EECS 16A Designing Information Devices and Systems I
Spring 2022 Midterm 1

PRINT your student ID: ________________________________

PRINT AND SIGN your name: ________________, ________________ ________________
(last name) (first name) (signature)

PRINT the time of your discussion section and your GSI(s) name: ________________________________

PRINT the student IDs of the person sitting on your right: ________________ and left: ________________

1. HONOR CODE

If you have not already done so, please copy the following statements into the box provided for the honor code on your answer sheet, and sign your name.

I will respect my classmates and the integrity of this exam by following this honor code. I affirm:

- *I have read the instructions for this exam. I understand them and will follow them.*
- *All of the work submitted here is my original work.*
- *I did not reference any sources other than my one reference cheat sheet.*
- *I did not collaborate with any other human being on this exam.*

2. Tell us about something that makes you happy (1 point) *All answers will be awarded full credit.*

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3. Information Storage (26 points)

(a) (10 points) Your team plans to build a database that stores information as vectors $\vec{v}_s \in \mathbb{R}^3$. Due to system constraints, Ayush, an engineer on your team, mentions that it'll be easiest to store these vectors as linear combinations of

$$\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, \vec{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{w}_3 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$$

For each of the following vectors $\vec{v}_i$, state if it can be written as a linear combination of $\vec{w}_1, \vec{w}_2, \vec{w}_3$. If so, find the coefficients. If not, explain why.

$$\vec{v}_1 = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\vec{v}_1 =$ ______ $\vec{w}_1 +$ ______ $\vec{w}_2 +$ ______ $\vec{w}_3$ or No solution

$\vec{v}_2 =$ ______ $\vec{w}_1 +$ ______ $\vec{w}_2 +$ ______ $\vec{w}_3$ or No solution

$\vec{v}_3 =$ ______ $\vec{w}_1 +$ ______ $\vec{w}_2 +$ ______ $\vec{w}_3$ or No solution

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(b) (2 points) Consider a matrix $\mathrm{M} \in \mathbb{R}^{3\times3}$ formed by the column vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ from part (a). What is the rank of M?

Rank =______

(c) (10 points) To manipulate the vectors in your database, you can multiply them by $\mathrm{M} \in \mathbb{R}^{3\times3}$. Suppose $\vec{v}_j$ is some vector in the database and

$$\mathrm{M} = \begin{bmatrix} 3 & -1 & 5 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

We generate $\vec{v_{new}} = \mathrm{M}\vec{v}_j$. Can Dahlia find a new matrix P that reverses this operation such that $\mathrm{P}\vec{v_{new}} = \vec{v}_j$? If so, find this new matrix. If not, why not?

$$\mathrm{P} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$$

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(d) (4 points) Now given a new matrix

$$\mathrm{M}_{\text{new}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 2 & \alpha \end{bmatrix}$$

what value of α makes M_{new} have a rank of 2?

$\alpha =$ ______

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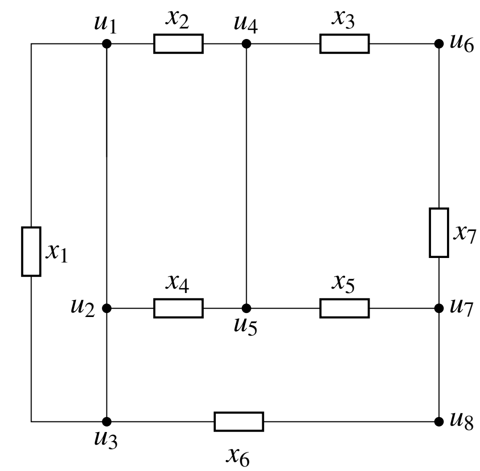
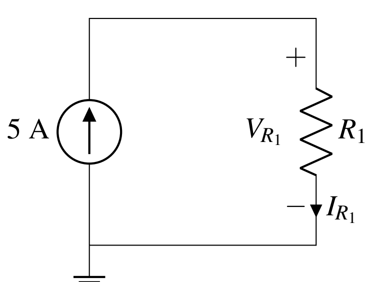
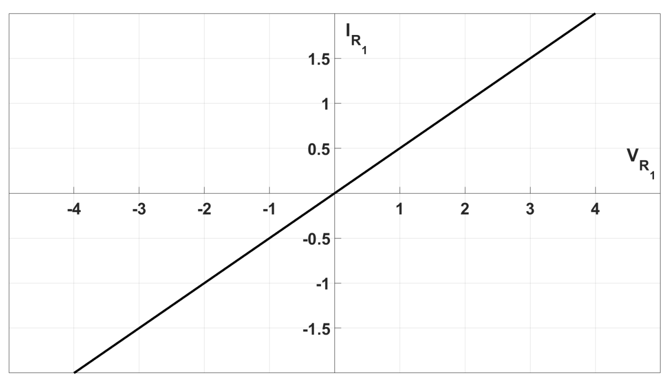
4. Circuits (18 points)

(a) (6 points) List which labelled nodes (u_1, u_2,..) are equivalent (ex: $u_h = u_i = u_j$). If the labelled node is unique (not equivalent to any other labelled node), do not list it.



