Midterm 1 Review

EECS16A Spring 2020

Slides by Chris Mitchell, Nick Werblun and Sang Min Han

Modified for Spring 2020

Outline

- Matrix Transformations and Gaussian Elimination
- Linear (In)dependence and Span
- Vector Spaces and Subspaces, Span and Basis
- Column Space, and Null Space
- Eigenvalues and Eigenvectors
- PageRank
- Derivations and Proofs
- Tips
- Practice exam problems

Matrix Transformations and Gaussian Elimination

Matrix Transformations

- We can view **A**x as a linear operation **A** applied to x
- Geometric operations
  - Rotation
  - Scaling
  - Reflecting
- Order of matrix multiplication matters!

Gaussian Elimination

- I have a system of equations and I want to know the solution
  - Doing it by hand is hard though
- Solution: Build an augmented matrix and use Gaussian Elimination
- See Note 1B for more

$$\begin{bmatrix} 2x & + & 4y & + & 2z & = & 8 \\ x & + & y & + & z & = & 6 \\ x & - & y & - & z & = & 4 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 2 & 4 & 2 & 8 \\ 1 & 1 & 1 & 6 \\ 1 & -1 & -1 & 4 \end{array}\right]$$

Gaussian Elimination

- Goal: Reduced Form
  - When reduced, you can read the solutions right out of the matrix (if they exist)
- Apply basic row operations to get into this form or reach a stopping condition
- Pivot: Having a nonzero value in a column where all values left of it are 0.
  - In reduced form, all values above and below a pivot are also 0
  - NOTE: Not all matrices will have a pivot in every column

Pivot

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Pivot

NO Pivot

$x_1 = -2$

$x_2 = -4$

$x_3 = ???$

What??

Gaussian Elimination

Examples (once in reduced form)

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Pivot in every row and column, exact sol

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

0=0, inf. solutions

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

0=1, no solutions

Linear (In)dependence and Span

Linear (In)dependence - Definition

- If a set of vectors is **linearly dependent**, that means one of them can be formed from a linear combination of the others (falls in the span of the other vectors) and is *redundant*.
- If a set of vectors is **linearly independent**, on the other hand, then no vector in that set can be constructed using linear combinations of the other ones.
- One way to determine this is to solve the equation: $a_1\overrightarrow{v_1} + a_2\overrightarrow{v_2} + \cdots a_n\overrightarrow{v_n} = 0$

  If the only solution is alpha_1, ... , alpha_n = 0, then the set of vectors {v_1, …, v_n} is linearly independent.

$$a_1\overrightarrow{v_1} + a_2\overrightarrow{v_2} + \cdots a_n\overrightarrow{v_n} = 0 \Rightarrow a_1 = a_2 = \cdots a_n = 0$$

Linear (In)dependence

Examples: Linearly dependent?

$$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix}, \begin{bmatrix} 5 \\ -10 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Linear (In)dependence

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YES

$$-2\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \\ -8 \end{bmatrix}$$

Thus the third vector is redundant information

Linear (In)dependence

NOTE: This boils down to essentially solving a system of equations. If we want to find the coefficients, we can do it by inspection, or:

$$a_1 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix} + a_3 \begin{bmatrix} 5 \\ -10 \\ -8 \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{ccc|c} 2 & 3 & 5 & 0 \\ -1 & -4 & -10 & 0 \\ 1 & -2 & -8 & 0 \end{array}\right]$$

Linear (In)dependence

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Thus the third vector is redundant information

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

NO. They're INDEPENDENT. The only way to solve

$$a_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + a_2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$

Is if BOTH coefficients are 0

Span

- Span (Noun) = All linear combinations of a set of vectors {v1, …, vk}
  - Span({v1, .. vk}) = a1*v1 + … + ak*vk for any scalars a1, …, ak

Vector Spaces and Subspaces
Basis and Dimension

Vector Space (formally)

- Definition: A set of vectors V, scalars F, that satisfy the following vector addition and scalar multiplication operations properties (from Note 7)

Exercise: Try to show these properties hold for V = R^n, F = R

- Vector Addition
  - Associative: $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$ for any $\vec{v},\vec{u},\vec{w}\in\mathbb{V}$.
  - Commutative: $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ for any $\vec{v},\vec{u}\in\mathbb{V}$.
  - Additive Identity: There exists an additive identity $\vec{0}\in\mathbb{V}$ such that $\vec{v}+\vec{0}=\vec{v}$ for any $\vec{v}\in\mathbb{V}$.
  - Additive Inverse: For any $\vec{v}\in\mathbb{V}$, there exists $-\vec{v}\in\mathbb{V}$ such that $\vec{v}+(-\vec{v})=\vec{0}$. We call $-\vec{v}$ the additive inverse of $\vec{v}$.
  - Closure under vector addition: For any two vectors $\vec{v},\vec{u}\in\mathbb{V}$, their sum $\vec{v}+\vec{u}$ must also be in $\mathbb{V}$.
- Scalar Multiplication
  - Associative: $\alpha(\beta\vec{v})=(\alpha\beta)\vec{v}$ for any $\vec{v}\in\mathbb{V}$, $\alpha,\beta\in\mathbb{R}$.
  - Multiplicative Identity: There exists $1\in\mathbb{R}$ where $1\cdot\vec{v}=\vec{v}$ for any $\vec{v}\in\mathbb{V}$. We call 1 the multiplicative identity.
  - Distributive in vector addition: $\alpha(\vec{u}+\vec{v})=\alpha\vec{u}+\alpha\vec{v}$ for any $\alpha\in\mathbb{R}$ and $\vec{u},\vec{v}\in\mathbb{V}$.
  - Distributive in scalar addition: $(\alpha+\beta)\vec{v}=\alpha\vec{v}+\beta\vec{v}$ for any $\alpha,\beta\in\mathbb{R}$ and $\vec{v}\in\mathbb{V}$.
  - Closure under scalar multiplication: For any vector $\vec{v}\in\mathbb{V}$ and scalar $\alpha\in\mathbb{R}$, the product $\alpha\vec{v}$ must also be in $\mathbb{V}$.

