Welcome to Module 3!

- MT2 scores released soon

- Course Notes, same as module 1: I'll update

Module 1: Focus on modeling, using tools from linear algebra.

Module 2: Analysis & Design of physical systems.

Module 3: On "optimization" and "machine learning".

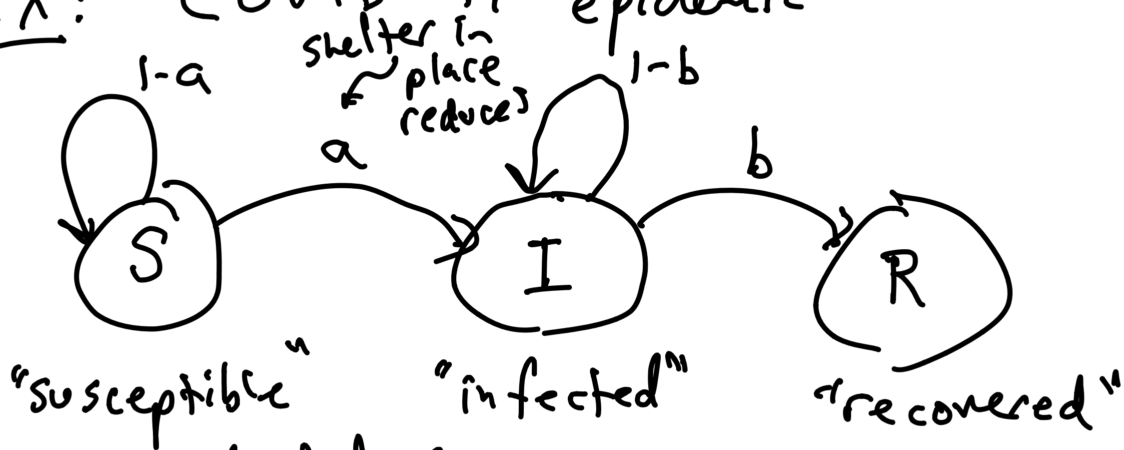
Module 1: How to solve consistent systems of LEQs of the form $A\vec{x} = b$.

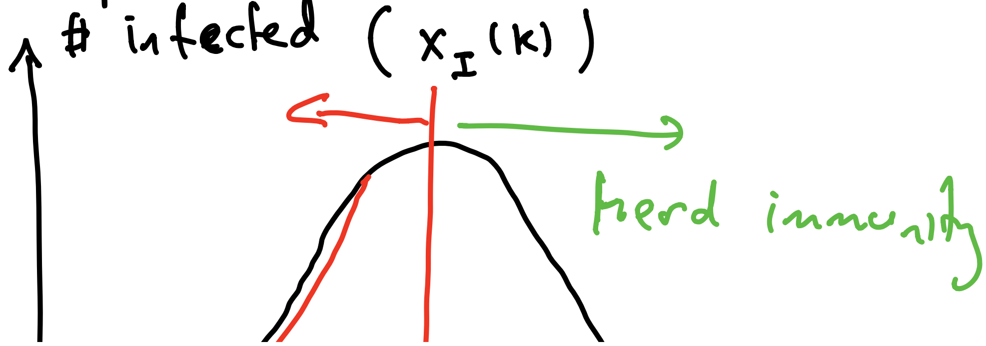
unknowns

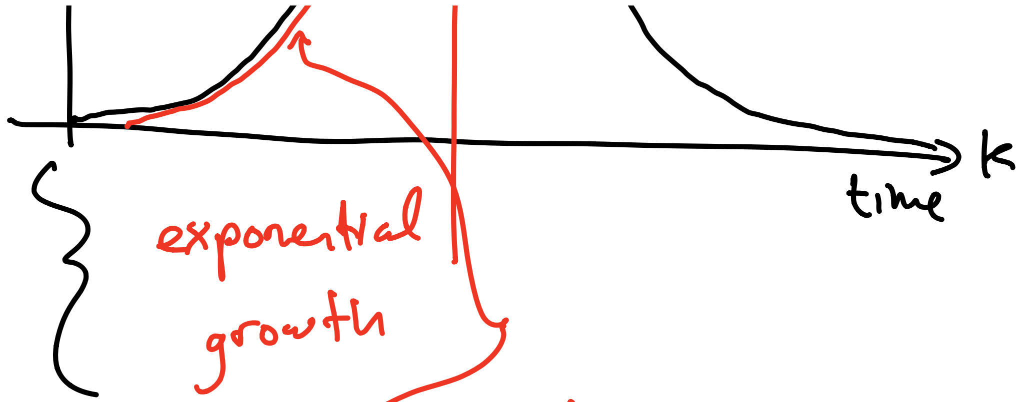
Module 3: Approximately solving <u>inconsistent</u> systems of LEQs of form $A\vec{x} \approx \vec{b}$.

inconsistent $\Rightarrow$ no choice of $\vec{x}$ which gives equality.

