponding current through the device. You get the following measurements:

V	I
$-\frac{1}{3}$ V	-1 A
0.5 V	1 A
1 V	9 A

Given that your ammeter may have some amount of error, find the best estimate of k_0 and k_1 using least squares. Show your work.

More space is provided on the next page if needed.

Solution: To apply least squares to solve for k_0 and k_1, we want to model the problem as $\mathbf{A}\begin{bmatrix}k_0\\k_1\end{bmatrix} \approx \vec{b}$. Using the given I-V relationship, we can fit this form by setting the rows of $\mathbf{A}$ equal to $\begin{bmatrix}1 & 1/V\end{bmatrix}$ for each value of V and setting the corresponding entry of $\vec{b}$ equal to the current measured at that voltage. This gives us the equation

$$\begin{bmatrix}1 & -3\\1 & 2\\1 & 1\end{bmatrix}\begin{bmatrix}k_0\\k_1\end{bmatrix} \approx \begin{bmatrix}-1\\1\\9\end{bmatrix}$$

From here, we can apply the least squares formula $\begin{bmatrix}\hat{k_0}\\\hat{k_1}\end{bmatrix} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b}$.

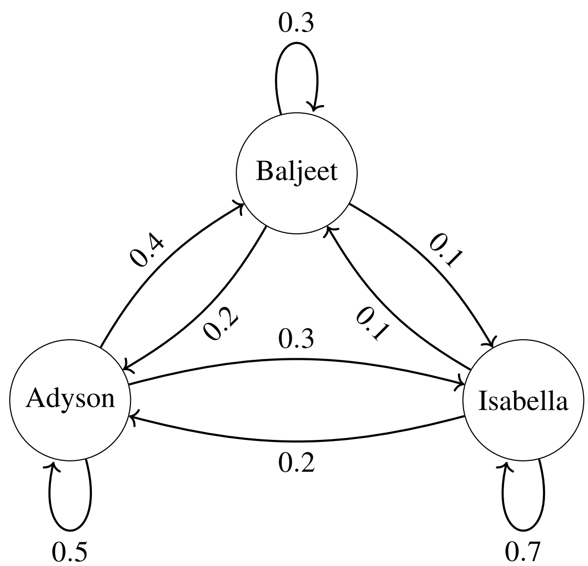
$$\begin{aligned}
\begin{bmatrix}\hat{k_0}\\\hat{k_1}\end{bmatrix} &= \left(\begin{bmatrix}1 & 1 & 1\\-3 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & -3\\1 & 2\\1 & 1\end{bmatrix}\right)^{-1}\begin{bmatrix}1 & 1 & 1\\-3 & 2 & 1\end{bmatrix}\begin{bmatrix}-1\\1\\9\end{bmatrix}\\
&= \left(\begin{bmatrix}3 & 0\\0 & 14\end{bmatrix}\right)^{-1}\begin{bmatrix}9\\14\end{bmatrix}\\
&= \begin{bmatrix}1/3 & 0\\0 & 1/14\end{bmatrix}\begin{bmatrix}9\\14\end{bmatrix}\\
&= \begin{bmatrix}3\\1\end{bmatrix}
\end{aligned}$$

Therefore, our final estimates of k_0 and k_1 are 3 and 1, respectively.

7. Danville Doozy (9 points)

Aniruddh has been watching a lot of Phineas and Ferb lately, and has decided to use his EECS 16A knowledge to learn more about the denizens of Danville.

(a) (3 points) Phineas and Ferb have finally built Croquet Y8. In this version, there are lots and lots of balls instead of just one. Let's say the players are Baljeet, Adyson, and Isabella, and the number of balls in each players possesion can be denoted by x_B, x_A, and x_I, respectively. Suppose we can represent the movement of the balls via the following state transition diagram.



The current number of balls in each player's posession at time-step t is given by the state vector $\vec{x}[t]$ defined as:

$$\vec{x}[t] = \begin{bmatrix} x_B[t] \\ x_A[t] \\ x_I[t] \end{bmatrix} = \begin{bmatrix} \text{number of balls in Baljeet's possesion at time } t \\ \text{number of balls in Adyson's possesion at time } t \\ \text{number of balls in Isabella's possesion at time } t \end{bmatrix}$$

Explicitly write out the transition matrix $\mathbf{T}$ from the provided diagram such that $\mathbf{T}\,\vec{x}[t] = \vec{x}[t+1]$.

