EECS 16A
1/28/2020

- Gaussian Elimination
  - # of solutions to LEQs
- Vectors, Matrices & Operations

$$m \text{ eq}^{n} \left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1 \\ \qquad\qquad\qquad \vdots \;\; = \;\; \vdots \\ \underbrace{a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n}_{n \text{ unknowns.}} = b_m \end{array} \right.$$

If a system of LEQs has a solution, it is said to be consistent.

" " " . . . . does not have a sol$\underline{n}$, it is said to be inconsistent.

free variables

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & 0 & 4 \\ 0 & 1 & 2 & 0 & 5 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

basic variables

augmented matrix representing sys. of LEQs, it is in rref.

FACT: A system of LEQs is inconsistent if and only if its augmented matrix in rref has a row equal to $[0,0,0,\dots 0|1]$.

Why? If matrix contains $[0,0\dots 0|1] \Rightarrow 0=1$ absurd!

What if matrix in rref does not contain $[0,\dots 0|1]$?

$$\begin{aligned} x_1 \quad + 3x_3 &= 4 \\ x_2 + 2x_3 &= 5 \\ x_4 &= 6 \end{aligned}$$

Then a solution can be obtained be setting free variables as you like, and solving for basic variables.

• Number of $\underline{\text{Sol}}$ns for system of LEQs:

- no solution $\iff$ inconsistent $\iff$ augmented matrix in rref contains $[0, 0 \cdots 0 \mid 1]$

- infinitely many solutions $\iff$ consistent, ≥ 1 free variables.

- unique solution $\iff$ consistent, and no free variables.

<u>Example</u>:

$$\begin{aligned} 2y + 3z &= 2 \\ x + y \quad &= 1 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 0 & 2 & 3 & 2 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

swap(R_1, R_2)

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 2 & 3 & 2 \end{array}\right]$$

$R_2 \leftarrow \frac{1}{2}R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 3/2 & 1 \end{array}\right]$$

$R_1 \leftarrow R_1 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3/2 & 0 \\ & 1 & 3/2 & 1 \end{array}\right]$$

$$\begin{aligned} x \quad - 3/2\, z &= 0 \\ y + 3/2\, z &= 1 \end{aligned}$$

Infinitely many solutions. Solutions of the form

$x = 3/2\, z$

$y = 1 - 3/2\, z$ for any choice of z.

<u>Ex:</u>

$$x+y=2$$
$$x-y=1$$
$$2x-2y=2$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 2 & -2 & 2 \end{array}\right]$$

$R_2 \leftarrow R_2 - R_1, \quad R_3 \leftarrow R_3 - 2R_1$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -1 \\ 0 & -4 & -2 \end{array}\right]$$

$R_2 \leftarrow -\frac{1}{2} R_2$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1/2 \\ 0 & -4 & -2 \end{array}\right]$$

$R_3 \leftarrow R_3 + 4R_2$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{array}\right]$$

$$\downarrow R_1 \leftarrow R_1 - R_2$$

$x = 3/2$

$y = 1/2$.

$$\left[\begin{array}{cc|c} 1 & 0 & 3/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{array}\right]$$

Unique soln!

<u>Ex:</u> $x + y + z = 1$

$x + y + z = 2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{array}\right]$$

$$\downarrow R_2 \leftarrow R_2 - R_1$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Def: A *vector* is an ordered list of numbers.

E.g. $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

Def: A matrix is a rectangular array of numbers:

m rows n cols.

$$A = \begin{array}{c c} & \overset{n}{\longleftrightarrow} \\ m \updownarrow & \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \end{array} \in \mathbb{R}^{m \times n}$$

a_{ij} = element of matrix A

row column

rows cols.

