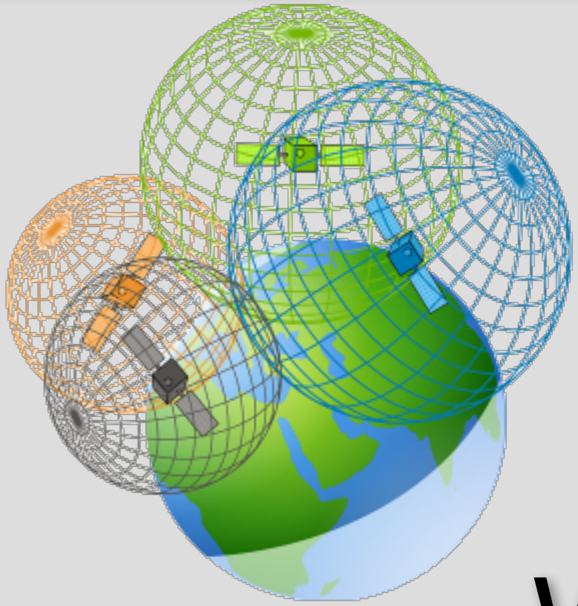
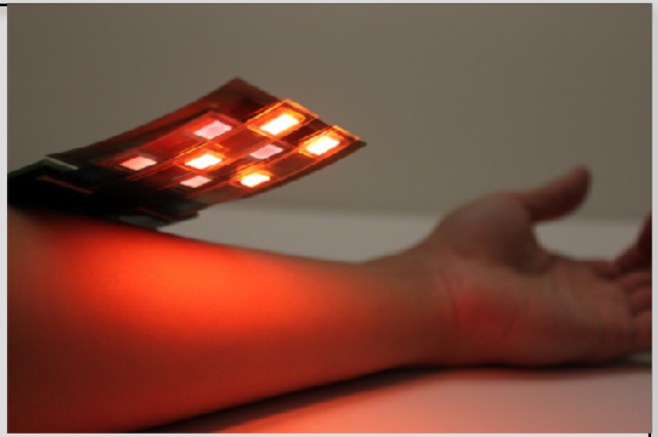




Welcome to EECS 16A!