(b) (6 points) For the circuit below, determine V_{R_1} and I_{R_1}. You are given the I-V curve of R_1.





$V_{R_1} =$ ______

$I_{R_1} =$ ______

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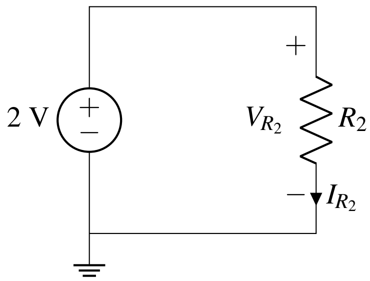
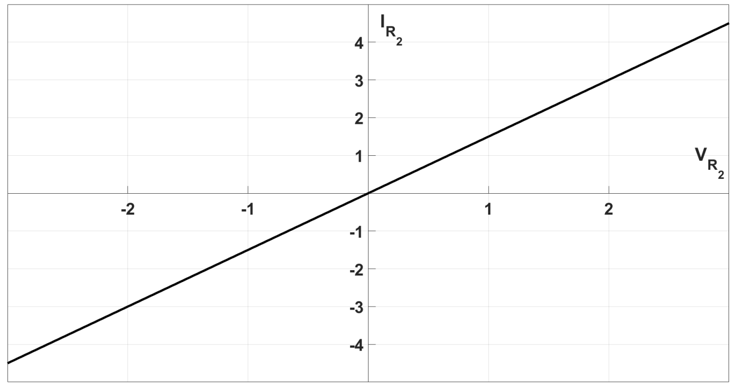
(c) (6 points) For the circuit below, determine V_{R_2} and I_{R_2}. You are given the I-V curve of R_2.





$V_{R_2} =$ ______

$I_{R_2} =$ ______

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5. (Gaussian) Eliminate Your Options (6 points)

We have a system of equations in the form of a matrix vector equation $A\vec{x} = \vec{b}$. We know the following about A:

- $A \in \mathbb{R}^{3\times 4}$
- The first and second **columns** of A are not scalar multiples of each other.
- The third **column** is a linear combination of the first and second **columns**.

Determine which of the possible augmented matrices could represent the result of performing Gaussian Elimination on A to reach the **Row Echelon Form**. Please justify your answer for each matrix. (Note: asterisks represent any real number)

$$\text{(a)} \left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 1 & * & * \end{array}\right] \quad \text{(b)} \left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * \end{array}\right] \quad \text{(c)} \left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & * \end{array}\right]$$

$$\text{(d)} \left[\begin{array}{cccc|c} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & * \end{array}\right] \qquad \text{(e) None of the above}$$

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6. Matrix Multiplications (20 points)

(a) (4 points) The matrix A $\in \mathbb{R}^{500x501}$ is shown below

$$A = \begin{bmatrix} a_{1,1} & \cdots & a_{1,501} \\ \vdots & \ddots & \vdots \\ a_{500,1} & \cdots & a_{500,501} \end{bmatrix}$$

Given another matrix B $\in \mathbb{R}^{501x500}$, what are the dimensions of the matrix AB?

$AB \in \mathbb{R}^{\_\_\_\_ \times \_\_\_\_}$

(b) (4 points) What are the dimensions of $((A^T A)B)^T$?

$((A^T A)B)^T \in \mathbb{R}^{\_\_\_\_ \times \_\_\_\_}$

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(c) (6 points) Given that the elements of matrix A and B follow the pattern:

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \end{bmatrix} \qquad a_{i,j} = \begin{cases} i & i = j \\ 0 & i \neq j \end{cases}$$

$$B = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 1 & \cdots & 1 \\ 1 & 1 & 3 & \cdots & 1 \\ \vdots & \vdots & \vdots & \cdots & \end{bmatrix} \qquad b_{k,l} = \begin{cases} k & k = l \\ 1 & k \neq l \end{cases}$$

Find the element in the 4^{th} row and 4^{th} column of the matrix multiplication (AB). In other words, what is $(AB)_{4,4}$?

$(AB)_{4,4} =$ ______

(d) (6 points) What is $(AB)_{4,5}$?

$(AB)_{4,5} =$ ______

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7. Geometric Transformations (16 points)

(a) (6 points) Write an expression for the transformation matrix that would reflect a vector across the line $y = -x$ and then rotate them by 45 degrees counterclockwise. Write your answer as some combination of the matrices below (ex: A*B).

