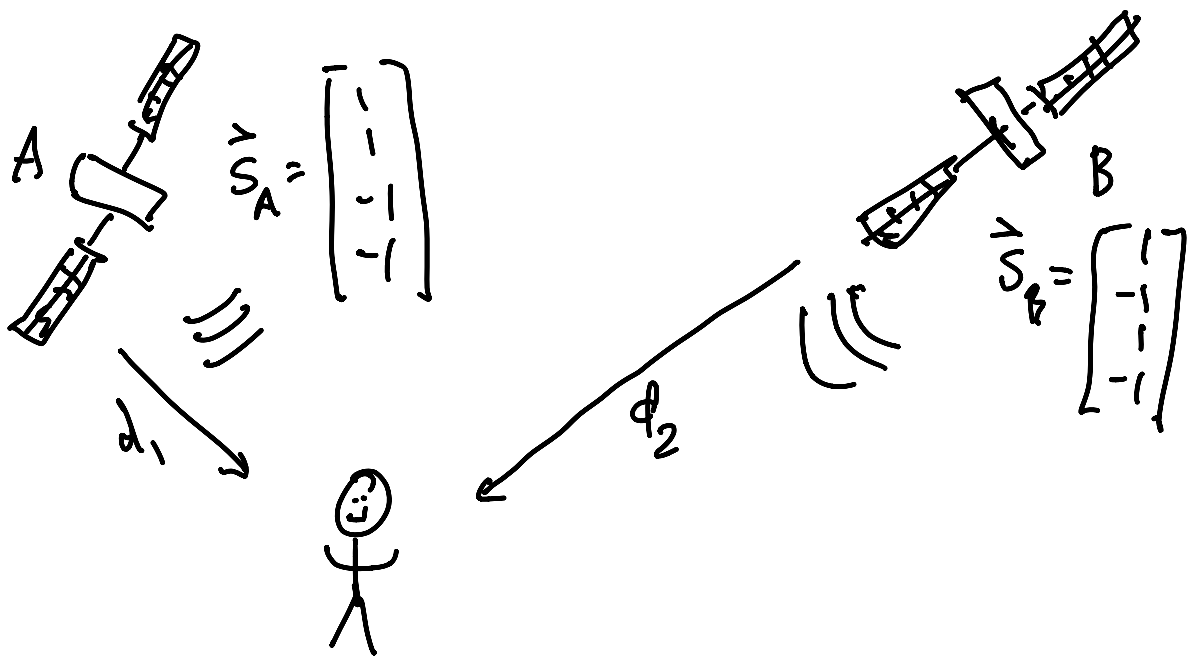
- ***Last Time***: orthogonality & inner products applied to classification/detection.

- ***Today***: More on crosscorrelation, Trilateration setup, intro to ***least squares***.



Two design problems:

① Interference management → use orthogonal signatures!

② ***Timing Estimation*** → use ***cross-correlation***

Time (k)

k=0 k=1 k=2 . . .

sat. A: $\cdots 0 \quad 0 \quad 1 \quad 1 \quad -1 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \cdots$

sat. B: $\cdots 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad -1 \quad 1 \quad -1 \quad 0 \quad 0 \quad 0 \cdots$

$r[k]: \cdots 0 \quad 0 \quad 1 \quad 1 \quad 0 \quad -2 \quad 1 \quad -1 \quad 0 \quad 0 \quad 0 \quad 0 \cdots$

received signal = superposition of tx signals shifted by respective delays.

$$\vec{S}_A^{\,T} = [1 \quad 1 \quad -1 \quad -1]$$

$r[k]$

k=0

$$\cdots [0 \quad [0 \quad [1 \quad [1 \quad] \quad 0] \quad -2] \quad 1] \quad -1 \quad 0 \quad 0 \quad 0 \cdots$$

$[1 \quad 1 \quad -1 \quad -1]$ ~ S_A shifted left by 2
right by -2

$1\cdot 0 + 1\cdot 0 + (-1)1 + (-1)\cdot 1$
$= -2$

$[1 \quad 1 \quad -1 \quad -1]$ ~ S_A shifted left by 1
" right by -1

$0\cdot 1 + 1\cdot 1 - 1\cdot 1 + 0\cdot(-1) = 0$

$$\begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix}$$ ← S_A shifted left by 0 " right by 0