A is an $m \times n$ matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \xleftrightarrow[\text{same!}]{\text{not the}} \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right]$$

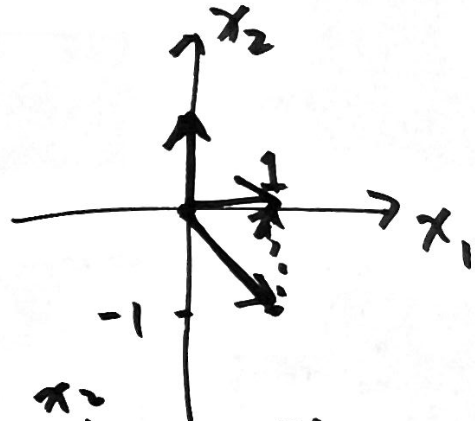
specifically interpreted as a compact representation of a system of LEQs.

Matrix-Vector representation of System of LEQs.

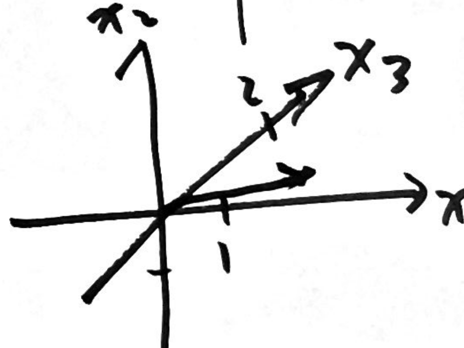
$$\underbrace{\begin{aligned} a_{11}x_1 + a_{12}x_2 + &\cdots a_{1n}x_n = b_1 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + &\cdots a_{mn}x_n = b_m \end{aligned}}_{A\vec{x} = \vec{b}.} \Rightarrow \underbrace{\begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}_{\vec{b}}$$

Vectors:

Ex: $\begin{bmatrix} 1 \\ -1 \end{bmatrix} \in \mathbb{R}^2$



Ex: $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$



operations: If $\vec{x}, \vec{y}$ have the same dimensions (e.g. $\in \mathbb{R}^n$)

Addition:

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

Ex: $\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Equality: $\vec{x} = \vec{y}$ if $x_i = y_i \ \forall i$.

Ex: $\vec{x} = 0$, $x_i = 0 \ \forall i$.

denotes special all-zero vector.

$$\vec{x} + \vec{y} = \vec{y} + \vec{x}$$

$$\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}.$$

$$\vec{x} + 0 = \vec{x}$$

$$\vec{x} + (-\vec{x}) = \vec{0}$$

Multiplication: scalar-vector multiplication.

$$\alpha \in \mathbb{R} \Rightarrow \alpha \vec{x} = \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix}.$$

Vector-Vector multiplication:

inner-product:

$$\underbrace{[y_1 \cdots y_n]}_{\text{row-vector}} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}}_{\text{column vector}} := x_1 y_1 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i .$$

Note: "Transpose" operation flips orientation of a vector. i.e., if $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix}$, $\vec{x}^T := [x_1 \cdots x_n]$

so, if $\vec{x}, \vec{y} \in \mathbb{R}^n$ (both column vectors)

$$\vec{y}^T \vec{x} = \vec{x}^T \vec{y} = \text{inner product btwn } \vec{x}, \vec{y}.$$

Also: outer-product

(column vector) x (row vector) = (matrix)

Caution:

$$\left.\begin{array}{l}\text{(row vector)(row vector)}\\ \text{(column vector)(column vector)}\end{array}\right\} \text{not allowed!}$$

Matrix-vector multiplication

If $\vec{x} \in \mathbb{R}^n$ and $A \in \mathbb{R}^{m \times n}$, we can define product $A\vec{x}$ as:

$$A\vec{x} = \underbrace{\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{\vec{x}} = \overbrace{\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots \quad + a_{mn}x_n \end{bmatrix}}^{A\vec{x}}$$

← j^{th} entry is inner product of j^{th} row of A with $\vec{x}$.

$$= \underbrace{x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}}_{\text{linear combination of columns of } A.}$$

$$A\vec{x} = \vec{b}$$