Solution:

$$T = \begin{bmatrix} B \to B & A \to B & I \to B \\ B \to A & A \to A & I \to A \\ B \to I & A \to I & I \to I \end{bmatrix} = \begin{bmatrix} 0.3 & 0.4 & 0.1 \\ 0.2 & 0.5 & 0.2 \\ 0.1 & 0.3 & 0.7 \end{bmatrix}$$

(b) (2 points) Suppose instead that the transition matrix, **T** is:

$$\begin{bmatrix} 0.2 & 0.0 & 0.8 \\ 0.2 & 0.6 & 0.4 \\ 0.1 & 0.4 & 0.3 \end{bmatrix}$$

Is this matrix **conservative**? Justify.

Solution: No, the matrix is not conservative since the columns do not sum to 1. This means that if 1 unit of material enters a node, 1 unit doesn't necessarily exit. We may call the node leaky, for example.

(c) (4 points) Now suppose there is a transition matrix **B** with eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = \frac{1}{2}$, and corresponding eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$, and $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

Given $\vec{x} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$, does $\lim_{n\to\infty} \mathbf{B}^n \vec{x}$ converge? If so, what does it converge to?

Solution: We can represent the vector $\vec{x}$ as a linear combination of the eigenvectors.

$$\vec{x} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{x} = 2\vec{v}_1 + \vec{v}_3$$

We can then easily analyze the convergence using the corresponding eigenvalues.

$$\lim_{n\to\infty} \mathbf{B}^n \vec{x} = \lim_{n\to\infty} \mathbf{B}^n [2\vec{v}_1 \vec{+} \vec{v}_3]$$

$$= \lim_{n\to\infty} [2(\lambda_1)^n \vec{v}_1 + (\lambda_3)^n \vec{v}_3]$$

$$= \lim_{n\to\infty} [2(1)^n \vec{v}_1 + (\frac{1}{2})^n \vec{v}_3]$$

$$= 2\vec{v}_1 + 0$$

$$= \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$$

Thus as $n \to \infty$ the system converges to $\begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$.

8. Plane Repair (9 points)

Congratulations! You've just been hired as the lead of the airplane repair division at Boeing. As your first task, the company wants you to help design their repair queueing system. The setup is as follows:

Each plane has a profile $\vec{v} \in \mathbb{R}^2$ that describes it, where the the first entry of $\vec{v}$ represents the the number of thousand miles the plane has flown and the second entry of $\vec{v}$ represents the priority of the aircraft. For example, the profile

$$\vec{v} = \begin{bmatrix} 650 \\ 2.5 \end{bmatrix}$$

represents a plane that has flown 650,000 miles and has a priority of 2.5.

Like all other airplane companies, Boeing's queueing system takes the profile of all the planes that have come in for repair and orders them by their profile's norm squared ($\langle \vec{x}, \vec{x} \rangle$), from largest to smallest, to determine in which orders the planes need to be fixed. Your job is to help design an inner product $\langle \vec{x}, \vec{y} \rangle$ to achieve appropriate queueing behavior.

Luckily, you have a friend at Airbus who has worked on the same task years ago. The formula that Airbus uses looks as follows:

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \mathbf{M} \vec{y}, \text{ where } \mathbf{M} = \begin{bmatrix} \frac{1}{100} & \frac{1}{10} \\ \frac{1}{10} & 2 \end{bmatrix}$$

(a) (3 points) If two planes, Plane 1 with profile $\vec{v}_1 = \begin{bmatrix} 30 \\ 3 \end{bmatrix}$ and Plane 2 with profile $\vec{v}_2 = \begin{bmatrix} 20 \\ 10 \end{bmatrix}$ come into Airbus' repair shop, using Airbus' formula, which of the planes would be repaired first? In other words, which plane has the larger norm squared ($\langle \vec{x}, \vec{x} \rangle$)?

More space is provided on the next page if needed.