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix};\ \mathbf{B} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix};\ \mathbf{C} = \begin{bmatrix} \cos(-45^\circ) & -\sin(-45^\circ) \\ \sin(-45^\circ) & -\cos(-45^\circ) \end{bmatrix};$$

$$\mathbf{D} = \begin{bmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{bmatrix}$$

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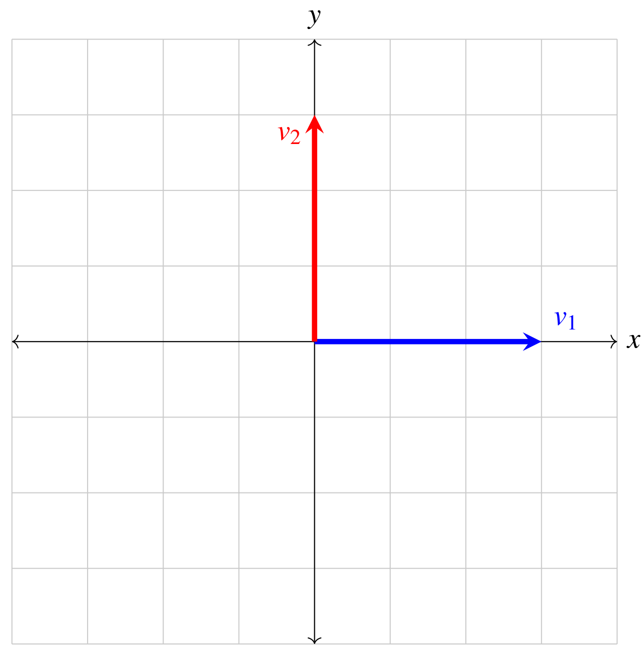
(b) (10 points) Consider a new transformation matrix T shown below.

$$T = \begin{bmatrix} -\cos(-60^\circ) & \sin(-60^\circ)) \\ \sin(-60^\circ) & \cos(-60^\circ) \end{bmatrix}$$

What transformation does T represent? Write your answer in terms of degrees rotated and/or reflection over an axis. Graph how this matrix transforms $\vec{v}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$. Do you best to approximate when necessary. All reasonable answers will be accepted.



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8. Nullspace (22 points)

(a) (6 points) Consider the matrix below. What is the set of vectors that span the nullspace of A?

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -1 & -6 \end{bmatrix}$$

null(A) = span$\left\{ \qquad \qquad \right\}$

(b) (10 points) Consider the matrix below. What is the set of vectors that span the nullspace of A?

$$\mathbf{A} = \begin{bmatrix} 3 & -6 & 2 \\ -2 & 4 & 2 \end{bmatrix}$$

null(A) = span$\left\{ \qquad \qquad \right\}$

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(c) (6 points) Consider the following matrix:

$$\mathbf{A} = \begin{bmatrix} (1-x) & 2 \\ 0 & (6+x) \end{bmatrix}$$

Find all values of x for which A has a non-trivial nullspace.

$x =$ ____

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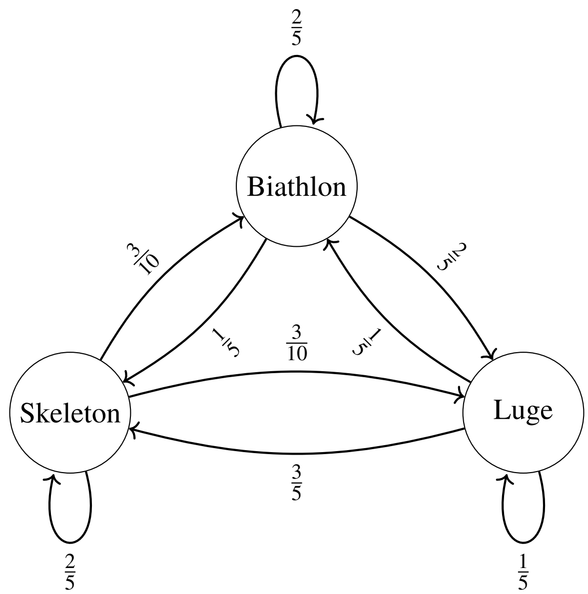
9. Cooling Off at The Olympics (18 points) A number of Berkeley students tuned into the Winter Olympics. You want to analyze the dynamics of the viewer traffic.

(a) (4 points) You are able to construct the following transition diagram.