$1\cdot 1 + 1\cdot 1 - 1\cdot 0 - 1\cdot(-2) = 4$

$$\begin{bmatrix} 1 & 1 & -1 & -1 \end{bmatrix}$$ ← S_A shifted left by -1 right by $+1$

$1\cdot 1 + 0\cdot 1 - 1(-2) - 1\cdot 1 = 2$

Cross-Correlation: For signals $x[k]$, $y[k]$, $k \in \mathbb{Z}$

$$\mathrm{Corr}_x(y)[k] = \sum_{i=-\infty}^{\infty} x[i]\, y[i-k]$$

$$\mathrm{Corr}_r(S_A)[k] = \sum_{i=-\infty}^{\infty} \underline{r[i]}\; \underline{S_A[i-k]}$$

$S_A[k] =$

$\ldots\ 0\ \ 0\ \ 1\ \ 1\ \ -1\ \ -1\ \ 0\ \ 0\ \ 0\ \ 0$

$k=0$

inner product of r with S_A shifted to right by k.

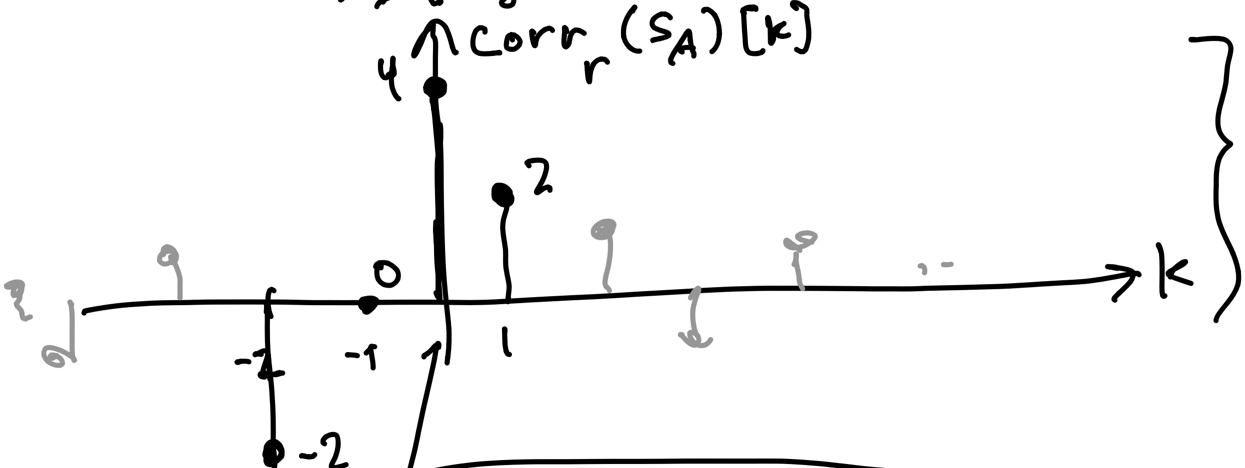
$$\mathrm{corr}_r(S_A)[-2] = -2$$

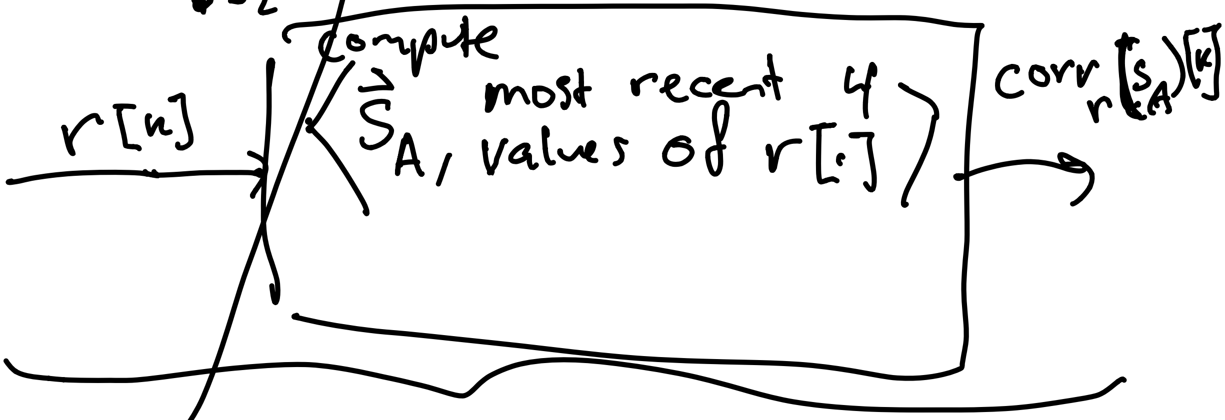
$$\mathrm{corr}_r(S_A)[-1] = 0$$

$$\mathrm{corr}_r(S_A)[0] = 4$$

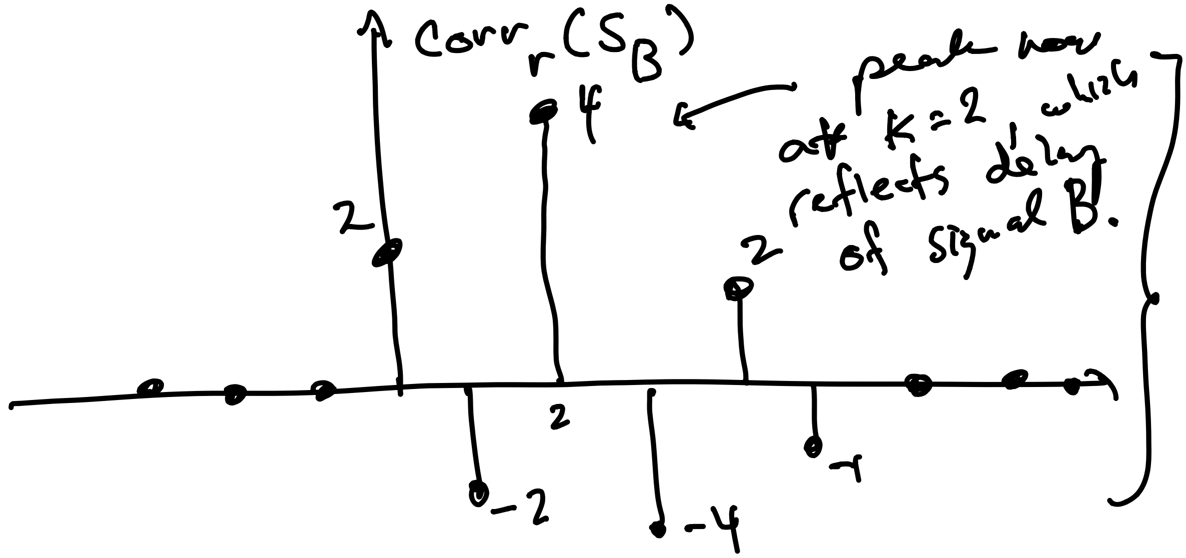
$$\mathrm{corr}_r(S_A)[1] = 2$$

