EECS 16A Designing Information Devices and Systems I
Spring 2022 Homework 12

This homework is due April 22, 2021, at 23:59.
Self-grades are due April 25, 2021, at 23:59.

Submission Format
Your homework submission should consist of **one** file.

- `hw12.pdf`: A single PDF file that contains all of your answers (any handwritten answers should be scanned).

Submit the file to the appropriate assignment on Gradescope.

1. **Reading Assignment**

For this homework, please review Note 20 (Op-Amp Current Source and Circuit Design), and read Note 21 (Inner Products and GPS). You are always encouraged to read beyond this as well.

2. **Inner Product Properties**

Learning Goal: *The objective of this problem is to exercise useful identities for inner products.*

Our definition of the inner product in $\mathbb{R}^n$ is:

$$\langle \vec{x}, \vec{y} \rangle \ = \ x_1 y_1 + x_2 y_2 + \ldots + x_n y_v \ = \vec{x}^{\mathsf{T}} \vec{y}\ , \quad \text{for any } \ \vec{x}, \vec{y} \in \mathbb{R}^n$$

Prove the following identities in $\mathbb{R}^n$:

(a) $\langle \vec{x}, \vec{y} \rangle = \langle \vec{y}, \vec{x} \rangle$

(b) $\langle \vec{x}, \vec{x} \rangle = \|\vec{x}\|^2$

(c) $\langle -\vec{x}, \vec{y} \rangle = -\langle \vec{x}, \vec{y} \rangle$.

(d) $\langle \vec{x}, \vec{y} + \vec{z} \rangle = \langle \vec{x}, \vec{y} \rangle + \langle \vec{x}, \vec{z} \rangle$

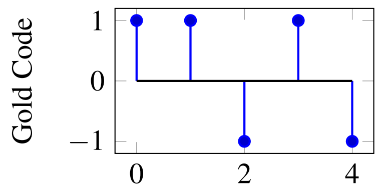
(e) $\langle \vec{x} + \vec{y}, \vec{x} + \vec{y} \rangle = \langle \vec{x}, \vec{x} \rangle + 2\langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{y} \rangle$

3. ***Gold*en Positioning System**

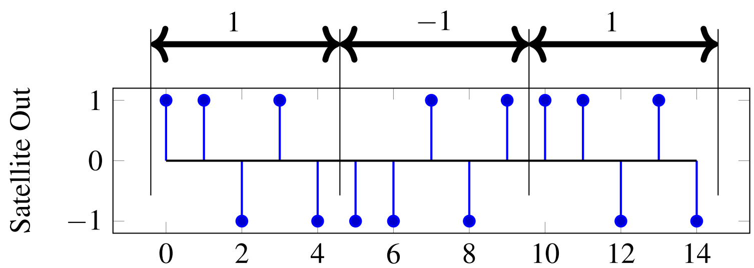
Learning Goal: *This problem is meant to help practice computing cross-correlation. It also covers the concept of trilateration.*

In this problem we will explore how real GPS systems work, and touch on a few aspects of implementing GPS receivers.

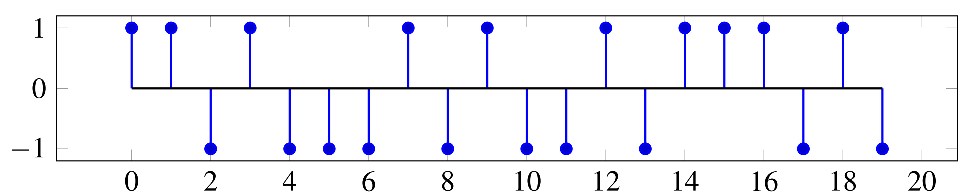
A Gold code is a sequence of 1's and -1's that has a high autocorrelation at a shift of 0, and small autocorrelations otherwise. Every GPS satellite has a unique Gold code assigned to it, and users are aware of the Gold code used by each satellite. The plot below shows a Gold code of length 5.