Solution:

$$||v_1||^2 = \langle v_1, v_1 \rangle = v_1^T M v_1 = [30 \quad 3] \begin{bmatrix} \frac{1}{100} & \frac{1}{10} \\ \frac{1}{10} & 2 \end{bmatrix} \begin{bmatrix} 30 \\ 3 \end{bmatrix} = \frac{1}{100}(30)^2 + \frac{2}{10}(30)(3) + 2(3)^2 = 9 + 18 + 18 = 45$$

$$||v_2||^2 = \langle v_2, v_2 \rangle = v_2^T M v_2 = [20 \quad 10] \begin{bmatrix} \frac{1}{100} & \frac{1}{10} \\ \frac{1}{10} & 2 \end{bmatrix} \begin{bmatrix} 20 \\ 10 \end{bmatrix} = \frac{1}{100}(20)^2 + \frac{2}{10}(20)(10) + 2(10)^2 = 4 + 40 + 200 = 244$$

Thus, the Plane 2 would be processed first.

(b) (6 points) You decide to take inspiration from Airbus and design your own inner product of the form $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \mathbf{A} \vec{y}$ where $\mathbf{A}$ takes the form $\mathbf{A} = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$

What values must a_1 and a_2 be such that, for $\vec{x} = \begin{bmatrix} 10 \\ 2 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} 20 \\ 1 \end{bmatrix}$, $\langle \vec{x}, \vec{x} \rangle = \langle \vec{y}, \vec{y} \rangle = 5$?

Solution: We can set up a system of equations to solve this problem:

$$\langle x, x \rangle = [10 \quad 2] \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} 10 \\ 2 \end{bmatrix} = 10^2 a_1 + 2^2 a_2 = 100 a_1 + 4 a_2 = 5$$
$$\langle y, y \rangle = [20 \quad 1] \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix} \begin{bmatrix} 20 \\ 1 \end{bmatrix} = 20^2 a_1 + 1^2 a_2 = 400 a_1 + 1 a_2 = 5$$

Multiplying the second equation by 4 we get

$$100 a_1 + 4 a_2 = 5$$
$$1600 a_1 + 4 a_2 = 20$$

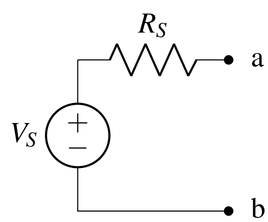
Subtracting (1) from (2) we get

$$1500 a_1 = 15$$

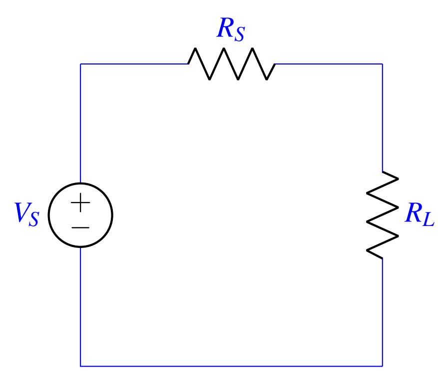
This gives us $a_1 = \frac{1}{100}$, plugging back in gets us $a_2 = 1$

9. Equivalents (11 points)
Suppose you are building a new house and are trying to source the correct lights to use.

(a) (3 points) The building is powered by a constant voltage source, modeled by the following circuit. You plan for your light (modeled by a load resistor R_L) to be connected across terminals a and b. If your wiring can handle up to 5mA (milli = 10^{-3}) across terminals a and b, what is the range of resistance values R_L that you can connect across those terminals, where $V_S = 10V$ and $R_S = 1\text{k}\Omega$ (kilo = 10^3)?