$$\mathrm{corr}_r(S_A)[2] = \ldots$$

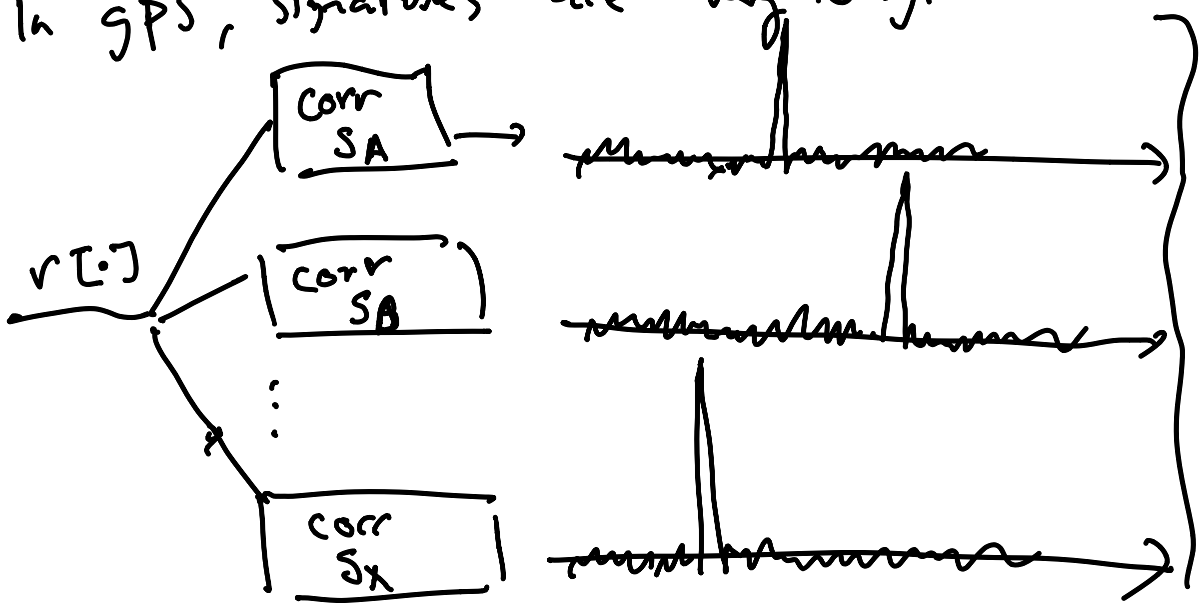




peak at $k=0$ tells us r is "most similar" to S_A at time $k=0$

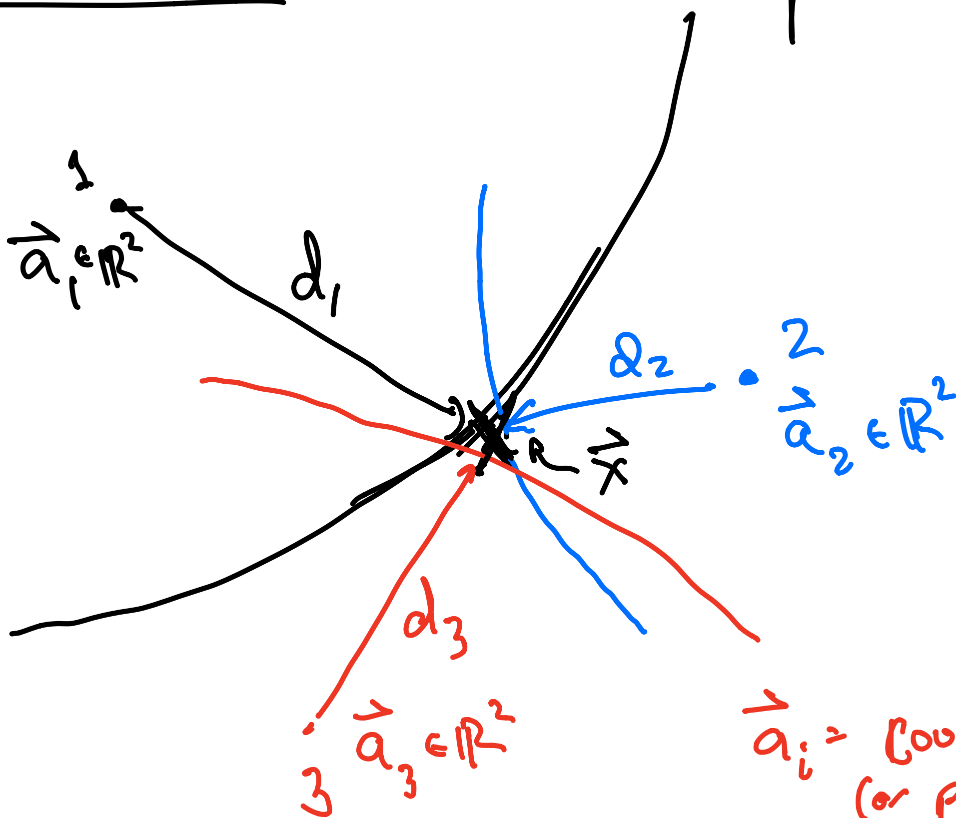


In GPS, signatures are very long.



Punchline: correlation is just a sequence of inner products. Finding when correlation is maximum tells us when signals are "most aligned", which reveals the delay we want to estimate.

