EEC S 16A
2/11/2020

- Inverse wrap-up
- Vector Spaces & Subspaces
- Nullspace and Column space

- Lecture pace survey
- HW 2

①

<u>Thm</u>: If A is invertible, then there exists a unique solution to $A\vec{x}=\vec{b}$ for ~~all~~ any choices of $\vec{b}$.

① Let's show a solution exists.

$$I\cdot\vec{b} = A(A^{-1}\vec{b}) = \vec{b}$$, so $\vec{x}_0 = A^{-1}\vec{b}$ is a solution

$\cancel{A^{-1}A\vec{x} = A^{-1}\vec{b}}$

② uniqueness? Let $\vec{x}_1$ be any solution to $A\vec{x}=\vec{b}$

$$\vec{x}_1 = (A^{-1}A)\vec{x}_1 = A^{-1}\vec{b}$$

$A \in \mathbb{R}^{n\times n}$ is invertible $\iff$ A has LI columns

$\iff$ A has LI rows

$\iff A\vec{x}=\vec{b}$ has unique soln ~~is consistent~~ for any $\vec{b}$

$\iff A\vec{x}=\vec{0}$ has unique soln.

[A has trivial nullspace]

$\iff$ A has nonzero "determinant".

Vector Spaces:

A vector space $\mathcal{V}$ over $\mathbb{R}$ is a set of elements of $\mathcal{V}$ (called vectors) satisfying the following axioms:

vector addition: associative: $\vec{u}+(\vec{v}+\vec{w})=(\vec{u}+\vec{v})+\vec{w}$ $\forall\, \vec{u},\vec{v},\vec{w}\in\mathcal{V}$

commutative: $\vec{u}+\vec{v}=\vec{v}+\vec{u}$

additive identity: $\exists\ \vec{0}\in\mathcal{V}$ s.t. $\vec{v}+\vec{0}=\vec{v}$ $\forall\,\vec{v}\in\mathcal{V}$.

additive inverse: For each $\vec{v}\in\mathcal{V}$ there exists $-\vec{v}\in\mathcal{V}$ s.t. $\vec{v}+(-\vec{v})=\vec{0}$.

closure: $\vec{u}+\vec{v}\in\mathcal{V}$ if $\vec{u},\vec{v}\in\mathcal{V}$.

Scalar multiplication:

Associative: $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$ $\alpha, \beta \in \mathbb{R}$

Mutiplicative identity: exists $1 \in \mathbb{R}$ s.t. $1 \cdot \vec{v} = \vec{v}$.

Distributive in vector addition: $\alpha(\vec{u}+\vec{v}) = \alpha\vec{u} + \alpha\vec{v}$.

Distributive in scalar addition: $(\alpha+\beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$

Closure: if $\vec{v} \in V$ then $\alpha\vec{v} \in V$ $\alpha \in \mathbb{R}$.

Examples of vector spaces (over $\mathbb{R}$):

$\mathbb{R}^n$, set of continuous functions $f: \mathbb{R} \to \mathbb{R}$.

set of functions $f: \mathbb{R}^n \to \mathbb{R}$ s.t. $\int |f|^2 dx < \infty$.

Def: Given a vector space $\mathcal{V}$, a set of vectors $\{\vec{v}_1, \cdots \vec{v}_n\}$ is called a basis for $\mathcal{V}$ if it satisfies:

1) $\{\vec{v}_1 \cdots \vec{v}_n\}$ are linearly independent.

2) $\mathcal{V} = \text{span}\{\vec{v}_1, \cdots \vec{v}_n\}$.

In other words, any $\vec{v} \in \mathcal{V}$ can be written as a unique linear combination of basis vectors

$$\vec{v} = \sum \alpha_i \vec{v}_i$$

Examples of Bases:

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$

also called "natural basis".

$\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ is a basis for $V = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ 0 \end{bmatrix} \middle| \, v_1, v_2 \in \mathbb{R} \right\}$

Thm.: Any basis for a vector space V contains the same number of vectors.

won't prove here...

Def: The dimension of a vector space V is equal to the number of vectors in any basis for V.

Examples:

$\dim(\mathbb{R}^3) = ?$

basis for $\mathbb{R}^3 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

3 vectors $\Rightarrow \dim(\mathbb{R}^3) = 3$.

Ex: $V = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_1+x_2 \end{bmatrix} \middle| \; x_1, x_2 \in \mathbb{R} \right\}$

$\dim(V) = ?$

basis for $\mathcal{V} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

$\dim(\mathcal{V}) = 2$.

Q: what is dimension of $\mathcal{V} = \{\vec{0}\}$.

$\dim(\mathcal{V}) = 0$ is by convention.