Solution: Connecting a load resistor R_L across terminals a and b gives us the following circuit:



Ohm's Law states that $\frac{V}{R} = I$. In this circuit, we can use the equivalent resistance $R = R_S + R_L$ because the resistors are in series. From the problem, the current should not exceed 0.005 A.

$$\frac{V_S}{R} \leq 0.005$$
$$\frac{V_S}{R_S + R_L} \leq 0.005$$
$$\frac{10}{1000 + R_L} \leq 0.005$$

Thus, $R_L \geq 1\text{k}\Omega$

(b) (8 points) Rather than the simplified circuit above, your wiring is actually made up of the following circuit components where $V_S = 10V$, $R = 1\Omega$

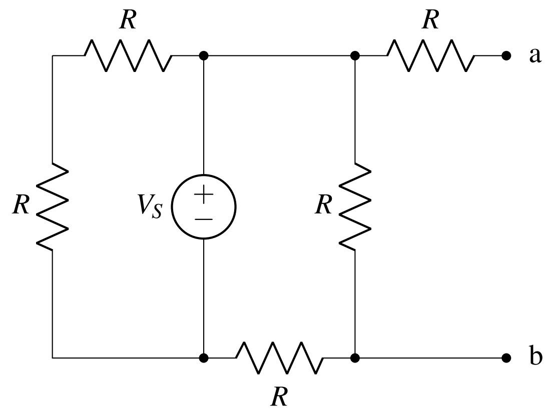
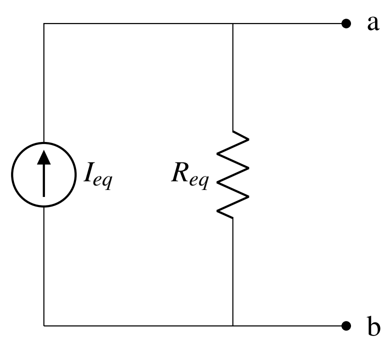


Figure 1

Provide values for I_{eq} and R_{eq} for the circuit below that will result in the same $I-V$ characteristics across **nodes a and b** as the circuit in Figure 1.



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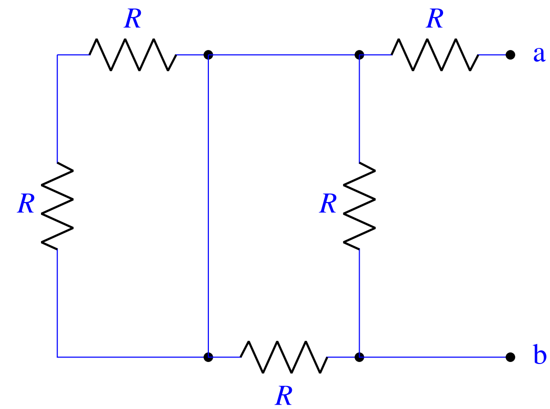
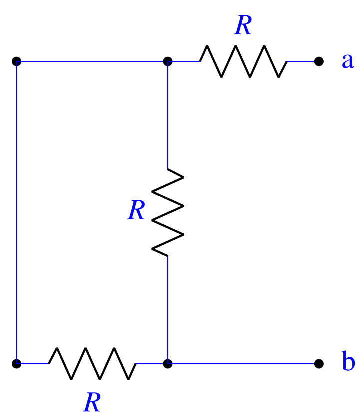
Solution: $I_{eq} = \frac{10}{3}$ A and $R_{eq} = \frac{3}{2}\Omega$.
Notice that this question is essentially asking for the Norton equivalent circuit. Start by finding $R_{eq} = R_{no}$ by zeroing out the voltage source. Zeroed out, the voltage source becomes a wire as shown below.



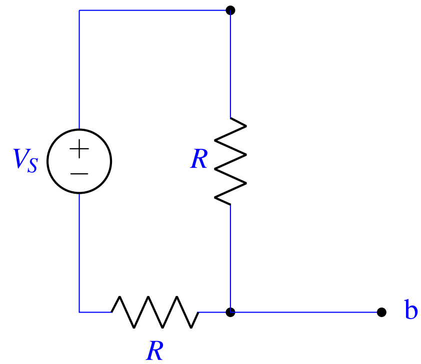
There will no longer be a curent going through the two left most resistors. Thus, we can redraw the circuit to exclude those two resistors.



Now the equivalent resistanace is two resistors in parallel in series with another resistor.

$$R_{eq} = R \parallel R + R = \frac{1}{2}R + R = \frac{3}{2}R = \frac{3}{2}\Omega$$

Now we need to find the equivalent current. Recall that $I_{no} = V_{th}/R_{no}$. Having already found R_{no}, we can find I_{no} by finding V_{th}. Note that the potential at node a is simply V_s. The potential at node b can be found through voltage division. This can be seen more clearly below:



$$V_b = \frac{R}{R+R} V_s = \frac{1}{2} V_S$$

Now that we have both V_a and V_b, we can find $V_{ab} = V_{eq} = V_{th}$.