Trilateration: How do we go from distance to position?



$\vec{a}_i$ = coordinates (or position) of sat. i

(known by design)

$$\left.\begin{aligned} &① \ \|\vec{x}-\vec{a}_1\|^2 = d_1^2 \\ &② \ \|\vec{x}-\vec{a}_2\|^2 = d_2^2 \\ &\quad \|\vec{x}-\vec{a}_3\|^2 = d_3^2 \end{aligned}\right\} \text{what I've measured.}$$

Q: How do we determine $\vec{x}$?

$$\textcircled{1}\ \|\vec{x}\|^2 - 2\langle \vec{x}, \vec{a}_1\rangle + \|\vec{a}_1\|^2 = d_1^2$$
$$\textcircled{2}\ \|\vec{x}\|^2 - 2\langle \vec{x}, \vec{a}_2\rangle + \|\vec{a}_2\|^2 = d_2^2$$
$$\textcircled{3}\ \|\vec{x}\|^2 - 2\langle \vec{x}, \vec{a}_3\rangle + \|\vec{a}_3\|^2 = d_3^2$$

3 Equns in 2 unknowns

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

These equations are quadratic, not linear!
Eliminate quadratic term by substracting

② − ①
③ − ①

$$\begin{cases} 2\langle (\vec{a}_1 - \vec{a}_2), \vec{x} \rangle = \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 - d_1^2 + d_2^2 \\ 2\langle (\vec{a}_1 - \vec{a}_3), \vec{x} \rangle = \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 - d_1^2 + d_3^2 \end{cases}$$

2 LEQs in 2 unknowns

$$\underbrace{\begin{bmatrix} 2(\vec{a}_1 - \vec{a}_2)^T \\ 2(\vec{a}_1 - \vec{a}_3)^T \end{bmatrix}}_{A \in \mathbb{R}^{2\times 2}} \vec{x} = \begin{bmatrix} \|\vec{a}_1\|^2 - \|\vec{a}_2\|^2 - d_1^2 + d_2^2 \\ \|\vec{a}_1\|^2 - \|\vec{a}_3\|^2 - d_1^2 + d_3^2 \end{bmatrix}$$

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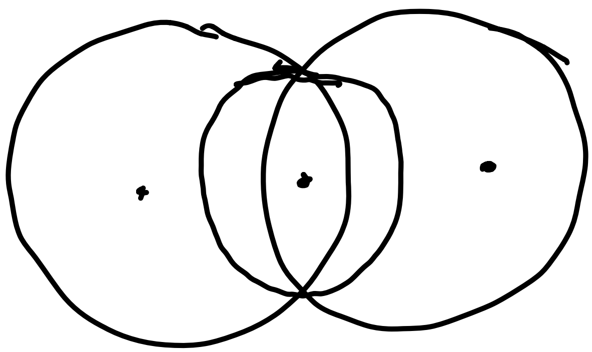
Solution will be unique provided cols/rows of A are linearly independent



