EECS 16A

2/4/2020

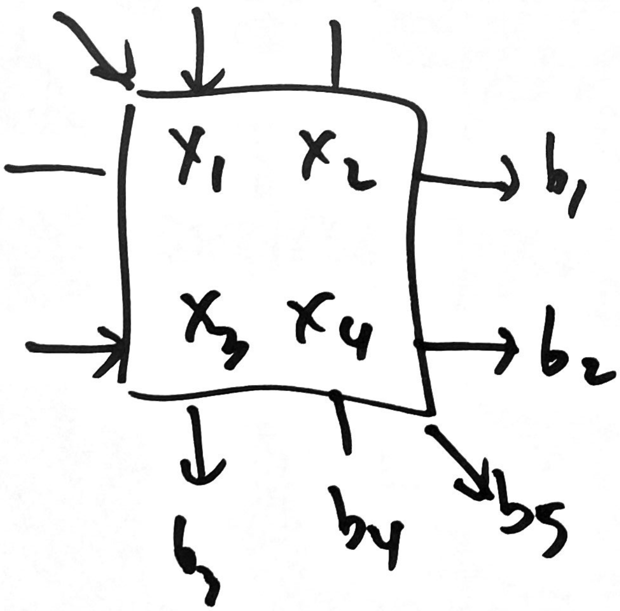
- More on Linear Dependence, w/ Proofs.
- Linear Transformations, Matrix-Matrix Mult.
- Inverses (if time).

Corollary: Let $A\vec{x}=\vec{b}$ be a consistent system ~~set~~ of LEQs. There are infinitely many solutions iff columns of A are linearly dependent.

Def: A set of vectors $\{\vec{v}_1 \cdots \vec{v}_n\}$ is linearly independent if $\sum_{i=1}^{n} \alpha_i \vec{v}_i = 0 \implies \alpha_i = 0 \ \forall i$.

Corollary': Let $A\vec{x}=\vec{b}$ be a consistent system of LEQs. Then unique solution iff columns of A are linearly independent.

Application: Tomography



$$\begin{array}{lcl} x_1 + x_2 & & = b_1 \\ & x_3 + x_4 & = b_2 \\ x_1 & + x_3 & = b_3 \\ \quad x_2 & + x_4 & = b_4 \\ x_1 & + x_4 & = b_5 \end{array}$$

$$A\vec{x} = \vec{b}$$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

Pf: If infinitely many solutions, then certainly 2 distinct solutions $\vec{x}_1, \vec{x}_2$, $\vec{x}_1 \neq \vec{x}_2$.

$$A\vec{x}_1 = \vec{b}$$

$$A\vec{x}_2 = \vec{b}$$

$$\underbrace{A(\vec{x}_1 - \vec{x}_2)}_{\text{nonzero linear comb. of cols. of } A} = 0$$

$$A = \begin{bmatrix} | & | & & | \\ \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \\ | & | & & | \end{bmatrix}$$

$\to \sum \alpha_i \vec{a}_i = 0.$ $\quad \alpha_i = (\vec{x}_{1i} - \vec{x}_{2i})$

{not all α_i's $= 0$ } Def$^{\underline{n}}$ of linear dependence. $\quad [\ \alpha = \vec{x}_1 - \vec{x}_2\ ]$

cols of A lin. dep. $\iff$ exists $\vec{\alpha} \neq \vec{0}$ s.t. $A\vec{\alpha} = 0$