<u>Ex</u>: COVID-19 epidemic







$x_I(k) \approx e^{\beta k}$

Growth rate β is a function of the model parameters $\underbrace{a, b}$.

we don't know these in real-life.

What do we know?

Observe # deaths

Observe # infected individuals (assuming widespread testing)

Say, we observe i_k infected individuals on day $k \geq 0$. $k = 0$ = date of first infection

infection

Model predicts:

$$i_k \approx e^{\beta k} \qquad 0 \le k \le T$$

unknown rate of epidemic growth.

$$0 \cdot \beta \approx \log i_0$$
$$1 \cdot \beta \approx \log i_1$$
$$\vdots$$
$$T \cdot \beta \approx \log i_T$$

$$\begin{bmatrix} 0 \\ 1 \\ 2 \\ \vdots \\ T \end{bmatrix} \beta = \begin{bmatrix} \log i_0 \\ \vdots \\ \vdots \\ \log i_T \end{bmatrix}$$

generally inconsistent system of LEQs.

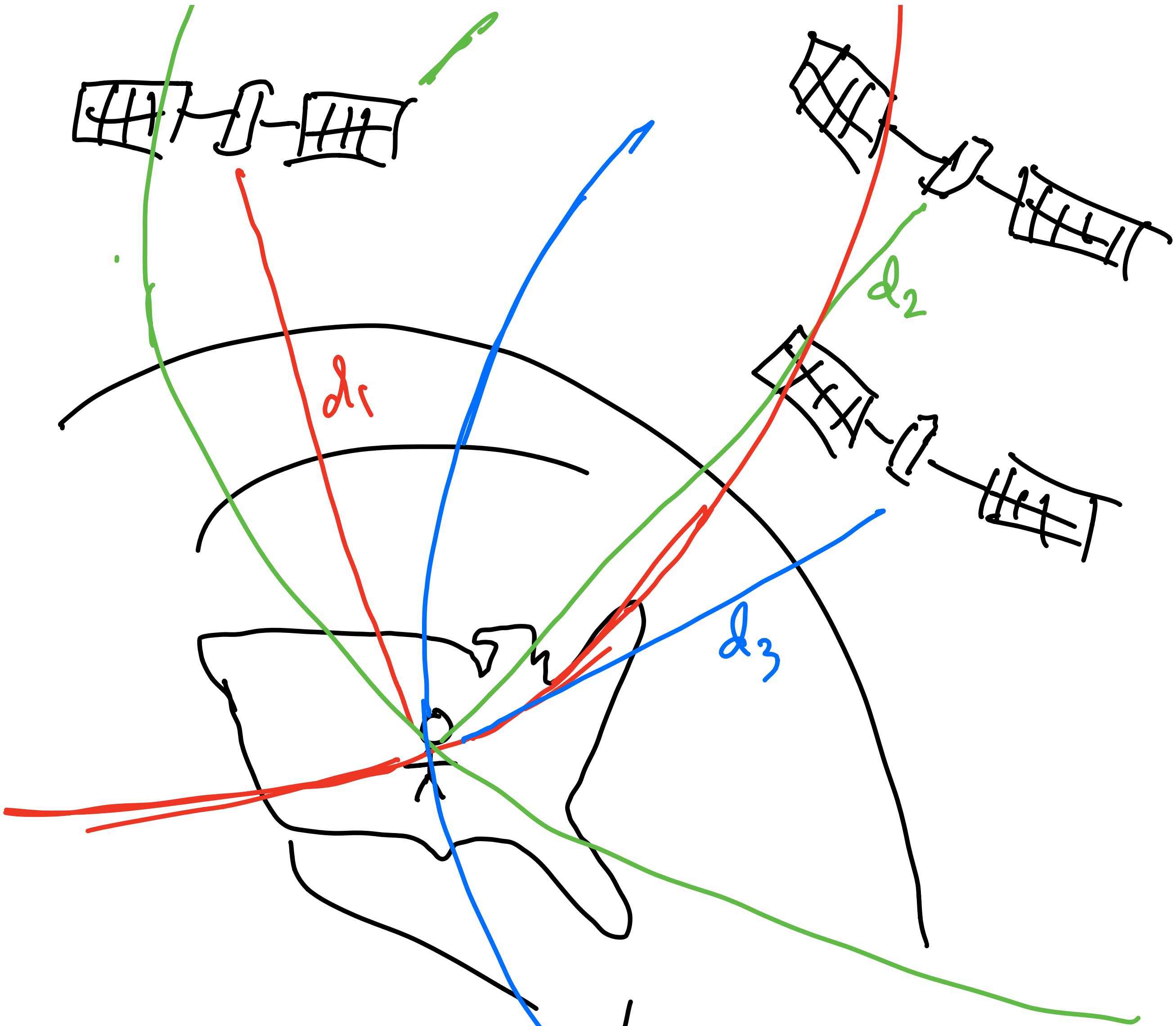
ML Problems: ~~Basically~~ Often boil down to inferring model parameters from observed data.