Each GPS satellite has a message that it transmits by modulating the Gold code. When the satellite is transmitting a 1, it sends just the Gold code sequence. When the satellite is transmitting a -1, it sends -1 times the Gold code. For example, if a satellite were transmitting the message [1, -1, 1], it would transmit the following:



(a) Suppose you receive the following from a GPS satellite that has the same Gold code as above. **What message is the satellite transmitting?**



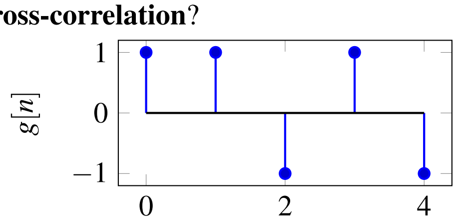
(b) In order to find the message being sent by the satellite, the receiver will find the linear cross-correlation of the received signal with a replica of the satellite Gold code.

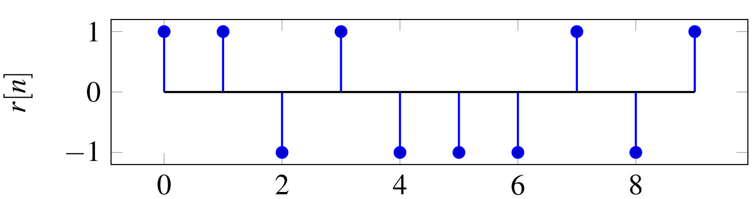
We need to find the **linear cross-correlation** of the signals shown below given by

$$\text{corr}_{\vec{r}}(\vec{g})[k] = \sum_{i=-\infty}^{\infty} r[i]g[i-k]$$

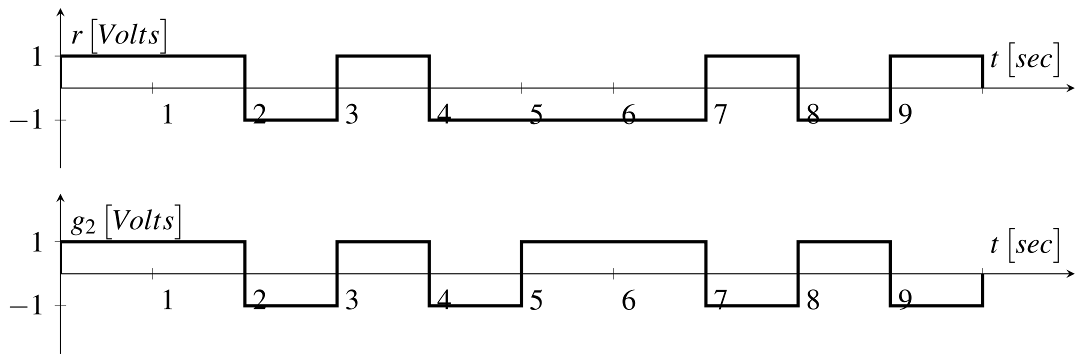
where $r[n]$ is the received signal and $g[n]$ is the Gold code sequence. *Note that neither of these signals is periodic in this part.*

Plot the values of corr$_{\vec{r}}(\vec{g})[k]$ for $-1 \leq k \leq 7$. What is the significance of the peaks in the linear cross-correlation?





(c) Real GPS receivers have specialized hardware to perform cross-correlation using circuits. However, since these transmissions are continuous signals instead of discrete values, we will model the received signal $r(t)$ and the Gold code signal $g_2(t)$ as square waves, as shown in the plot below. Notice that $g_2(t)$ shows two periods of the Gold code.



An essential hardware block to implementing a GPS correlator is *Multiply and Integrate*. The Multiply and Integrate block takes in two inputs, then integrates the product of the two inputs over time. For example, the output of the *Multiply and Integrate* block given the above two inputs would be:

$$y(t) = \int_0^t r(\tau) g_2(\tau) d\tau$$

where $y(t)$ is the circuit output at time t. **Draw $y(t)$ as a function of time, for $t = 0$ to $t = 10$ sec.**

(d) Receivers also need to use the received data to calculate the position of the satellite. Each receiver will receive data from k satellites. Each satellite transmits the time, S_i, at which it started sending the message, where i is the index of the satellite, and $1 \leq i \leq k$. The receiver knows the time, T_i, at which each message arrives. You may assume the receiver and transmitter clocks are synchronized perfectly. Let c represent the speed of the signal.

Find an expression for d_i, the distance between the receiver and the i^{th} satellite, in terms of S_i, T_i, and other relevant parameters.