2 possibilities ~~if $A\vec{x} = \vec{b}$ doesn't have unique solution.~~

1) $A\vec{x} = \vec{b}$ has no solution

2) $A\vec{x} = \vec{b}$ has ~~infinitely many~~ a solution.

~~~~

If #1) true, then conclusion also holds.

So, let assume #2). i.e, there is solution $\vec{x}_0$: $A\vec{x}_0 = \vec{b}$.

For any $\beta \in \mathbb{R}$ $A(\underbrace{\vec{x}_0 + \beta\vec{\alpha}}) = A\vec{x}_0 + \beta A\vec{\alpha} = \vec{b}$

infinitely many vectors of this form by varying β.
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Q: How to check whether $\{\vec{a}_1, \cdots \vec{a}_n\}$ are linearly indep / dep?

A: Check whether $A\vec{x} = \vec{0}$ has unique/inf solutions

$$\begin{bmatrix} | & & | \\ \vec{a}_1 & \cdots & \vec{a}_n \\ | & & | \end{bmatrix}$$

Linear Transformations:

A function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation if it satisfies

1) Homogeneity: $f(\alpha\vec{x}) = \alpha f(\vec{x})$ $\forall \alpha \in \mathbb{R}$, $\vec{x} \in \mathbb{R}^n$
2) Additivity (aka superposition):
$f(\vec{x}+\vec{y}) = f(\vec{x}) + f(\vec{y})$ $\forall \vec{x}, \vec{y} \in \mathbb{R}^n$.

<u>Example</u>: Every matrix $A \in \mathbb{R}^{m \times n}$ defines a linear transformation via

$$f(\vec{x}) = A\vec{x}$$

why? $f(\alpha\vec{x}) = A(\alpha\vec{x}) = \alpha A\vec{x} = \alpha f(\vec{x})$ ✓

$f(\vec{x}+\vec{y}) = A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = f(\vec{x}) + f(\vec{y})$ ✓

In <u>discussion</u>:

E.g. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ rotates vectors in $\mathbb{R}^2$ by θ rad.

E.g. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ reflects vectors in $\mathbb{R}^2$ about horizontal axis.

<u>Fact</u>: Every linear transformation $f: \mathbb{R}^n \to \mathbb{R}^m$ can be represented in terms of matrix-vector multiplication. i.e.,

$$f(\vec{x}) = A\vec{x} \quad \text{for some } A \in \mathbb{R}^{m \times n}.$$

<u>Why?</u>

$$f(\vec{x}) = \begin{bmatrix} f_1(\vec{x}) \\ \vdots \\ f_m(\vec{x}) \end{bmatrix} = \begin{bmatrix} \vec{a}_1^T \vec{x} \\ \vec{a}_2^T \vec{x} \\ \vdots \\ \vec{a}_m^T \vec{x} \end{bmatrix} = \underbrace{\begin{bmatrix} - a_1^T - \\ - a_2^T - \\ \vdots \\ - a_m^T - \end{bmatrix}}_{A} \vec{x}.$$

$f_i: \mathbb{R}^n \to \mathbb{R}$ is linear function,

Operations on Matrices

- scalar multiplication : $\alpha \in \mathbb{R}$ $A \in \mathbb{R}^{m \times n}$

$$\alpha A = \begin{bmatrix} \alpha a_{11} & \cdots & \alpha a_{1n} \\ \vdots & & \\ \alpha a_{m1} & \cdots & \alpha a_{mn} \end{bmatrix}$$

a_{ij}'s = entries of A.

- addition : $A, B \in \mathbb{R}^{m \times n}$

$$A+B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ \vdots & & & \\ a_{m1}+b_{m1} & \cdots & \cdot & a_{mn}+b_{mn} \end{bmatrix}$$

$$= B+A$$

- Transpose

$A \in \mathbb{R}^{m \times n}$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & & a_{mn} \end{bmatrix} \in \mathbb{R}^{n \times m}$$

columns of A^T are rows of A

Matrix-Matrix Multiplication.

$$A \in \mathbb{R}^{m \times n} \qquad B \in \mathbb{R}^{n \times p}$$

same!

$$AB = \underbrace{\begin{bmatrix} — \vec{a}_1^T — \\ \vdots \\ — \vec{a}_m^T — \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_p \\ | & | & & | \end{bmatrix}}_{B} = \begin{bmatrix} \vec{a}_1^T \vec{b}_1 & \vec{a}_1^T \vec{b}_2 \cdots & \vec{a}_1^T \vec{b}_p \\ \vdots & & \\ \vec{a}_m^T \vec{b}_1 & \cdots & \vec{a}_m^T \vec{b}_p \end{bmatrix}$$

$\vec{a}_i \in \mathbb{R}^n$, $\vec{b}_i \in \mathbb{R}^n$

$$(AB)_{ij} = \vec{a}_i^T \vec{b}_j$$

j^{th} col of B

i^{th} row of A

$$AB = \begin{bmatrix} | & | & & | \\ A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_p \\ | & | & & | \end{bmatrix}$$

<u>Example</u>:

$$\underbrace{\begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{B} = \begin{bmatrix} 2\cdot1+3\cdot4 & 2\cdot2+4\cdot4 \\ 3\cdot1+1\cdot3 & 3\cdot2+1\cdot4 \end{bmatrix} = \begin{bmatrix} 14 & 20 \\ 6 & 10 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 6 \\ 18 & 16 \end{bmatrix}$$

Note: $AB \neq BA$ matrix-matrix mult. is <u>not</u> commutative!

Matrix-Matrix Mult. <u>is</u> associative

$$A(BC) = (AB)C$$

<u>Ex:</u> $\vec{x}, \vec{y} \in \mathbb{R}^n$

<u>outer product</u> $\vec{x}\vec{y}^T = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} [y_1 \cdots y_n] = \begin{bmatrix} x_1y_1 \cdots x_1y_n \\ \\ x_ny_1 \cdots x_ny_n \end{bmatrix}$

inner product $\vec{x}^T\vec{y} = \sum x_i y_i$

<u>Ex:</u> $\underline{I} = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & \\ & & & 1 \end{bmatrix} \in \mathbb{R}^{n \times n}$ = "identity matrix".

$A \in \mathbb{R}^{n \times n}$ $\qquad AI = IA = A.$