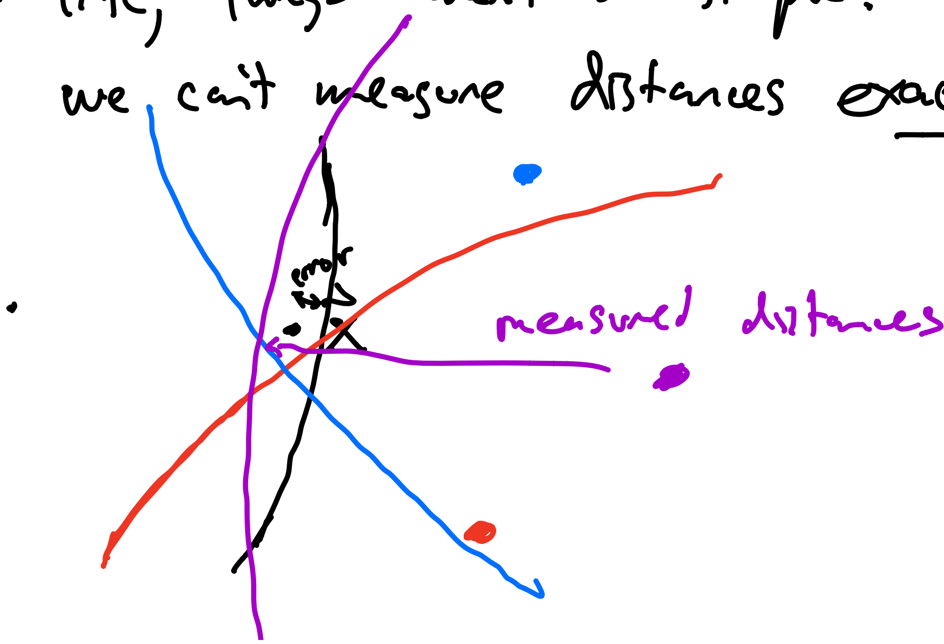
not unique solution

need more satellites

<u>Trilateration</u>: take known distances from known locations, and determine your location by solving system of LEQs.

In real life, things aren't so simple!

Reason: we can't measure distances <u>exactly</u>



This measurement error will lead to an inconsistent system of linear equations (in general).

So, this motivates the problem of trying to "approximately" solve inconsistent systems of LEQs.

Least Squares

Goal: approximately solve an inconsistent system of equations

$$\underline{\underline{A}}\vec{x} \approx \underline{\vec{b}}.$$

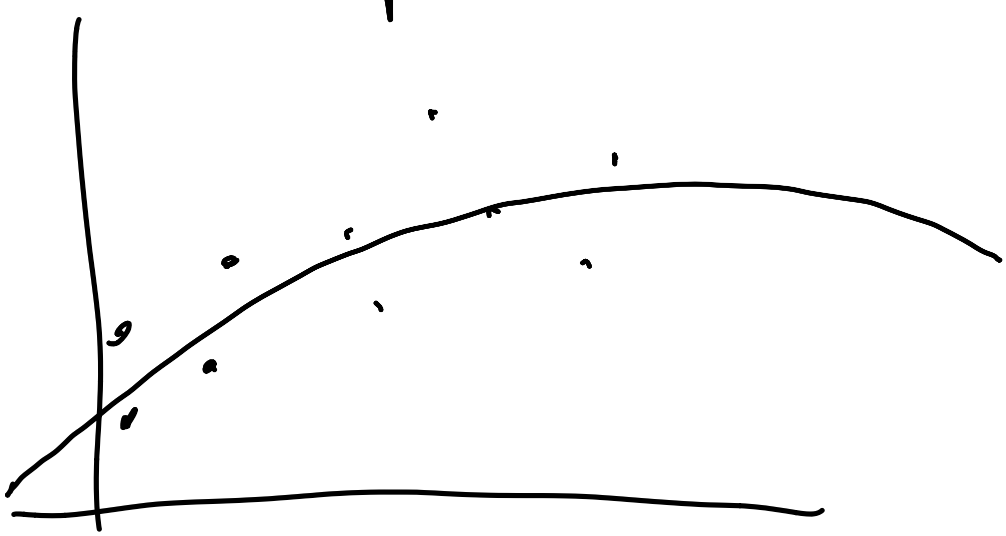
Least Squares Problem:

$$\min_{\vec{x}} \|A\vec{x} - \vec{b}\|^2$$

Assumption: Henceforth, assume Euclidean inner product / norm.

Note: For general inner products and their induced norms, we obtain a "weighted least squares" problem. Concepts are similar.

