EECS 16A
1/30/2020

(1)

- (More) Matrix-vector multiplication
- Linear dependence, span, proofs.

Matrix-vector multiplication.

$A \in \mathbb{R}^{m \times n}$, $\vec{x} \in \mathbb{R}^n$.

$$A\vec{x} = x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

columns of A

$$A\vec{x} = \begin{bmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{bmatrix}$$

looks like LHS of sytem of LEQs that we considered before.

"Matrix-Vector form" of sys. of LEQs

$$A\vec{x} = \vec{b}$$

<u>Ex:</u>

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 0 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}.$$

<u>Ex:</u>

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \cdot & 0 & \cdots & 0 \\ \vdots & & \ddots & & & 0 \\ 0 & 0 & \cdots & 0 & & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \overset{\vec{e}_1}{\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}} + x_2 \overset{\vec{e}_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}} + \cdots + x_n \overset{\vec{e}_n}{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}}$$

$$\underbrace{}_{n \times n \text{ matrix}} \left(\begin{matrix}\text{Identity} \\ \text{matrix } I\end{matrix}\right) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

Matrix-vector multiplication applied to model "dynamical" systems.

$\vec{x}(n)$ = "state" of our system at time n

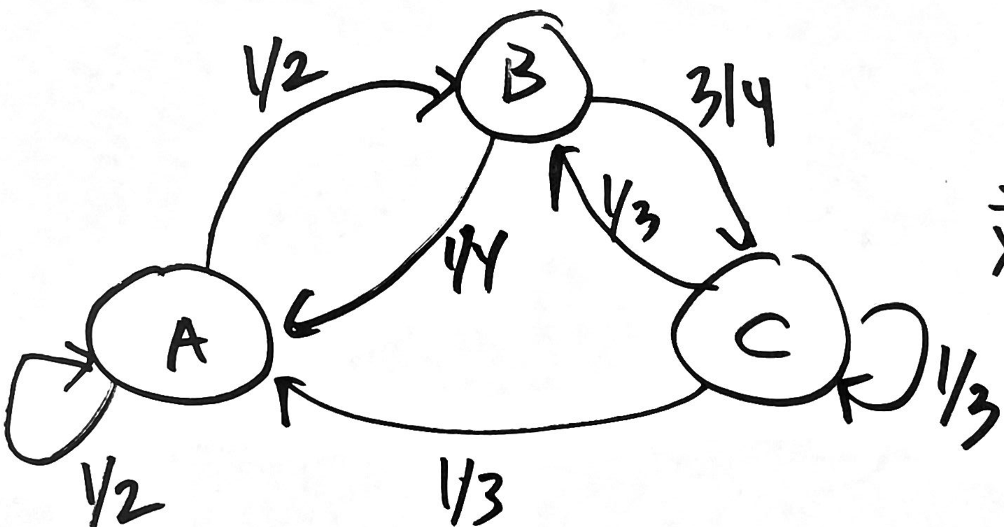
= Example: $\vec{x} = \begin{pmatrix} x\text{-position} \\ y\text{-position} \\ z\text{-position} \\ x\text{-velocity} \\ y\text{-velocity} \\ z\text{-velocity} \end{pmatrix}$ } state of airplane.

Ex: web traffic $\vec{x} = \begin{pmatrix} \#\text{ users on site } 1 \\ \#\text{ users on site } 2 \\ \vdots \\ \#\text{ users on site } n \end{pmatrix}$

Usual model for dynamical system:

$$\vec{x}(n+1) = A\vec{x}(n) \overbrace{\left[+u(n)\right]}^{\text{optional}}$$

prototypical example: pumps and reservoirs



$$\vec{x}(n) = \begin{bmatrix} x_A(n) \\ x_B(n) \\ x_C(n) \end{bmatrix}$$

$x_A(n)$ = # gallons in reservoir A at time n.

$$x_A(n+1) = \tfrac{1}{2}x_A(n) + \tfrac{1}{4}x_B(n) + \tfrac{1}{3}x_C(n)$$

$$x_B(n+1) = \tfrac{1}{2}x_A(n) + \tfrac{1}{3}x_C(n)$$

$$x_C(n+1) = \tfrac{3}{4}x_B(n) + \tfrac{1}{3}x_C(n).$$

$$\vec{x}(n+1) = A\vec{x}(n)$$

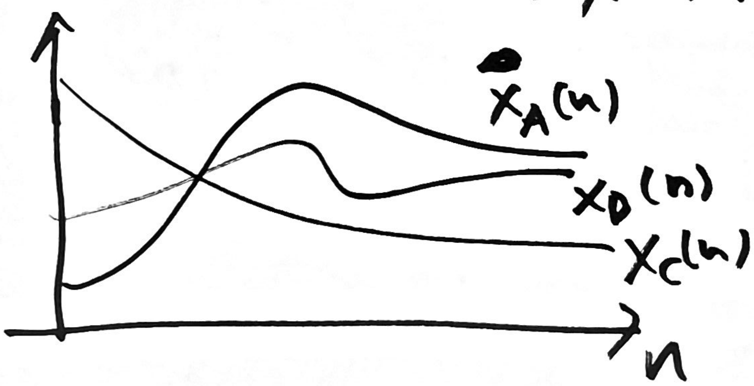
$$A = \begin{bmatrix} 1/2 & 1/4 & 1/3 \\ 1/2 & 0 & 1/3 \\ 0 & 3/4 & 1/3 \end{bmatrix}$$

$$\vec{x}(n+1) = A\vec{x}(n) = A\left(A\vec{x}(n-1)\right)$$

$$= \vdots$$

$$= AAA\cdots A\vec{x}(1).$$

$$= A^n\vec{x}(1).$$



Back to systems of LEQs.

