EECS 16A
2/20

- Eigenvalues/vectors
- Imaging lab
- Determinants

Module 1 Wrap-up!

①

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \det(A - \lambda I) = (\lambda - 5)(\lambda + 1)$$

$$\lambda_1 = 5 \quad \lambda_2 = -1.$$

$$E_5 = \text{span}\left\{ \underbrace{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}}_{\vec{v}_1} \right\} \qquad E_{-1} = \text{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{\vec{v}_2} \right\}.$$

Note: Any $\vec{x} \in \mathbb{R}^2$ can be written as

$$\vec{x} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2.$$

$$\overbrace{A \cdot A \cdot A \cdots A}^{k}$$

$$A^k \vec{x} = A^k(\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2) = \alpha_1 5^k \vec{v}_1 + \alpha_2 (-1)^k \vec{v}_2.$$

Consider $\vec{x}(k) = A\vec{x}(k-1)$

$$\vec{x}(0) = \begin{bmatrix}1\\2\end{bmatrix} = 2\vec{v}_1 + 0\cdot\vec{v}_2$$

$$\vec{x}(100) = 5^{100}\begin{bmatrix}1\\2\end{bmatrix}$$ "unstable" system behavior

$$\vec{x}(0) = \begin{pmatrix}3\\-3\end{pmatrix} = 3\vec{v}_2$$

$$\vec{x}(100) = (-1)^{100}\begin{pmatrix}3\\-3\end{pmatrix}$$

When we design systems, we generally want systems to have "stable" behavior

$$\vec{x}(n) = \underbrace{A}_{\text{design}} \vec{x}(n-1)$$

i.e. all eigenvalues have magnitude < 1.

(or ≤ 1, which may give rise to some nontrivial steady state).

Thm: Let $A \in \mathbb{R}^{n\times n}$ have distinct eigenvalues $\lambda_1 \cdots \lambda_n$. Let $\vec{v}_1 \cdots \vec{v}_n$ be nonzero eigenvectors corresp. to $\lambda_1 \cdots \lambda_n$.

Then $\{\vec{v}_1, \vec{v}_2 \cdots \vec{v}_n\}$ are linearly indep.

Pf (for 2x2 case):

Know: $A\vec{v}_1 = \lambda_1 \vec{v}_1$, $A\vec{v}_2 = \lambda_2 \vec{v}_2$ | $\vec{v}_1, \vec{v}_2 \neq 0$, $\lambda_1 \neq \lambda_2$

WTS: $\vec{v}_2$ not scalar multiple of $\vec{v}_1$

Lets suppose that $\vec{v}_1 = \alpha \vec{v}_2$.

$$\lambda_1 \vec{v}_1 = A\vec{v}_1 = A\alpha\vec{v}_2 = \alpha A\vec{v}_2 = \alpha\lambda_2\vec{v}_2 = \lambda_2\vec{v}_1$$

$$\Rightarrow \lambda_1 = \lambda_2 \quad \rightarrow\!\leftarrow$$

"contradiction".