Vector **Sub**space

- To verify that a vector space V is a subspace of R^n (or any other vector space) check that it is:
  - A subset of R^n
  - If $\overrightarrow{v_1}, \overrightarrow{v_2}$ are in V then $\overrightarrow{v_1} + \overrightarrow{v_2}$ are also in V (closed under *vector addition*)
  - If $\overrightarrow{v_1}$ is in V then $a\overrightarrow{v_1}$ is also in V for any $a \in \mathbb{R}$ (closed under *scalar multiplication*)
  - Contains the zero vector

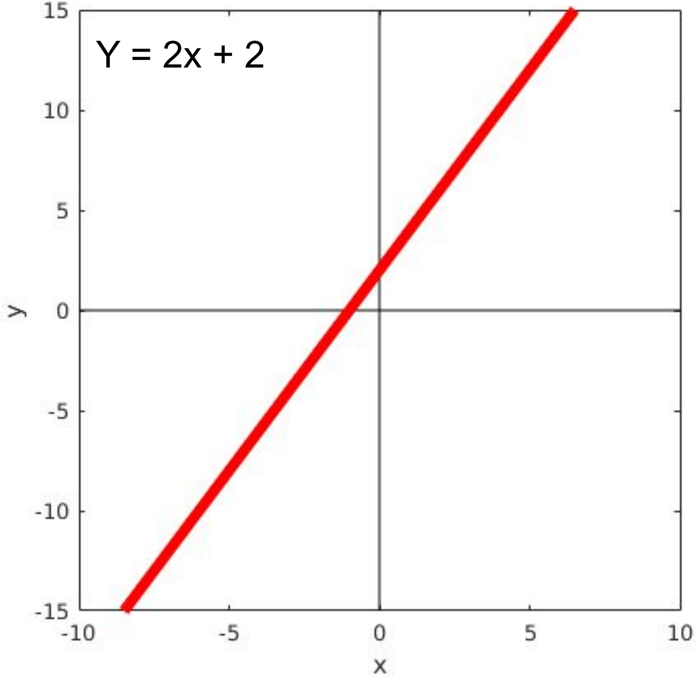
Vector **Sub**space

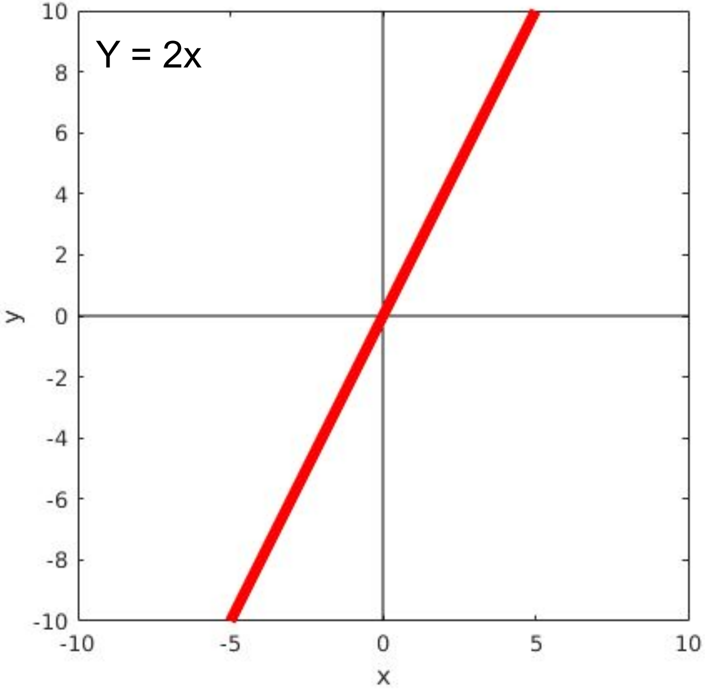
- The previous conditions imply that V itself is a vector space and so if you take a linear combination of any vectors (v_1, v_2) in V, then the resulting vector (w) stays in V.

$$a_1\overrightarrow{v_1} + a_2\overrightarrow{v_2} = \overrightarrow{w}$$

- **Note:** We refer to R^n as the parent vector space of the subspace V. Also, V is a subspace of itself.