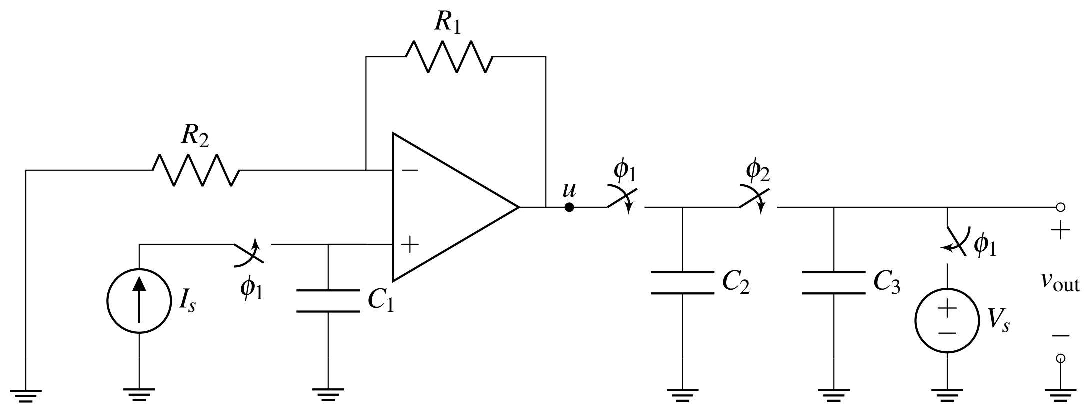
$$V_{ab} = V_a - V_b = V_s - \frac{1}{2} V_S = \frac{1}{2} V_S$$
$$V_{ab} = 5V = V_{eq}$$

Finally, we can find I_{eq}.

$$I_{eq} = \frac{V_{eq}}{R_{eq}} = \frac{5}{\frac{3}{2}} = \frac{10}{3} \text{ A.}$$

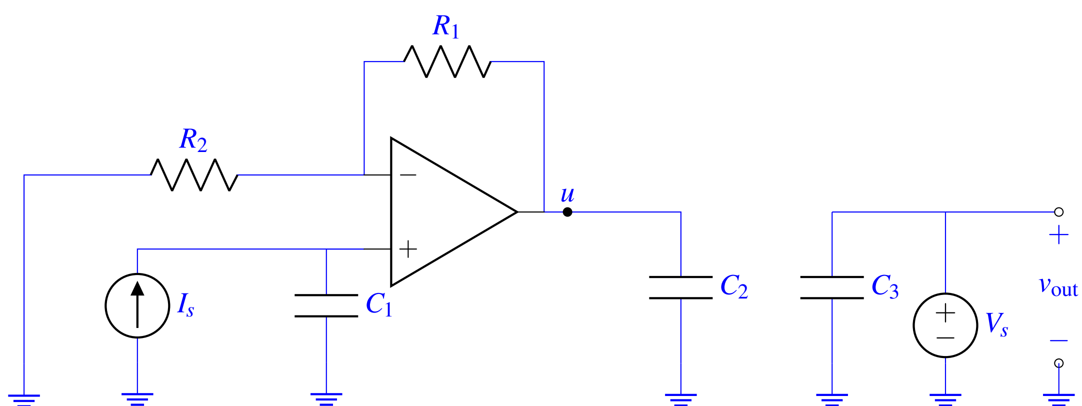
10. Charge Sharing with Op Amps (10 points)

For the following circuit, assume that all capacitors are initially uncharged, every switch is open, and I_s and V_s are constant sources. During phase 1: ϕ_1 switches are closed and the ϕ_2 switch is open. During phase 2: the ϕ_2 switch is closed and ϕ_1 switches are open.



(a) (6 points) At time $t = 0$, phase 1 starts. Compute $u^{\phi_1}(t)$, the voltage at node u during phase 1 in terms of t, I_s, C_1, C_2, C_3, R_1, R_2, and/or V_s.