(e) Each satellite's position in 3D space is (u_i, v_i, w_i), where $1 \leq i \leq k$. The receiver position is given by (x, y, z). We need a linear system of equations the receiver can use to solve for its position, $\vec{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Due to limitations of the hardware, the receiver can only handle **linear** systems of equations. **How many satellites must the receiver get data from to solve for the receivers position?**

4. Inner Products

For each of the following functions, show whether it defines an inner product on the given vector space. If not, give a counterexample.

(a) For $\mathbb{R}^2$:

$$\langle \vec{p}, \vec{q} \rangle = \vec{p}^T \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \vec{q}$$

(b) For $\mathbb{R}^2$:

$$\langle \vec{p}, \vec{q} \rangle = \vec{p}^T \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix} \vec{q}$$

(c) For $\mathbb{R}^2$:

$$\langle \vec{p}, \vec{q} \rangle = \vec{p}^T \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \vec{q}$$

(d) For $\mathbb{R}^{2x2}$, the space of all 2x2 real matrices, the *Frobenius* inner product is defined as:

$$\langle A, B \rangle_F = Tr(A^T B)$$

Where A and B are 2x2 real matrices, and Tr represents the *Trace* of a matrix, or the sum of its diagonal entries. Prove that the Frobenius inner product is valid over $\mathbb{R}^{2x2}$.

5. Cauchy-Schwarz Inequality

Learning Goal: *The objective of this problem is to understand and prove the Cauchy-Schwarz inequality for real-valued vectors.*

The Cauchy-Schwarz inequality states that for two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$:

$$|\langle \vec{v}, \vec{w} \rangle| = |\vec{v}^T \vec{w}| \leq \|\vec{v}\| \cdot \|\vec{w}\|$$

In this problem we will prove the Cauchy-Schwarz inequality for vectors in $\mathbb{R}^2$.

Take two vectors: $\vec{v} = r \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and $\vec{w} = t \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$, where $r > 0$, $t > 0$, θ, and ϕ are scalars. Make sure you understand why any vector in $\mathbb{R}^2$ can be expressed this way and why it is acceptable to restrict $r, t > 0$.

(a) In terms of some or all of the variables r, t, θ, and ϕ, what are $\|\vec{v}\|$ and $\|\vec{w}\|$? *Hint: Recall the trig identity:* $\cos^2 x + \sin^2 x = 1$

(b) In terms of some or all of the variables r, t, θ, and ϕ, what is $\langle \vec{v}, \vec{w} \rangle$? *Hint: The trig identity* $\cos(a)\cos(b) + \sin(a)\sin(b) = \cos(a-b)$ *may be useful.*

(c) Show that the Cauchy-Schwarz inequality holds for any two vectors in $\mathbb{R}^2$. *Hint: consider your results from part (b). Also recall* $-1 \leq \cos x \leq 1$ *and use both inequalities.*

(d) Note that the inequality states that the inner product of two vectors must be less than *or equal to* the product of their magnitudes. What conditions must the vectors satisfy for the equality to hold? In other words, when is $\langle \vec{v}, \vec{w} \rangle = \|\vec{v}\| \cdot \|\vec{w}\|$?

6. Orthonormal Matrices

Definition: A matrix $U \in \mathbb{R}^{n \times n}$ is called an orthornomal matrix if $U^{-1} = U^T$ and each column of U is a unit vector.

Orthornomal matrices represent linear transformations that preserve angles between vectors and the lengths of vectors. Rotations and reflections, useful in computer graphics, are examples of transformations that can be represented by orthonormal matrices.

(a) Let U be an orthonormal matrix. For two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$, show that $\langle \vec{x}, \vec{y} \rangle = \langle U\vec{x}, U\vec{y} \rangle$, assuming we are working with the Euclidean inner product.

(b) Show that $\|U\vec{x}\| = \|\vec{x}\|$, where $\|\cdot\|$ is the Euclidean norm.

(c) How does multiplying $\vec{x}$ by U affect the length of the vector? That is, how do the lengths of $\vec{Ux}$ and $\vec{x}$ compare?

7. Homework Process and Study Group

Who did you work with on this homework? List names and student ID's. (In case you met people at homework party or in office hours, you can also just describe the group.) How did you work on this homework? If you worked in your study group, explain what role each student played for the meetings this week.