Are these subspaces of V = $\mathbf{R}^2$?





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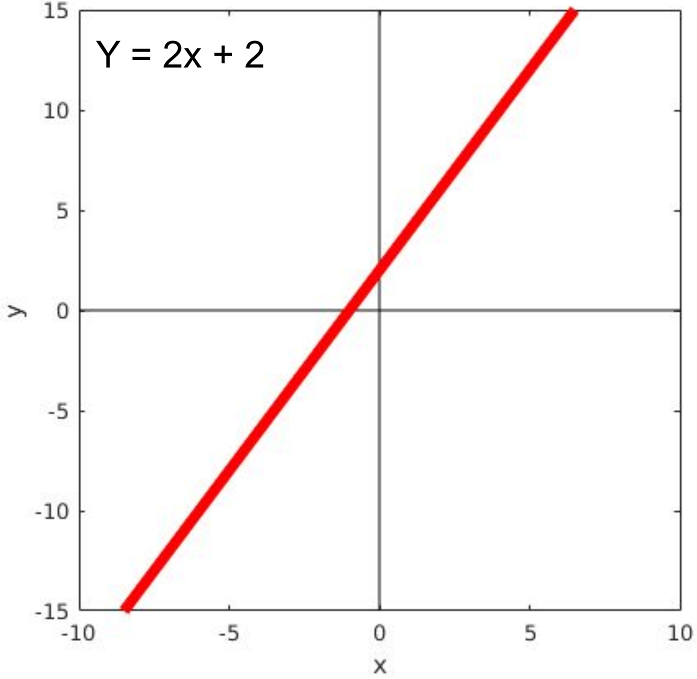
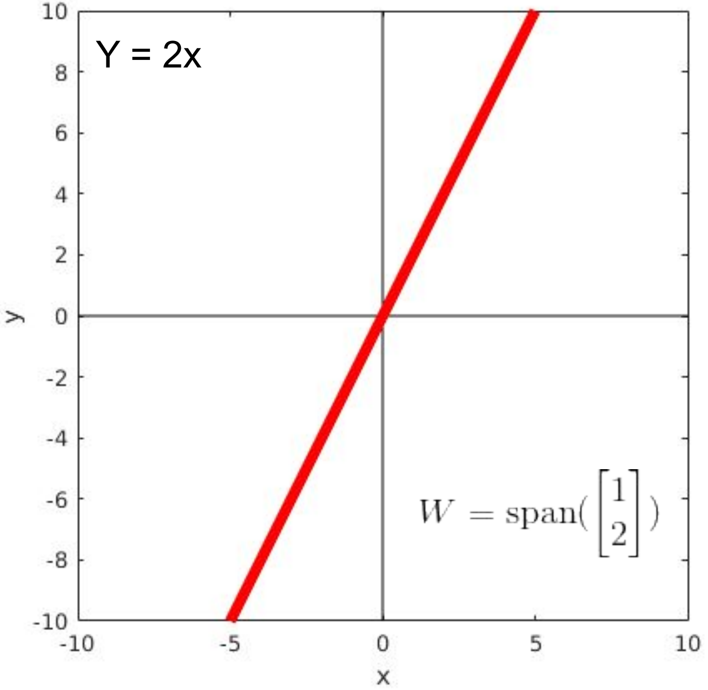


No. Not closed under vector addition or scalar multiplication and doesn't contain the zero vector.



Yes. The line contains the zero vector. We can take linear combinations of points on the line and stay on the line.

Span

- Span (Noun) = All linear combinations of a set of vectors {v1, …, vk}
  - Span({v1, .. vk}) = a1*v1 + … + ak*vk for any scalars a1, …, ak

- Span (Verb) / Spanning (Adj)
  - A list of vectors {v1,..,vk} spans a vector space, V, if every vector in V is in the span({v1,..,vk})

Basis and Dimension

- Basis: a set of vectors that are spanning the space + linearly independent
  - A minimum, spanning set of vectors for a given vector space
- Dimension: number of vectors in the basis
  - All possible bases for a vector space contain the same number of vectors
  - The number of vectors in the basis is the dimension
- Helpful fact: any set of n linearly independent vectors in a n-dimensional vector space is a basis for that space.

Column Space, and Null Space

Column Space

- Let **A** have columns $A_1, \ldots, A_n$.
- Column Space(**A**) = Span($\{A_1, \ldots, A_n\}$)
- Range(**A**) = What you can "reach" with that matrix
  - AKA all y such that there exists an x where **A**x = y
  - AKA Column space!
- These ideas are equivalent:
  - Column Space(**A**) = span($\{A_1, .., A_n\}$) = Range(**A**)
- The dimension of Range(**A**) is called the rank of A.

Null Space

- Null(**A**) is the set of vectors that map to a zero output:
  - Null(**A**) = {x such that **A**x = 0}
- This is related to the idea of linear independence:
  - If columns of **A** are linearly independent, then Null(**A**) = {0}, i.e. only the zero vector is in the null space for an n x n matrix **A**
  - If columns of **A** are linearly dependent, then the nullspace is a subspace of dimension 1 or more

Why do Column Space(**A**) and Null(**A**) matter?