The current number of students watching each sport at time-step t is given by the state vector $\vec{x}[t]$ defined as:

$$\vec{x}[t] = \begin{bmatrix} x_B[t] \\ x_S[t] \\ x_L[t] \end{bmatrix} = \begin{bmatrix} \text{number of students watching Biathlon at time } t \\ \text{number of students watching Skeleton at time } t \\ \text{number of students watching Luge } t \end{bmatrix}$$

Explicitly write out the transition matrix $\mathbf{T}$ from the provided diagram such that $\vec{x}[t+1] = \mathbf{T}\,\vec{x}[t]$. Is the system conservative? Justify your answer.

$\mathrm{T} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}$

Conservative/Not conservative

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(b) (10 points) Suppose we have a different transistion matrix given by:

$$\mathbf{T}_2 = \begin{bmatrix} \frac{3}{4} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \\ 0 & \frac{1}{2} & 0 \end{bmatrix}$$

Does a steady state vector exist for this system? If so, identify a steady state vector $\vec{x}_{steady}$ such that $\mathbf{T}_2\vec{x}_{steady} = \vec{x}_{steady}$.

$\vec{x}_{steady} = \begin{bmatrix} \\ \\ \\ \end{bmatrix}$, or no steady state

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(c) (4 points) Say you know the state at time t, $\vec{x}[t]$, for the system described in part (b) with $\mathbf{T}_2$. For *any* given $\vec{x}[t]$, is it possible to find the previous state, $\vec{x}[t-1]$? **Justify**.

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10. Matrix Multiplication Proof (14 points)

(a) (4 points) Given that Matrix A is square and has linearly independent columns, which of the following is true?

i. A is full rank
ii. A has a trivial nullspace
iii. $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b}$
iv. A is invertible
v. The determinant of A is non-zero

(b) (10 points) Let two square matrices $M_1, M_2 \in \mathbb{R}^{2x2}$ each have linearly independent columns. Prove that $G = M_1 M_2$ also has linearly independent columns.

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(Answer continued from previous page.)

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11. Intersection and Union of Two Subspaces (12 points)

Let $U = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ and $V = \text{span}\left\{ \begin{bmatrix} -2 \\ -2 \end{bmatrix} \right\}$. For the statements in each of the subparts below, determine if the statement is true or false. Justify your reasoning.

(a) (6 points) Statement: The intersection, $U \cap V$ is a subspace of $\mathbb{R}^2$.

Note that the intersection is defined as $U \cap V = \{\vec{v} \mid \vec{v} \in U \text{ and } \vec{v} \in V\}$. This means that any vector $\vec{v}$ in subspace $U \cap V$ must be in both subspace U and subspace V.

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(b) (6 points) Statement: The union, $U \cup V$ is a subspace of $\mathbb{R}^2$.
Note that the union is defined as $U \cup V = \{\vec{v} \mid \vec{v} \in U \text{ or } \vec{v} \in V\}$. This means that any vector $\vec{v}$ in subspace $U \cup V$ must be in either subspace U or subspace V.

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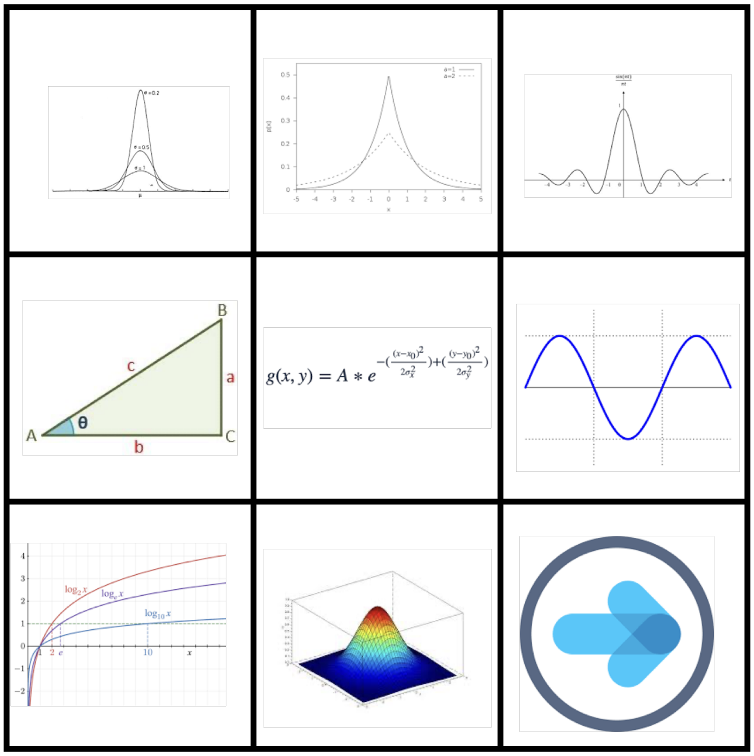
12. G.E. Game (0 points)



Gaussian Elimination: The game — not the algorithm. Eliminate those Gaussians!