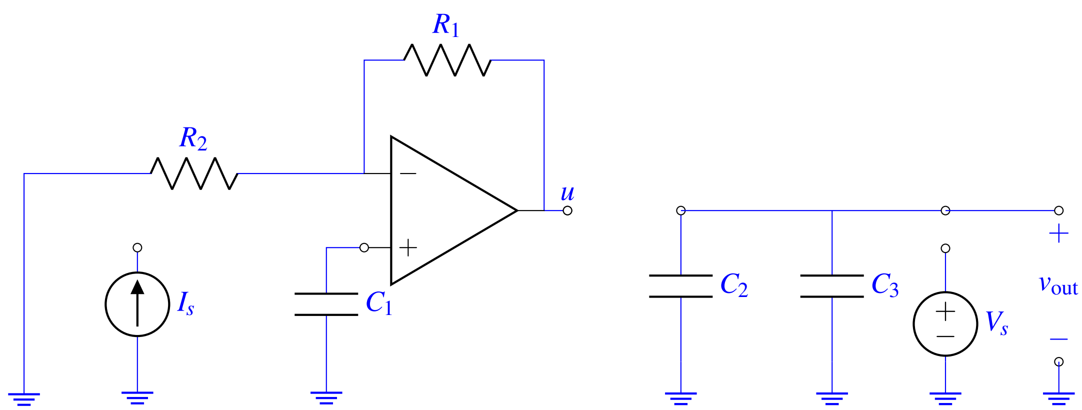
Solution: The circuit in phase 1:



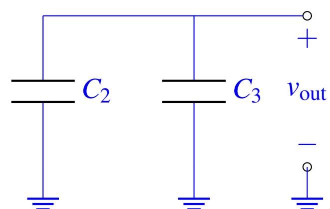
The op amp is in the non-inverting configuration, thus $u^{\phi_1}(t) = u^+(1 + \frac{R_1}{R_2})$. The non-inverting input voltage to the op amp is $u^+ = \frac{I_s}{C_1}t$. Therefore, $u^{\phi_1}(t) = \frac{I_s}{C_1}t(1 + \frac{R_1}{R_2})$

(b) (4 points) After τ seconds have elapsed, the switches labeled ϕ_1 open and the switch labeled ϕ_2 closes. Compute v_{out} in phase 2 in terms of $u^{\phi_1}(\tau)$, I_s, C_1, C_2, C_3, R_1, R_2, and/or V_s.

Solution: The circuit in phase 2:



Or more simply:



This problem can be solved with charge sharing. During phase 1, C_2 is charged to $V_{C_2}^{\phi_1} = u^{\phi_1}(\tau)$, thus $Q_{C_2}^{\phi_1} = C_2 u^{\phi_1}(\tau)$. During phase 1, C_3 is charged to $V_{C_3}^{\phi_1} = V_s$, thus $Q_{C_3}^{\phi_1} = C_3 V_s$.

During phase 2, C_2 has $V_{C_2}^{\phi_2} = v_{out}$, thus $Q_{C_2}^{\phi_2} = C_2 v_{out}$. During phase 2, C_3 has $V_{C_3}^{\phi_2} = v_{out}$, thus $Q_{C_3}^{\phi_2} = C_3 v_{out}$.

The charge conservation equation on the floating node between the capacitors (the node connected to v_{out}) is written as follows:

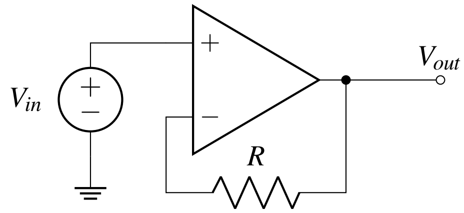
$$Q_{C_2}^{\phi_1} + Q_{C_3}^{\phi_1} = Q_{C_2}^{\phi_2} + Q_{C_3}^{\phi_2}$$

$$C_2 u^{\phi_1}(\tau) + C_3 V_s = C_2 v_{out} + C_3 v_{out}$$

$$\text{Therefore } v_{out} = \frac{C_2 u^{\phi_1}(\tau) + C_3 V_s}{C_2 + C_3}.$$