- One reason is that it helps us to understand solutions to Ax = b
- Existence of a solution
    - If b is not in the Column Space(**A**), then there is no solution to **A**x = b
- Uniqueness of a solution
    - Suppose b is in the Column Space(**A**), so there is at least one solution.
    - Then Null(**A**) tells us if that solution is unique or not.
    - If Null(**A**) is just the zero vector, then we have a unique solution to **A**x = b.
    - If Null(**A**) is a subspace of dimension 1 or more, we have infinite solutions to **A**x = b
        - Why? Pick any non-zero vector in Null(**A**). Call it v_null.
        - Let x = u be some solution to **A**x = b.
        - **A**(u + v_null) = b + 0 = b
        - So x = u + v_null is another solution to **A**x = b. We can scale v_null and get an infinite set of solutions to **A**x = b.

How to Find Null Space

- Begin with Gaussian Elimination for $\mathbf{A}x = 0$
- Look at constraints/equations in reduced form
- Identify free variables; write all other variables in terms of these
  - Select the variables that don’t have corresponding pivots
- Write out the solution vector in terms of the free variables
- Factor out the free variables so you have a linear combination of vectors, with the free variables as the scalars
- Columns being scaled by free variables form a basis
- See Dis 4A for examples

Example: Null Space of A

$$\mathbf{A} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

$$\mathbf{A}\vec{x} = \vec{0}$$

Step 1: Gaussian Elimination

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 2 & 5 & 8 & 0 \\ 3 & 6 & 9 & 0 \end{array}\right]$$

Step 2: Identify free variables (z in this case)

$$\left[\begin{array}{ccc|c} 1 & 4 & 7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Step 3: Write solution in terms of free parameters

$$z = t, t \in \mathbb{R}$$

$$y = -2t$$

$$x = -7z - 4y = t$$

Step 4: Factor free parameters - the constant vector(s) form a basis for N(A)

$$\vec{x} = \begin{bmatrix} t \\ -2t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Basis for N(A)

Step 5: Express N(A) using that basis

$$\boxed{N(A) = \text{span}\{\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\}}$$

Is it Invertible?

- Matrix or vector division does not exist!!!
- Only square matrices can have inverses (if they are invertible).
- **A** (n x n) is not invertible if it
  - it has linearly dependent rows (or columns)
  - has a non-trivial null space (a null space that doesn’t just contain zero)
  - has a zero eigenvalue (more on eigenvalues later)
  - has zero determinant (more on determinants later)
- If any of the statements in the above list are false, then **A** is invertible.

Determinants

- For this class, you need to know 2x2 determinants

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

- Determinants encode some very neat info about **square** matrices
- They are linked to the area (2D) of the parallelogram or volume (3D) or parallelepiped formed by the rows (or cols) of the **A** matrix
- Important for finding eigenvalues of a matrix **A**
- det(**A**) = 0 → non-empty Null(**A**)

Eigenvalues and Eigenvectors

Eigenvectors/Eigenvalues

$$A\vec{v} = \lambda\vec{v}$$

Eigenvectors/Eigenvalues (and Eigenspaces)

- An ***eigenvector*** of matrix **A** is a **non-zero** vector such that applying matrix **A** to it yields the **SAME** exact vector, but scaled by a constant.
- The constant that the vector is scaled by is the ***eigenvalue***
- We say that this ***eigenvector*** belongs to the ***eigenspace*** *corresponding to that* ***eigenvalue***.

$$A\vec{v} = \lambda\vec{v}$$

Finding Eigenvalues

$$A\vec{v} = \lambda\vec{v} \Rightarrow (A - \lambda I)\vec{v} = 0$$

1) Find the eigenvalues by **<u>solving</u>** the equation resulting from:

   $$\det(A - \lambda I) = 0$$

   It will be some polynomial whose roots are the eigenvalues

   <u>Note:</u> You can definitely have repeated eigen**<u>values</u>** (multiple roots)

Finding the *corresponding* Eigenvectors/Eigenspaces

1) Plug the eigenvalue λ we just found back into $(A - \lambda I)\vec{v} = 0$

2) Solve for $\vec{v}$

   In other words find the ***nullspace*** of the matrix $(A - \lambda I)$

3) The solution will be the ***eigenspace*** *corresponding to the* ***eigenvalue*** λ. Any **non-zero** vector in that space is an ***eigenvector*** *corresponding to the* ***eigenvalue*** λ.