Designing Information Devices and Systems I

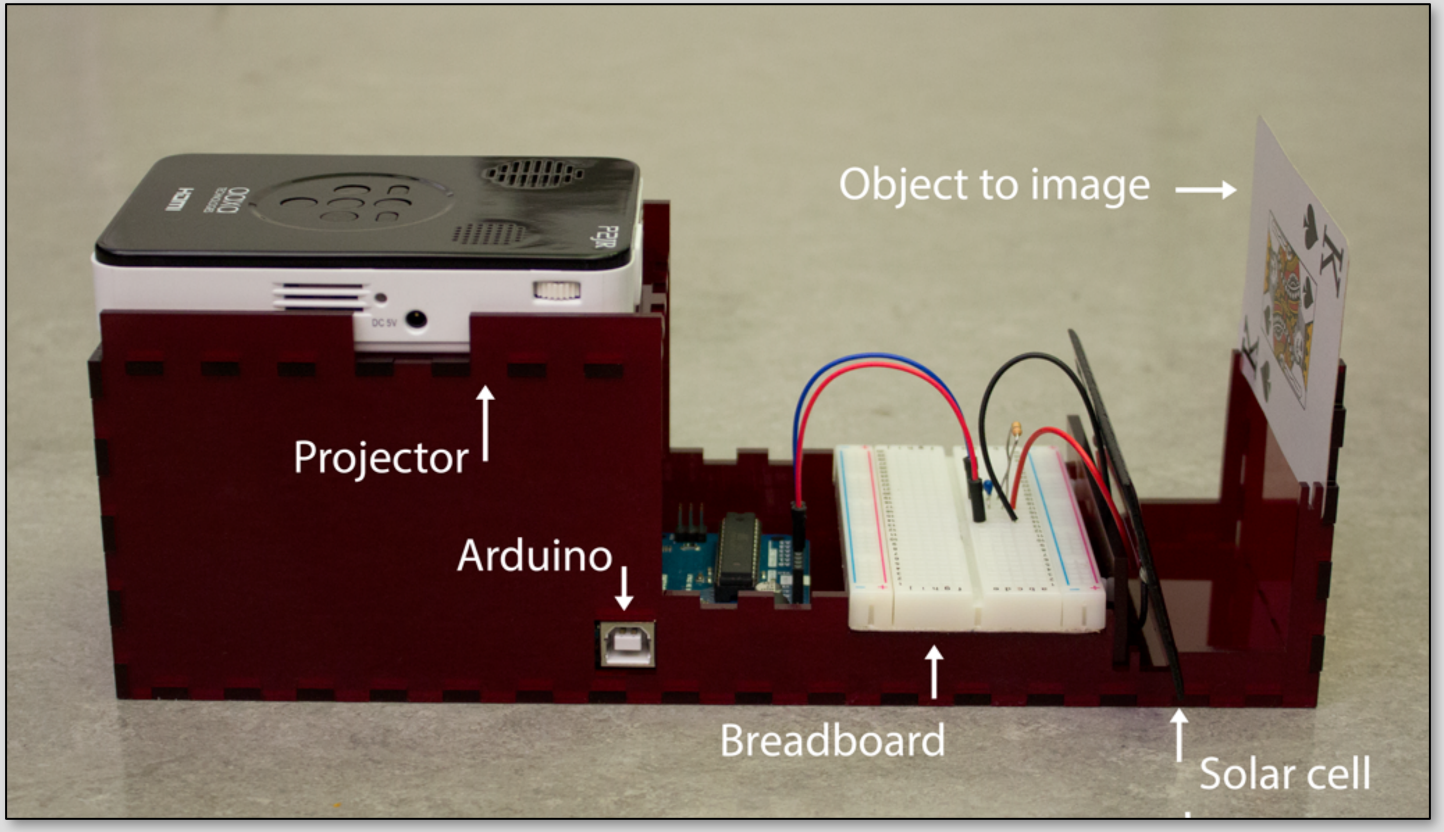
Ana Arias and Miki Lustig

Sp 2022

Lecture 0B

Tomography and Linear Equations

Module 1: Imaging



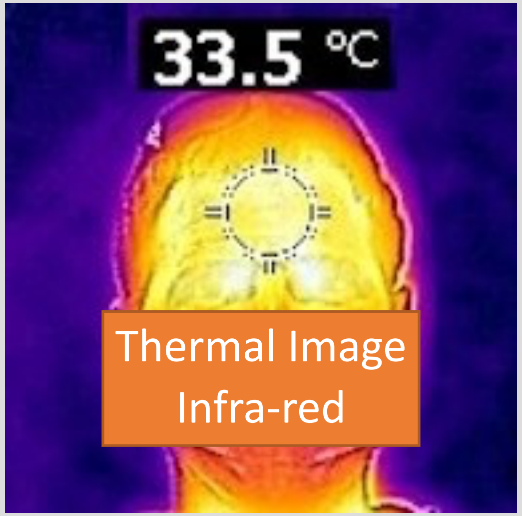
Image

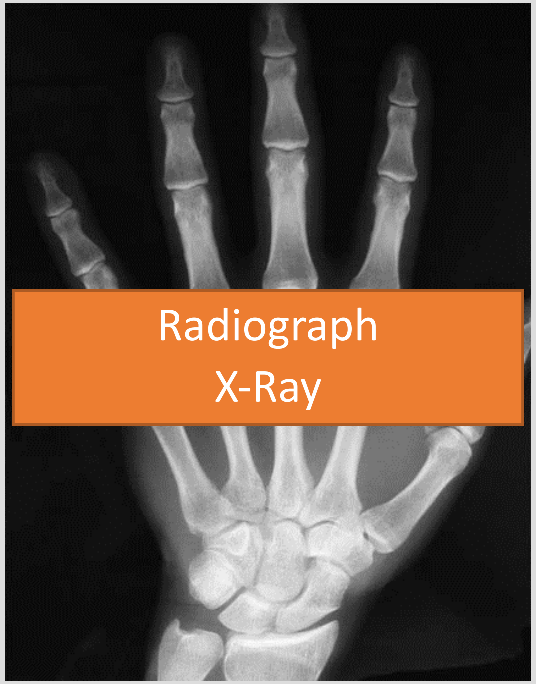
Merriam-Webster: A visual representation of something