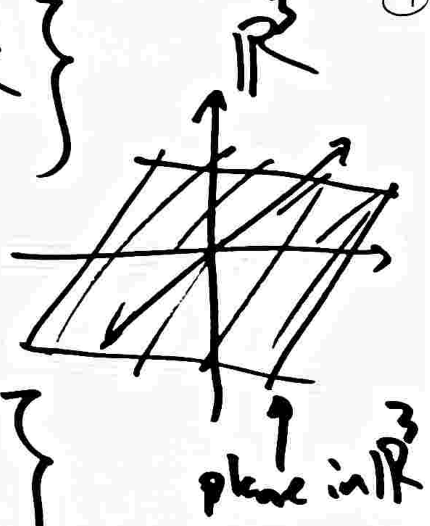
Def: Given a vector space $\mathcal{V}$, a set $\mathcal{U} \subseteq \mathcal{V}$ (subset of) is called a subspace, if for all $\alpha, \beta \in \mathbb{R}$ and $\vec{u}_1, \vec{u}_2 \in \mathcal{U}$,

$$\alpha\vec{u}_1 + \beta\vec{u}_2 \in \mathcal{U}.$$

<u>Example:</u>

$$\mathcal{U} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_1+x_2 \end{bmatrix} \,\middle|\, x_1, x_2 \in \mathbb{R} \right\}$$

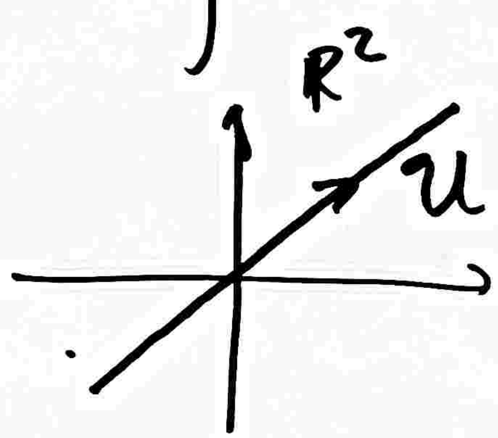
is a subspace of $\mathbb{R}^3$.



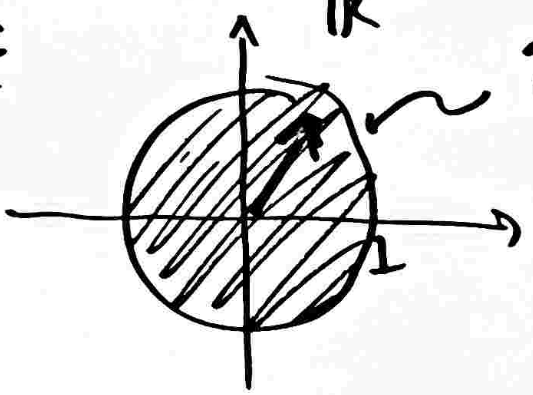
Example:

$$\mathcal{U} = \left\{ \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} \,\middle|\, \alpha \in \mathbb{R} \right\}$$

is a subspace of $\mathbb{R}^2$.



Ex:



← not a subspace of $\mathbb{R}^2$

Notes:

- A subspace $\mathcal{U} \subseteq \mathcal{V}$ is closed under operations of scalar multiplication and vector addition. (def$^{\underline{n}}$ restated)

- A subspace $\mathcal{U} \subseteq \mathcal{V}$ is also a vector space itself.

Example: If $\{\vec{v}_1, \dots \vec{v}_k\} \subset \mathcal{V}$ then $\mathcal{U} = \text{span}\{\vec{v}_1, \dots \vec{v}_k\}$ is a subspace of $\mathcal{V}$.

Why? If $\vec{u}_1, \vec{u}_2 \in \mathcal{U}$ then by def$^{\underline{n}}$ of span,

$$\vec{u}_1 = \sum \alpha_i \vec{v}_i \qquad \vec{u}_2 = \sum \beta_i \vec{v}_i$$

for choices of scalars α_i, β_i's.

Note:

$$a\vec{u}_1 + b\vec{u}_2 = \sum_i (a\alpha_i + b\beta_i)\vec{v}_i \in \text{span}\{\vec{v}_1, \dots \vec{v}_k\}.$$

There are several subspaces naturally associated to any matrix $A \in \mathbb{R}^{m \times n}$.

A can be thought of as representing a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

$$\begin{aligned}
\text{range}(A) &= \{A\vec{x} \mid \vec{x} \in \mathbb{R}^n\} \\
&= \text{span}\{\vec{a}_1, \dots \vec{a}_n\} \quad \text{cols of } A. \\
&= \text{"column space of } A\text{"} \\
&= C(A) \\
&= \text{a subspace of } \mathbb{R}^m.
\end{aligned}$$

Nullspace: Given a matrix $A \in \mathbb{R}^{m \times n}$ the nullspace of A, N(A) is defined as

$$N(A) = \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0} \}$$

Claim: N(A) is a subspace of $\mathbb{R}^n$.

Pf: $\vec{x}_1, \vec{x}_2 \in N(A)$ WTS $\alpha\vec{x}_1 + \beta\vec{x}_2 \in N(A)$

$$A(\alpha\vec{x}_1 + \beta\vec{x}_2) = \alpha \underbrace{A\vec{x}_1}_{0} + \beta \underbrace{A\vec{x}_2}_{0} = \vec{0}$$