Final Eigenvalue/Eigenvector Notes

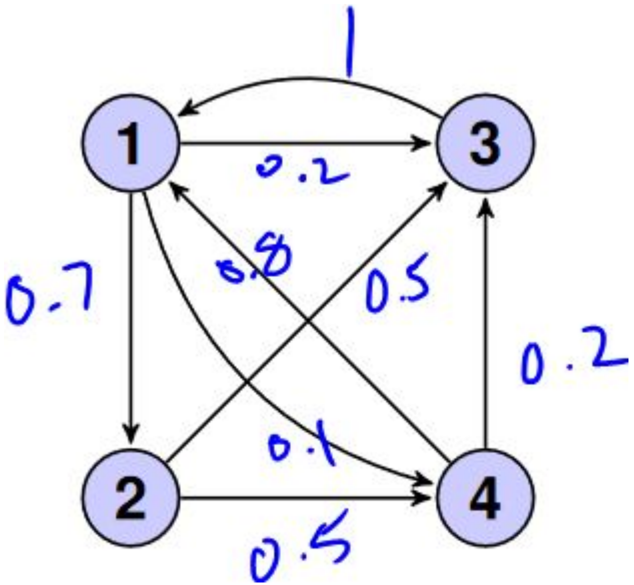
1) The 0 vector can never be an **eigenvector** (by definition)
2) To find the ***eigenvectors***, you first must find the ***eigenvalues***
3) Equation $\det(A - \lambda I) = 0$ gives us all ***eigenvalues*** λ (lambda) for which equation $(A - \lambda I)\vec{v} = 0$ has **non-zero** solutions (i.e. the nullspace of $(A - \lambda I)$ is non-trivial)
4) To find **all** pairs of eigenvalues/eigenvectors(spaces), plug in the ***eigenvalues*** (one at a time) into $(A - \lambda I)\vec{v} = 0$ and solve or find the null space of $(A - \lambda I)$ as outlined in the previous slide. That will be the *corresponding* ***eigenspace***

PageRank

Flow Matrices and Graphs

- A system that updates its state every “time step”
- Usually represented by nodes connected by directed edges with some “weight”
  - Weight can mean different things depending on the system.
    - Water tanks → percentage of water that goes to a different (or the same) tank
    - Pagerank → percentage of people that goes to a different webpage

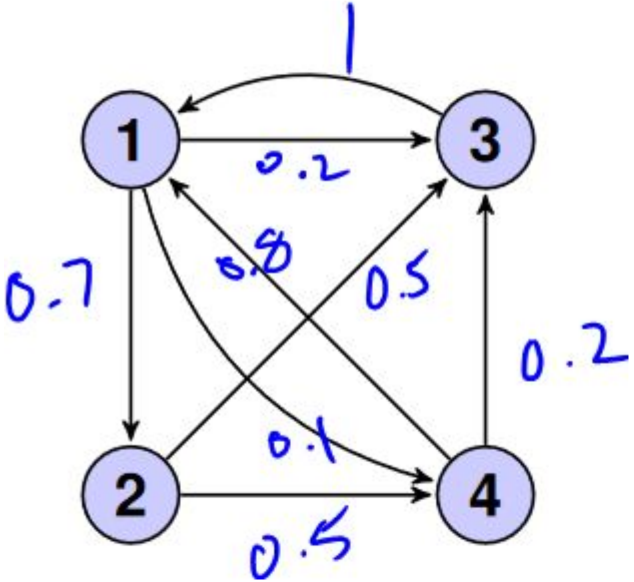
Flow Matrices and Graphs



Node 1 to all other nodes

$$T = \left[\qquad \qquad \right]$$

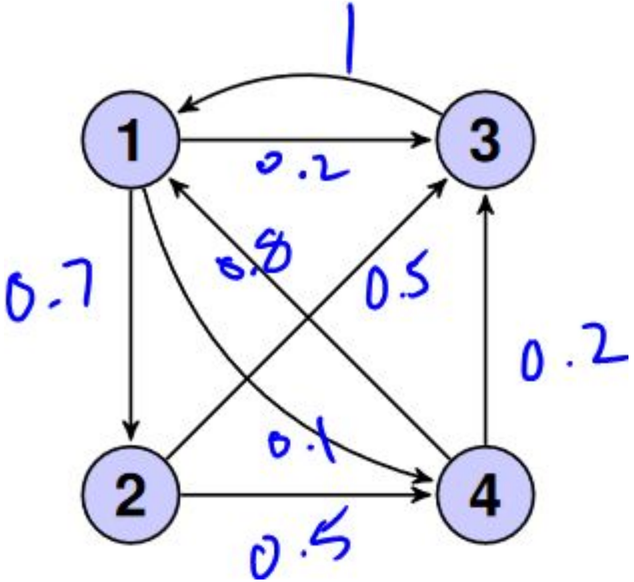
Flow Matrices and Graphs



$$T = \left[\qquad \qquad \qquad \right]$$

All other nodes to node 1

Flow Matrices and Graphs



Node 1 to all other nodes

$$T = \begin{bmatrix} 0 & 0 & 1 & 0.8 \\ 0.7 & 0 & 0 & 0 \\ 0.2 & 0.5 & 0 & 0.2 \\ 0.1 & 0.5 & 0 & 0 \end{bmatrix}$$

All other nodes to node 1

Flow Matrices and Graphs

- Conservative system?

  No “stuff” leaves as time progresses. A system is conservative if the columns sum to 1 --- Remind them of HW problem

  - A column with sum >1 has “stuff” entering from outside. NOT conserved, states go to infinity
  - A column with sum < 1 has “stuff” leaking out. States will trend towards 0.
- State transitions?
  - We have a state as a vector s[n]
  - Transition matrix **T** describes how the state changes at each time step.