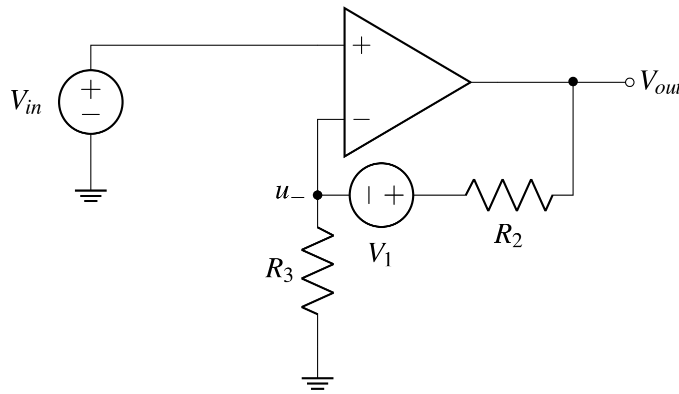
11. Amped Up On Op Amps! (11 points)

(a) (3 points) For the following circuit, find V_{out} in terms of V_{in} and/or R.



Solution: The circuit is in negative feedback so $u_+ = u_-$. Because $u_+ = V_{in}$ and $u_- = V_{out}$, we find $V_{out} = V_{in}$.

(b) (8 points) For the following circuit in negative feedback, find an expression for node V_{out} in terms of V_{in}, V_1, and any other necessary labelled circuit components.
Hint: You may want to start by performing KCL analysis at the u_- node and solving u_- in terms of V_{out} and other variables.



Solution:
Using the hint, we start by performing KCL analysis at node u_-:

$$\frac{V_{out} - (u_- + V_1)}{R_2} = \frac{u_-}{R_3}$$

$$\frac{V_{out}}{R_2} = \frac{u_-}{R_3} + \frac{u_-}{R_2} + \frac{V_1}{R_2}$$

Because the circuit is in negative feedback, $u_- = u_+ = V_{in}$.

$$\frac{V_{out}}{R_2} = \frac{V_{in}}{R_3} + \frac{V_{in}}{R_2} + \frac{V_1}{R_2}$$

$$V_{out} = \frac{R_2}{R_3}V_{in} + V_{in} + V_1$$

$$V_{out} = (\frac{R_2}{R_3} + 1)V_{in} + V_1$$

12. Eigenvalues Galore (11 points)

(a) (7 points) Prove that if a square matrix $\mathbf{A}$ is equal to $\mathbf{A}^2$ ($\mathbf{A} = \mathbf{A}^2$), then all of the eigenvalues of $\mathbf{A}$ are either 0 or 1.

Solution:

$$\begin{aligned} \mathbf{A}\vec{v} &= \lambda\vec{v} \\ A^2\vec{v} &= \mathbf{A}\lambda\vec{v} \\ &= \lambda\mathbf{A}\vec{v} \\ &= \lambda^2\vec{v} \\ \lambda^2\vec{v} &= \lambda\vec{v} \\ \lambda^2\vec{v} - \lambda\vec{v} &= 0 \\ (\lambda^2 - \lambda)\vec{v} &= 0 \end{aligned}$$

By definition of eigenvectors, $\vec{v}$ cannot be zero so

$$\begin{aligned} \lambda(\lambda - 1) &= 0 \\ \lambda &= 0, 1 \end{aligned}$$

(b) (4 points) Given matrix $\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & -2 \end{bmatrix}$, find the eigenvalues of $4((\mathbf{B}^{-1})^T)^4$.

Solution: 4, 36, and $\frac{1}{4}$.

$\mathbf{B}$ has eigenvalues 1, $\frac{1}{\sqrt{3}}$, and -2. If $\mathbf{B}$ has eigenvalues λ_i, $\mathbf{B}^{-1}$ has eigenvalues $\frac{1}{\lambda_i}$ so $\mathbf{B}^{-1}$ has eigenvalues 1, $\sqrt{3}$, and $-\frac{1}{2}$.

For any general matrix $\mathbf{C}$, Det($\mathbf{C}$) = Det($\mathbf{C}^T$) which means that $\mathbf{C}$ and $\mathbf{C}^T$ have the same eigenvalues. So $\mathbf{B}^{-1}$ and $(\mathbf{B}^{-1})^T$ have the same eigenvalues.