$$\vec{x}(k+1) = A\vec{x}(k) \quad , \quad A \in \mathbb{R}^{n\times n}$$

with distinct evals

$\lambda_1 \cdots \lambda_n$

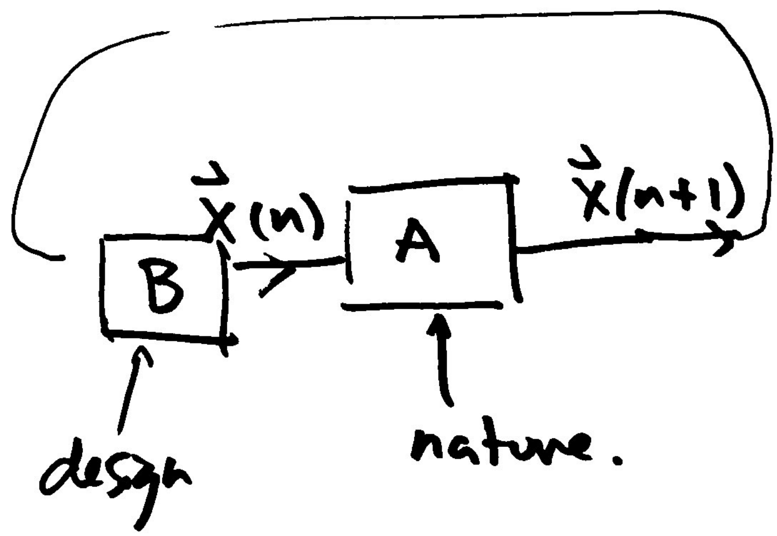
Q: $\vec{x}(100) = ?$

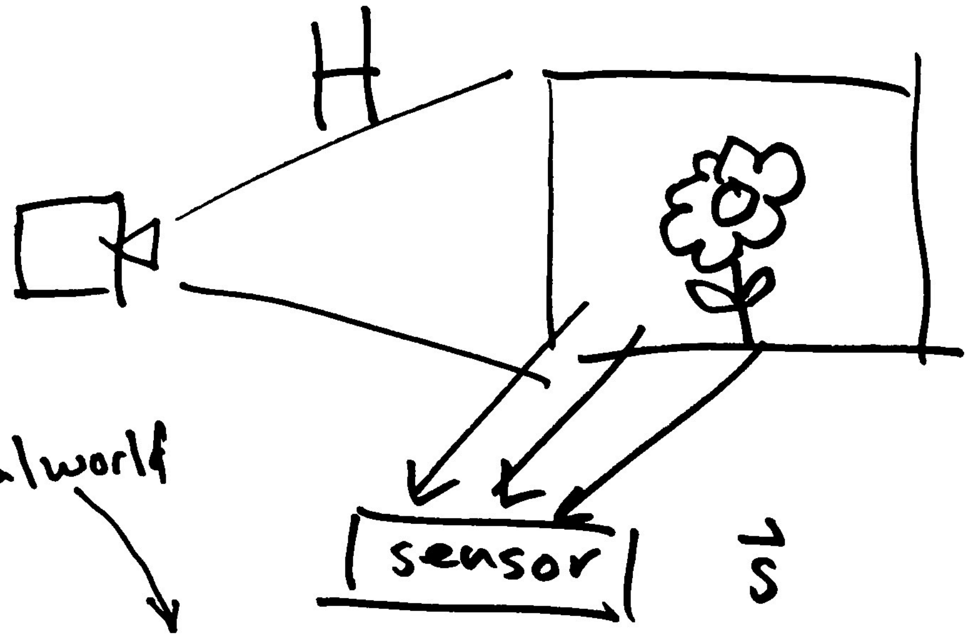
Step 1: write $\vec{x}(0) = \sum \alpha_i \vec{v}_i$. (why?)

Step 2:

$$\vec{x}(k) = A^k \vec{x}(0) = A^k \sum_{i=1}^{n} \alpha_i \vec{v}_i$$

$$= \sum_{i=1}^{n} \alpha_i A^k \vec{v}_i = \sum_{i=1}^{n} \alpha_i (\lambda_i)^k \vec{v}_i$$


$\vec{x}(n)$
$\vec{x}(n+1)$
B
A
design
nature.



$$\vec{s} = H\vec{i} + \overbrace{\underbrace{\vec{w}}_{\text{noise}}}^{\text{realworld}}$$

Perfect world: $\vec{i} = H^{-1}\vec{s}$.

(How to model "noise" see EECS 126)

$$\vec{s} = H\vec{i} + \vec{w}$$

$$\hat{\vec{i}} = H^{-1}\vec{s} = \vec{i} + \underbrace{H^{-1}\vec{w}}.$$

Q: What properties of H^{-1} would ensure $H^{-1}\vec{w}$ is "small"

A: Eigenvalues of H^{-1} should be "small".

Q: If H invertible, how do eigenvalues of H^{-1} relate to those of H?

Let $(\lambda, \vec{x})$ be e.val/e.vec for H.

$$H\vec{x} = \lambda\vec{x}$$

left-mult. by H^{-1}: $\vec{x} = \lambda H^{-1}\vec{x} \Longrightarrow H^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}$

$\Rightarrow (\frac{1}{\lambda}, \vec{x})$ is eval/evec for H^{-1}.

when $\lambda \neq 0$.

Claim: If H is invertible, then all eigenvalues are nonzero. (Why? $0 \neq \det(H) = \det(H - 0 \cdot I)$
$= p_H(0)$
$\Rightarrow$ 0 is not a root of p_H
$\Rightarrow$ 0 is not e-value of H.

<u>Summary</u>: If H is invertible with eigenvalues $\lambda_1 \cdots \lambda_n$, then H^{-1} has eigenvalues $\frac{1}{\lambda_1}, \cdots \frac{1}{\lambda_n}$.

<u>Also</u>: if $\vec{x}$ is an eigenvector of H for eigenvalue λ, then it is also an eigenvector of H^{-1} for eigenvalue $\frac{1}{\lambda}$.

Last observation of module 1.

$$p_A(\lambda) = \det(A - \lambda I) = \prod_{i=1}^{n} (\lambda_i - \lambda) \quad , \quad \lambda_i\text{'s} = \text{evals of } A.$$

$$\det(A) = p_A(0) = \prod \lambda_i$$

$\Rightarrow \det(A) =$ product of eigenvalues.

Ex: $\det(A^{-1}) = \prod_{i=1}^{n} \frac{1}{\lambda_i} = \frac{1}{\prod_{i=1}^{n} \lambda_i} = \frac{1}{\det(A)}$.