Flow Matrices and Graphs

- Next states? Previous states?
  - We have a state as a vector s[n] which represents how much stuff is at which node
  - Transition matrix **T** describes how the state changes at each time step.

$$T\vec{s}[n] = \vec{s}[n+1]$$

  - We can advance many timesteps by computing powers of **T**

$$T^N\vec{s}[n] = \vec{s}[n+N]$$

  - We can go backward in time to recover previous states IF **T** IS INVERTIBLE

$$T^{-1}\vec{s}[n+1] = \vec{s}[n]$$

Flow Matrices and Graphs

- Steady state?
- We're interested in a state where updating to the next step does not change the state vector
  - $$T\vec{s}[n] = \vec{s}[n+1] = \vec{s}[n]$$
  - Maybe it looks more familiar written this way…

$$A\vec{v} = \vec{v}$$

Flow Matrices and Graphs

- Steady state?
  - We’re interested in a state w
  - In other words…

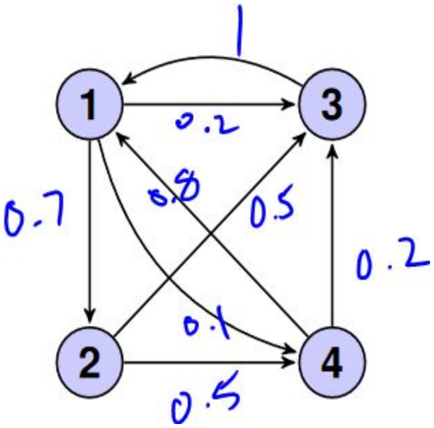
$$A\vec{v} = \lambda\vec{v}$$

$$\lambda = 1$$

Flow Matrices and Graphs

- We want to find the vector that does not change when **T** operates on it
- We can find it with a foolproof (not fun) method of taking **A** to the nth power times s[n], then take the limit as n approaches infinity
- OR we can find the steady state with linear algebra
  - We want the eigenvector that has an eigenvalue of 1. This vector will not change with **T**
  - Find the nullspace of (A-I)

Flow Matrices and Graphs



$$T = \begin{bmatrix} 0 & 0 & 1 & 0.8 \\ 0.7 & 0 & 0 & 0 \\ 0.2 & 0.5 & 0 & 0.2 \\ 0.1 & 0.5 & 0 & 0 \end{bmatrix}$$

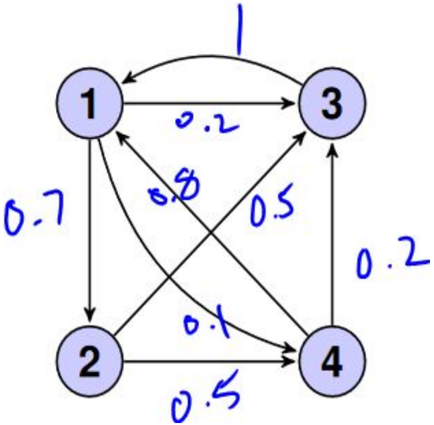
- There's an eigenvalue of 1. Thus a steady state exists

- The corresponding eigenvector is:

$$\overrightarrow{v_1} = \begin{bmatrix} 0.6897 \\ 0.482 \\ 0.441 \\ 0.3103 \end{bmatrix}$$

- There's just one last thing to do...

Flow Matrices and Graphs



$$T = \begin{bmatrix} 0 & 0 & 1 & 0.8 \\ 0.7 & 0 & 0 & 0 \\ 0.2 & 0.5 & 0 & 0.2 \\ 0.1 & 0.5 & 0 & 0 \end{bmatrix}$$

$$\overrightarrow{v_1} = \begin{bmatrix} 0.6897 \\ 0.482 \\ 0.441 \\ 0.3103 \end{bmatrix} \rightarrow \begin{bmatrix} 0.3584 \\ 0.2508 \\ 0.2293 \\ 0.1612 \end{bmatrix}$$

- These numbers don't make sense for our system. We need to "normalize" it so that the total sums to 1, representing 100%
- Divide each entry by the sum of all values
- Therefore at steady state, 35.84% of the stuff is in node 1, 25.08% in node 2, etc.

Normalize to get percentages of population!!