For $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, when does

$$A\vec{x} = \vec{b} \text{ have a solution?}$$

Note: Any solution must satisfy

$$x_1 \underbrace{\begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}}_{\vec{a}_1} + \cdots x_n \underbrace{\begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}}_{\vec{a}_n} = \vec{b}.$$

Solution exists if $\vec{b}$ can be expressed as a lin. combination of the columns of A.

Motivates following defⁿ:

<u>Def</u>: Let $\vec{V}_1, \vec{V}_2 \cdots \vec{V}_n$ be a collection of vectors in $\mathbb{R}^m$. Define the <u>Span</u>

$$\mathrm{Span}\{\vec{V}_1, \vec{V}_2 \cdots \vec{V}_n\} = \left\{ \sum_{i=1}^{n} \alpha_i \vec{V}_i \;\middle|\; \alpha_i \in \mathbb{R} \right\}$$

= set of <u>all</u> linear combinations of $\vec{V}_1, \cdots \vec{V}_n$.

<u>Ex</u>: $A\vec{x} = \vec{b}$ has a solution iff $\vec{b} \in \mathrm{Span}\{\vec{a}_1, \cdots \vec{a}_n\}$, where $\vec{a}_1 \cdots \vec{a}_n$ are columns of A.

Example:

$$\text{span}\left\{\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\} = \mathbb{R}^2.$$

Why? Any linear combination

"in"

$$x_1\begin{bmatrix}0\\1\end{bmatrix} + x_2\begin{bmatrix}1\\-1\end{bmatrix} \in \mathbb{R}^2$$

"is subset of".

$$\Rightarrow \text{span}\left\{\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\} \subseteq \mathbb{R}^2$$

To show other direction, need to show any vector in $\mathbb{R}^2$ can be realized as a lin. combination of ⋯.

$$x_1\begin{bmatrix}0\\1\end{bmatrix} + x_2\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}b_1\\b_2\end{bmatrix} \quad \text{for arbitrary } (b_1, b_2).$$

$\rightarrow$ is $\begin{bmatrix}0 & 1\\1 & -1\end{bmatrix}\vec{x} = \begin{bmatrix}b_1\\b_2\end{bmatrix}$ consistent?

In general, not easy to compute span explicitly.

Closely related to span is concept of linear dependence.

Def: A set of vectors $\{\vec{v}_1, \cdots \vec{v}_n\} \subset \mathbb{R}^m$ is linearly dependent if there exist scalars $\alpha_1 \cdots \alpha_n$, not all zero, such that

$$\sum_{i=1}^{n} \alpha_i \vec{v}_i = 0 \quad . \qquad (1)$$

Defn: A set of vectors $\{\vec{v}_1 \cdots \vec{v}_n\} \subseteq \mathbb{R}^m$ is linearly dependent if there is some index i such that

(2) $$\vec{v}_i = \sum_{j \neq i} \alpha_j \vec{v}_j \quad , \quad \text{for some scalars } \alpha_1 \cdots \alpha_n.$$

First "proof" of the course. WTS that Def ① & ② are equivalent.

① ⇒ ②: Assume $\vec{V}_1 \cdots \vec{V}_n$ are linearly dependent.

Then ① ensures existence of α_i's not all zero

s.t. $\sum \alpha_i \vec{v}_i = 0$. Say $\alpha_{i_0} \neq 0$ (know this to be true)

then above rearranges to $-\alpha_{i_0} \vec{V}_{i_0} = \sum_{j \neq i_0} \alpha_j \vec{V}_j$

Divide by $-\alpha_{i_0}$ ⇒ $\vec{V}_{i_0} = \sum_{j \neq i_0} \left(-\frac{\alpha_j}{\alpha_{i_0}} \right) \vec{V}_j$

⇒ ②.

2 $\Rightarrow$ 1: Start out assuming 2, i.e, for some index i,

$$\vec{v}_i = \sum_{j \neq i} \alpha_j \vec{v}_j$$

rearrange:

$$0 = \sum_{j \neq i} \alpha_j \vec{v}_j - \vec{v}_i$$

$$= \sum_{j=1}^{n} \alpha_j \vec{v}_j \quad , \quad \Rightarrow 1.$$

for $\alpha_j = -1$ if $j = i$

<u>Thm</u>: If the system of LEQs $A\vec{x}=\vec{b}$ has infinitely many solutions, then cols. of A are linearly dependent.

<u>Thm</u>: If cols of A are linearly dependent, then $A\vec{x}=\vec{b}$ does <u>not</u> have a unique solution.