Some flavors in this module:

- Classification — Q: How to tell whether picture is dog or cat?
- Estimation — Q: How to estimate model parameters from data.
- Prediction — Q: How to predict stocks tomorrow based on past performance.

Today: Introduce localization problem (GPS), which will be running example in this module.

Design Problem: GPS



24 satellites, ensures 8 satellites visible from any point.

Q: ① How do we estimate distance?
↳ estimation.

② which satellite am I hearing?

↳ Classification

③ How to synthesize imperfect estimates?

Note: All of these problems can be solved by clever applications of L.A.

Note: These same techniques underly all communication systems.

Inner Products:

For a real vector space $\mathcal{V}$,

a mapping $\underline{\vec{u}, \vec{v} \in \mathcal{V} \longrightarrow \langle \vec{u}, \vec{v} \rangle \in \mathbb{R}}$

is called an inner product if it satisfies:

1) Symmetry: $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$ ✓

2) Linearity in first argument: ✓

$$\langle \alpha \vec{u}, \vec{v} \rangle = \alpha \langle \vec{u}, \vec{v} \rangle \qquad \alpha \in \mathbb{R}$$

$$\langle \vec{u}+\vec{w}, \vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{w}, \vec{v} \rangle$$

3) Positive-definiteness: ✓

$$\langle \vec{v}, \vec{v} \rangle \geq 0 \text{ , with equality iff } \vec{v} = \vec{0}.$$

Big Idea: Why care about inner products?

They allow us to measure "similarity" between vectors.

Ex: "Euclidean" inner product on $\mathbb{R}^n$:

$\vec{x}, \vec{y} \in \mathbb{R}^n$, $\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y} = \sum x_i y_i$. ✓

(dot product)

Other examples of inner products:

Ex: $Q \in \mathbb{R}^{n \times n}$ is symmetric matrix $\hookrightarrow$ satisfies (1)

with positive eigenvalues. $\hookrightarrow$ satisfies (3).

$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T Q \vec{y}$ is an inner product on $\mathbb{R}^n$.

Ex: $Q = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $V = \mathbb{R}^2$

$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T Q \vec{y} = x_1 y_1 + 3 x_2 y_2$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

Ex: V = set of functions $f: \mathbb{R} \to \mathbb{R}$ such that $\int_{-\infty}^{+\infty} |f(x)|^2 dx < \infty$.

$$\langle f, g \rangle = \int_{-\infty}^{+\infty} f(x) g(x) dx$$

This plays a huge role in EE 120.

Norms: Any inner product induces a corresponding "norm":

$\|\vec{v}\| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$

$\|v\| := \sqrt{\langle v, v \rangle}$.

In general, norm $\|\cdot\|$ satisfies 3 properties

1) Homogeneity: $\|\alpha \vec{v}\| = |\alpha| \, \|\vec{v}\|$

$\alpha \in \mathbb{R}$.

2) Non-negative: $\|\vec{v}\| \geq 0$, with equality only if $\vec{v} = \vec{0}$.

3) Triangle Inequality:

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

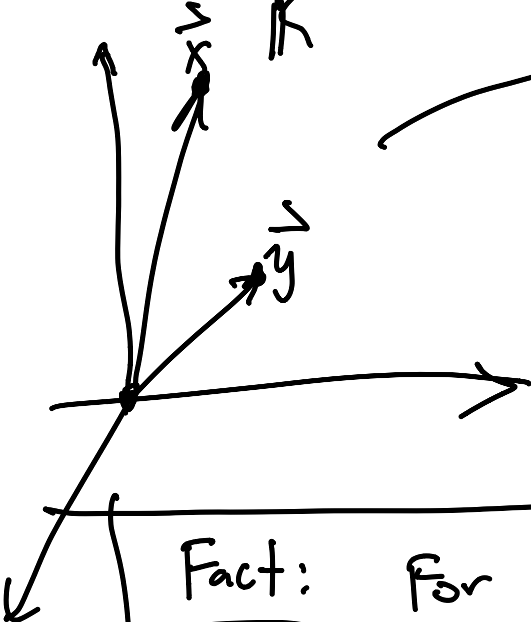
Note: Euclidean inner product induces norm:

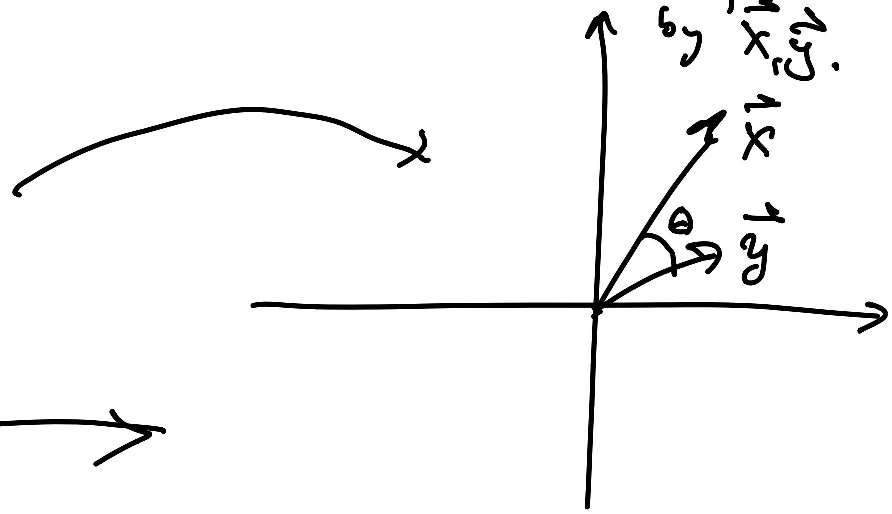
$$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \underbrace{\sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}}_{\text{Euclidean length}}$$

<u>Euclidean length.</u>

<u>Geometric Interpretation of Inner Product</u>

Lets stick with Euclidean Inner Product for now:



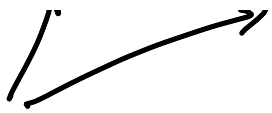


$\boxed{\begin{array}{l}\underline{\text{Fact:}} \text{ For Euclidean Inner Product} \\ \langle \vec{x}, \vec{y} \rangle = \|\vec{x}\| \, \|\vec{y}\| \cos\theta \end{array}}$

θ = angle between vectors $\vec{x}, \vec{y}$.

__Why?__ Can assume $\vec{x}, \vec{y} \neq \vec{0}$

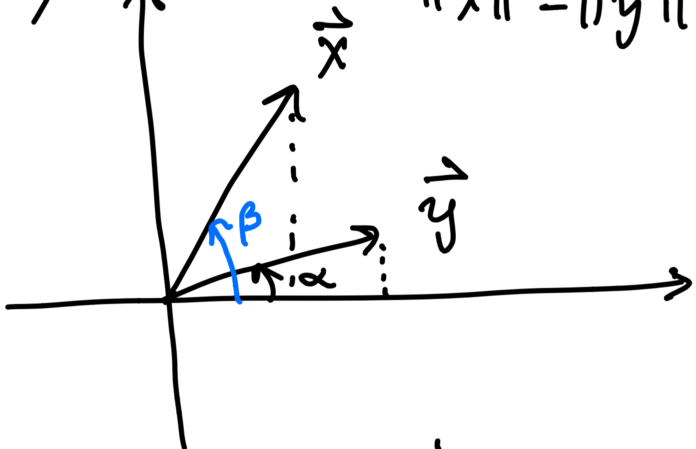
$$\left\langle \frac{\vec{x}}{\|\vec{x}\|}, \frac{\vec{y}}{\|\vec{y}\|} \right\rangle = \cos\theta$$



unit vectors in direction of $\vec{x}, \vec{y}$ resp.

E.g. $\left\| \frac{\vec{x}}{\|\vec{x}\|} \right\| = \frac{1}{\|\vec{x}\|} \|\vec{x}\| = 1.$

So, can assume $\|\vec{x}\| = \|\vec{y}\| = 1.$



$$\vec{x} = \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix} \qquad \vec{y} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$$

$$\langle \vec{x}, \vec{y} \rangle = \cos\beta \cos\alpha + \sin\beta \sin\alpha$$

$$= \cos(\underbrace{\beta - \alpha}_{\text{angle between } \vec{x} \text{ and } \vec{y}.})$$

trig identity

For general inner products, we can define a notion of angle btwn vectors using the same identity.

<u>Cauchy-Schwarz Inequality</u>

$$|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \, \|\vec{v}\|.$$

(For any inner product)