Derivations and Proofs - Tips and Tricks

- Start with the fundamental equations about what you know, **in math**. Try to slowly involve the things you care about showing. Precise definitions are important!
- Select a strategy based on the proof type/wording (constructive proof vs proof by contradiction)
- If stuck, try a small example
- If your proof becomes a bit too unwieldy, try to start over with a different strategy
- More proof tips on this piazza post of Prof. Courtade Piazza - Proofs

Derivations and Proofs - Example

Show that if a 2x2 matrix has distinct eigenvalues λ_i then the corresponding eigenvectors form a basis for R^2

Given: $A\vec{v_1} = \lambda_1 \vec{v_1}$
$A\vec{v_2} = \lambda_2 \vec{v_2}$
$\lambda_1 \neq \lambda_2$

Want to show: $\vec{v_1}, \vec{v_2}$ form a basis for R^2

$\Rightarrow$ $\vec{v_1}, \vec{v_2}$ are linearly independent

Strategy: Contradiction

Let's assume that $\vec{v_1}, \vec{v_2}$ can be linearly dependent, i.e $\vec{v_1} = a\vec{v_2}, a \neq 0$

$$A\vec{v_1} = A(a\vec{v_2}) = a(\overrightarrow{Av_2}) = a\lambda_2\vec{v_2}$$
$$A\vec{v_1} = \lambda_1\vec{v_1} = \lambda_1 a\vec{v_2} = a\lambda_1\vec{v_2}$$
$$\Rightarrow \lambda_1 = \lambda_2$$

Contradiction!

$\Rightarrow$ $\vec{v_1}, \vec{v_2}$ must be linearly independent

$\Rightarrow$ they span R^2

$\Rightarrow$ they form a basis

Practice Problems: True/False

a) If the augmented matrix of the system Ax = b has a pivot in the last column, then the system Ax = b has no solution.
b) If A and B are invertible 2×2 matrices, then $(AB)^{-1} = A^{-1}B^{-1}$
c) If A is a 3×3 matrix such that the system A**x** = 0 has only the trivial solution, then the equation A**x** = **b** is consistent for every **b** in R^3 .
d) If $T\,[x\ y]^T = [x\ 0]^T$, then $Nul(T) = span\{[1, 0]^T\}$
e) R^2 is a subspace of R^3
f) If Nul(A) = {0}, then A is invertible.
g) If {**v1, v2, v3**} are linearly independent vectors in R^n , then {**v1, v2**} is linearly independent as well.

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d) If $T\,[x\ y]^T = [x\ 0]^T$, then $\mathrm{Nul}(T) = \mathrm{span}\{[1, 0]^T\}$
e) R^2 is a subspace of R^3
f) If Nul(A) = {0}, then A is invertible.
g) If {**v1, v2, v3**} are linearly independent vectors in R^n , then {**v1, v2**} is linearly independent as well.

Practice Problems: True/False

a) If the augmented matrix of the system Ax = b has a pivot in the last column, then the system Ax = b has no solution.
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Practice Problem: Proofs

Spring ‘19 Question 4b

Let $\mathbf{U}$ and $\mathbf{V}$ be $n \times n$ matrices. If $\mathbf{UV} = \mathbf{0}$, prove that every vector in $\text{col}(\mathbf{V})$ is in $\text{nul}(\mathbf{U})$.

Eigenvalues and Eigenvectors(Fa18 Q6)

6. A Tropical Tale of Triumph: Does Pineapple Come Out on Top? (52 points)

(Based on a true story) During a discussion section, one of your TAs, Nick, makes the claim that pineapple belongs on pizza. Another TA, Elena, strongly disagrees. Naturally, a war starts and students begin to flock to the TA they agree with, switching discussion sections every week. Some students don't have an opinion and go to Lydia's section since she is neutral in the matter. As a 16A student, you want to analyze this war to see how it will play out.

(a) (6 points) You manage to capture the behavior of the students as a transition matrix, but want to visualize it. You've written out the transition matrix $\mathbf{M}$:

$$\mathbf{M} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0.25 & 0.5 & 1 \\ 0.25 & 0.5 & 0 \end{bmatrix}$$

such that

$$\begin{bmatrix} x_{\text{Elena}}[n+1] \\ x_{\text{Nick}}[n+1] \\ x_{\text{Lydia}}[n+1] \end{bmatrix} = \mathbf{M} \begin{bmatrix} x_{\text{Elena}}[n] \\ x_{\text{Nick}}[n] \\ x_{\text{Lydia}}[n] \end{bmatrix}.$$

(b) (10 points) Your friend Vlad tells you that your transition matrix **M** was wrong, and gives you a new transition matrix **S**, **which has a steady state**. In order to find who wins the war, you need to find how many students end up in each section after everything has settled. **Find a vector $\vec{x}$ that represents a steady state of S**.

$$\mathbf{S} = \begin{bmatrix} 0.2 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0.3 & 0 & 1 \end{bmatrix}$$

(c) (6 points) Your other friend Gireeja points out that the arguments are causing new people to join the sections and others to leave entirely. In other words, the system is not conservative! The new system can be modeled with a state transition matrix **A** that has the following eigenvalue/eigenvector pairings:

$$\lambda_1 = 1 : \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_2 = \frac{1}{2} : \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 2 : \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

You want the number of students in sections to stabilize. Which of the vectors below represent **steady states** of the system, i.e. $\vec{x}$ such that $\mathbf{A}\vec{x} = \vec{x}$? **Fill in the circle(s) to the left of these vector(s).**

○ $\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} 513 \\ 513 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix}$

○ $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 1026 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 1026 \\ 0 \end{bmatrix}$

(d) (6 Points) Assume we are still working with the same state transition matrix **A** as in part (c). Which of the vectors below represent **initial states** such that the number of students in the sections keeps growing? **Fill in the circle(s) to the left of these vector(s).**