If $\mathbf{B}$ has eigenvalues λ_i, $\mathbf{B}^4$ has eigenvalues λ_i^4 so $((\mathbf{B}^{-1})^T)^4$ has eigenvalues 1, 9, and $\frac{1}{16}$.
Thus, $4((\mathbf{B}^{-1})^T)^4$ has eigenvalues 4, 36, and $\frac{1}{4}$.

13. An Easier Way To Do Math Homework (10 points)

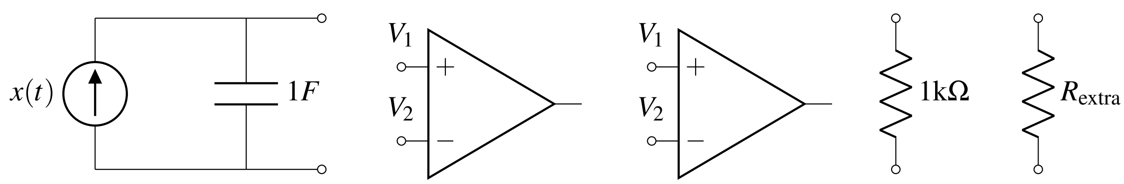
You're working on your Math 1B homework and you don't know how to calculate an integral. Instead, you decide to put your circuit skills to use to solve this problem!
The integral that you're trying to solve is of the form

$$-\frac{1}{5}\int_0^\tau x(t)\,dt$$

Your helpful lab TA, Raghav, gives you several circuit elements that you can use.

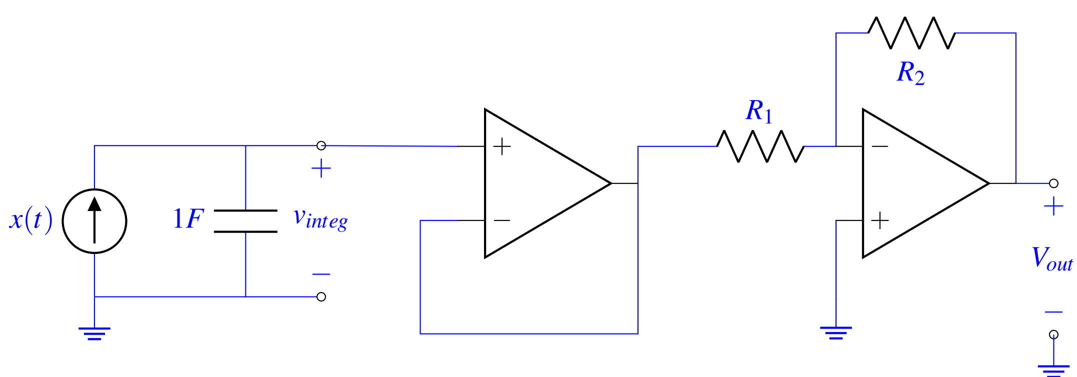
These elements are:

- A current source $I_s = x(t)$ amps in parallel with a capacitance of 1F.
- **Two** op-amps (assume that the supply voltages to the op-amps are provided).
- A resistor $R_{\text{fixed}} = 1\text{k}\Omega$.
- One additional resistor R_{extra} that can have **any value**. Be sure to specify the resistance you use.



Design a circuit to have the above output using the provided elements. Clearly specify the resistance of R_{extra} if used.

Solution:



The circuit consists of three main blocks: the current source and capacitor which integrate our signal, a buffer, and an inverting amplifier. The current source charges our capacitor resulting in:

$$v_{integ} = \frac{1}{1F}\int_0^\tau x(t)dt$$

Next, in order to scale and invert our signal we can chose $R_1 = 5R_2$. Since we have a fixed resistor of 1Ω, we pick $R_2 = R_{fixed} = 1\text{k}\Omega$ and $R_1 = R_{extra} = 5\text{k}\Omega$.

Our overall result becomes:

$$v_{out} = \frac{-R_2}{R_1} v_{integ} = \frac{-1}{5} \int_0^{\tau} x(t)dt$$

More space is provided on the next page if needed.