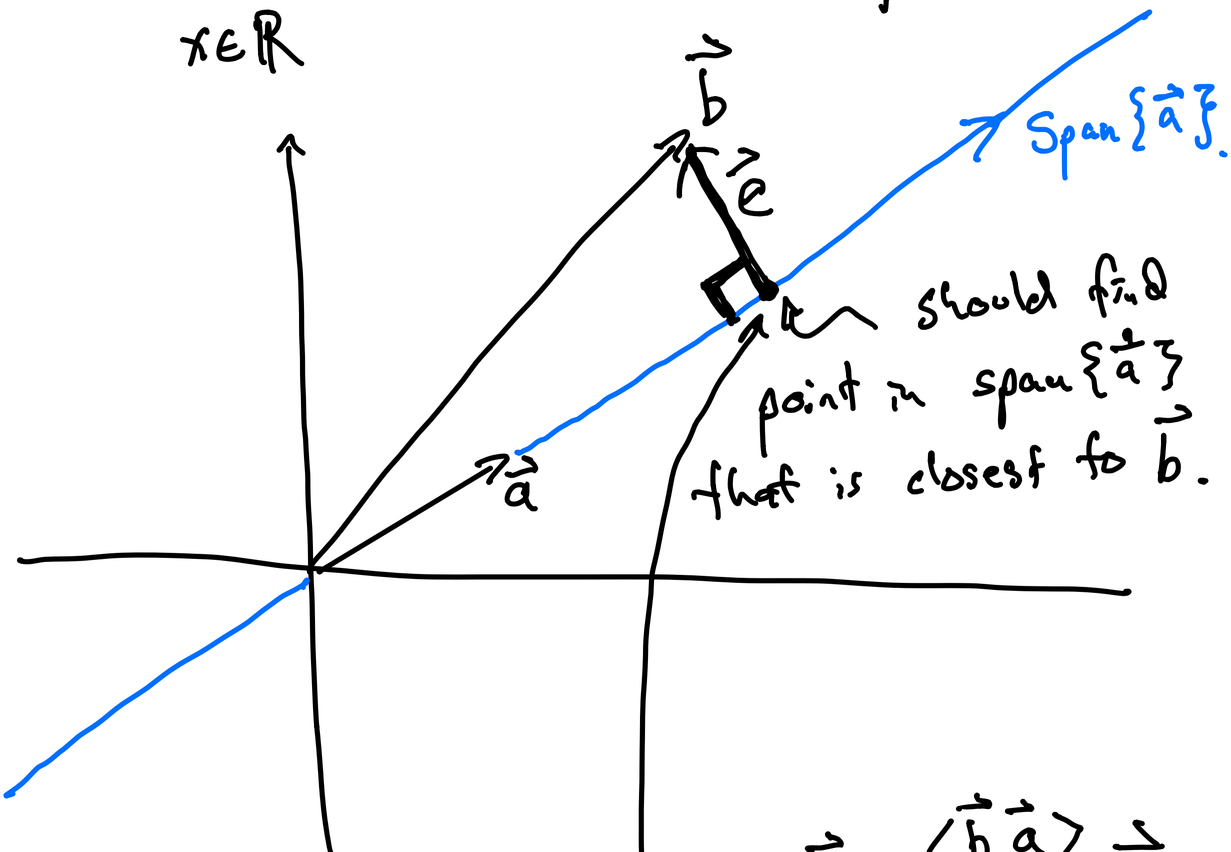
dwarf planet Ceres



Least Squares has alot of geometric intuition! Try to understand this first, equations second.

Start with a simpler problem

$$\min_{x\in\mathbb{R}} \|\vec{a}x - \vec{b}\|^2 = \min_{\vec{v}\in \mathrm{Span}\{\vec{a}\}} \|\vec{v} - \vec{b}\|^2$$



$$\mathrm{proj}_{\vec{a}}(\vec{b}) = \frac{\langle \vec{b}, \vec{a}\rangle}{\langle \vec{a}, \vec{a}\rangle}\vec{a}.$$

$\in \mathrm{Span}\{\vec{a}\}$

Key property is that

(✱) $\quad \vec{e} = \vec{b} - \mathrm{proj}_{\vec{a}}(\vec{b})$ is orthogonal to Span$\{\vec{a}\}$

Let's verify property (*):

$$\langle \vec{e}, \alpha \vec{a} \rangle = \alpha \langle \vec{b} - \mathrm{proj}_{\vec{a}}(\vec{b}), \vec{a} \rangle$$

$$= \alpha \left\langle \vec{b} - \frac{\langle \vec{b}, \vec{a} \rangle}{\langle \vec{a}, \vec{a} \rangle} \vec{a}, \vec{a} \right\rangle$$

$$= \alpha \left[\langle \vec{b}, \vec{a} \rangle - \frac{\langle \vec{b}, \vec{a} \rangle}{\cancel{\langle \vec{a}, \vec{a} \rangle}} \cancel{\langle \vec{a}, \vec{a} \rangle} \right]$$

$$= 0.$$