○ $\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} 513 \\ 513 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix}$

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(e) (6 points) Again assume we are still working with the same state transition matrix $\mathbf{A}$ as in part (c). Which of the vectors below represent **initial states** such that everyone leaves the system, i.e. $\lim_{n\to\infty} \mathbf{A}^n\vec{x} = \vec{0}$? **Fill in the circle(s) to the left of these vector(s).**

○ $\begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ○ $\begin{bmatrix} 513 \\ 513 \\ 0 \end{bmatrix}$ ○ $\begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix}$

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(f) (16 Points) Let us generalize the idea of convergence. Consider the following system:

$$\vec{x}[n+1] = \mathbf{T}\vec{x}[n]$$

where $\vec{x}$ is a vector with N elements and $\mathbf{T}$ is any $N \times N$ matrix unrelated to the previous parts. $\mathbf{T}$ has N distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$, and N associated eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_N$ such that $\mathbf{T}\vec{v}_i = \lambda_i \vec{v}_i$ for $1 \leq i \leq N$. Let $|\lambda_i| > 1$. Prove that there exists at least one initial state $\vec{x}[0]$ for this system such that it does not converge to a steady state.

From Fall 2016 MT 1

6. Directional Shovels (10 points)

Kody and Nara were found exceptional at taking measurements to figure out light intensities, and they were both granted admission to a graduate school. Unfortunately, they both supported their new school's football team while they were playing against Berkeley and angry Berkeley fans found them and left them in a room at an unknown location under the ground. As compassionate people, Berkeley fans left some tools in the room that can help them escape.

(a) Kody found a shovel in the room and figured that it can operate in the following directions: $\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$. Is it possible for them to escape to Berkeley by digging in the given directions to a point which is located at $\begin{bmatrix}3\\-2\\5\end{bmatrix}$ given that they are at point $\begin{bmatrix}0\\0\\0\end{bmatrix}$? If so, find the scalars that multiply the vectors such that they reach Berkeley.

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(b) While Kody was busy planning his escape to Berkeley, Nara found a pick-axe in the room that can operate in the following directions: $\left\{\begin{bmatrix}2\\2\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}3\\-2\\5\end{bmatrix}\right\}$. Nara is convinced that the axe she found is better, but Kody disagrees. Show that Kody's shovel can reach anywhere that Nara's pick-axe can.

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From Fall 2019 MT1

3. Campfire Smores (11 points)

Patrick and SpongeBob are making smores.

There are three ingredients: **Graham Crackers, Marshmallows, and Chocolate**. To make a smore, SpongeBob needs: s_g Graham Crackers, s_m number of Marshmallows, and s_c Chocolate.

Ingredients	**Amount Needed**
Graham Crackers (s_g)	10
Marshmallows (s_m)	14
Chocolate (s_c)	20

Table 3.1: SpongeBob's smore

They find out that these ingredients are only stored in bundles as below:

Lobster Pack (p_l)
6 graham crackers
4 marshmallows
2 chocolates

Mr. Krabs Pack (p_k)
2 graham crackers
2 marshmallows
1 chocolates

Squidward Pack (p_s)
3 graham crackers
3 marshmallows
5 chocolates

Gary Pack (p_g)
1 graham crackers
4 marshmallows
5 chocolates

Pearl Pack (p_p)
2 graham crackers
3 marshmallows
2 chocolates

Table 3.2: Amount of Ingredients per Bundle

Spongebob and Patrick need to know how many of each bundle to buy: number of "Lobster" Packs, p_l, number of "Mr. Krabs" Packs, p_k, number of "Squidward" Packs, p_s, number of "Gary" Packs, p_g, and number of "Pearl" Packs, p_p.

(a) (3 points) How many equations/constraints does the information in the problem provide you with?

(b) (4 points) Based on the information provided in Tables 3.1 and 3.2, **write** an equation of the form $\mathbf{A}\vec{p} = \vec{s}$ that SpongeBob can use to decide how many of each pack to buy. Here, $\vec{p} = \begin{bmatrix} p_l \\ p_k \\ p_s \\ p_g \\ p_p \end{bmatrix}$.

(c) (4 points) Now, the ingredients in the packets $(\mathbf{A})$ and Spongebob's receipe $(\vec{s})$ change. We have:

$\mathbf{A} = \begin{bmatrix} 1 & 1 & 3 & 2 & 2 \\ 0 & 1 & 3 & 0 & 2 \\ 1 & 3 & 9 & 2 & 6 \end{bmatrix}$, and $\vec{s} = \begin{bmatrix} 3 \\ 2 \\ 10 \end{bmatrix}$.

Find a $\vec{p}$ that satisfies $\mathbf{A}\vec{p} = \vec{s}$. If no solution exists, explain why not.

General Tips

If you get stuck on a question, move on and then come back.

Try to spend your time answering things you know well, and feel free to skip parts if you get stuck. You can always go back to the tougher parts if you have time.

Show your work

But also be aware that you can only assume things that were shown on homework and in class. You cannot reference some theorem that was not covered as your proof.

Good luck!