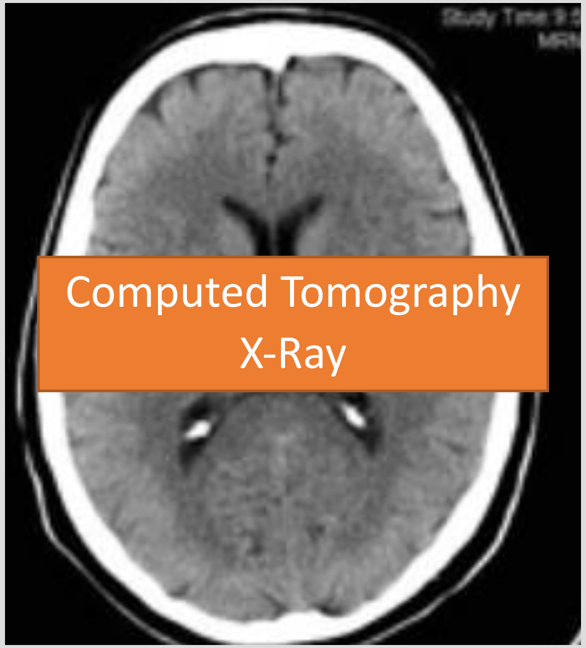
Imaging

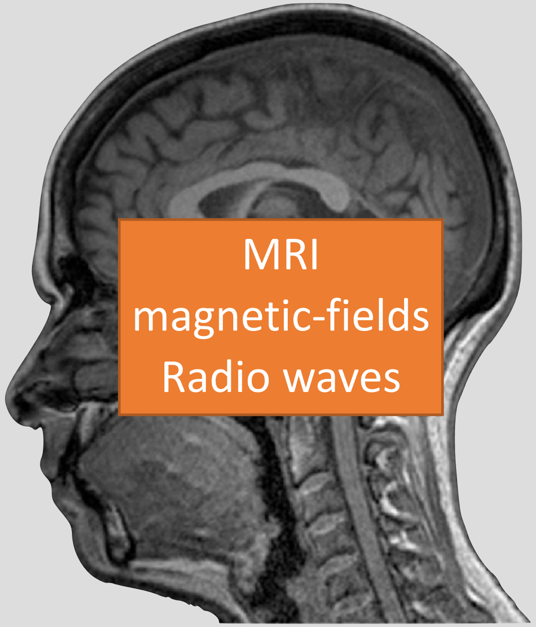
Merriam-Webster: the action or the process of producing an image

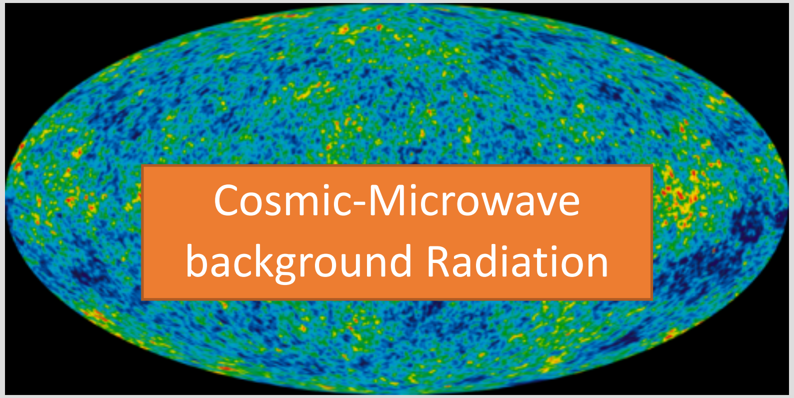
Different Images

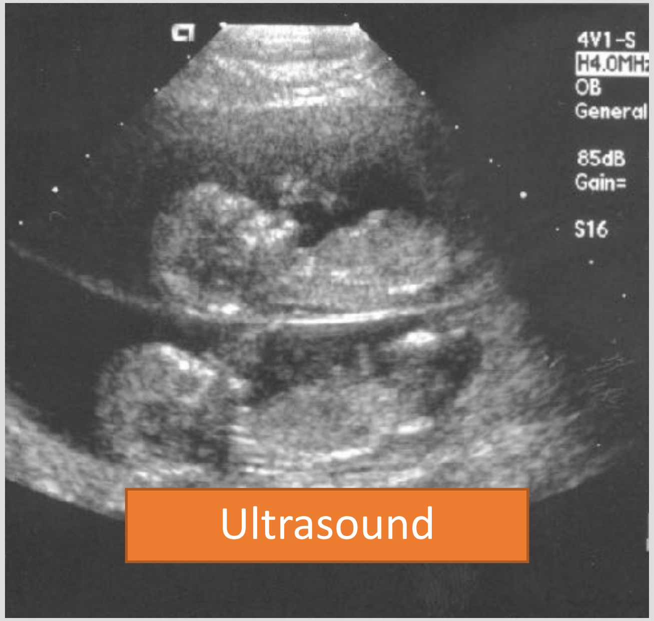


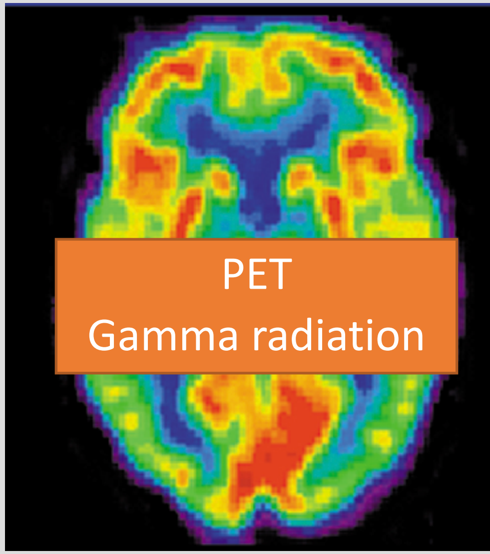




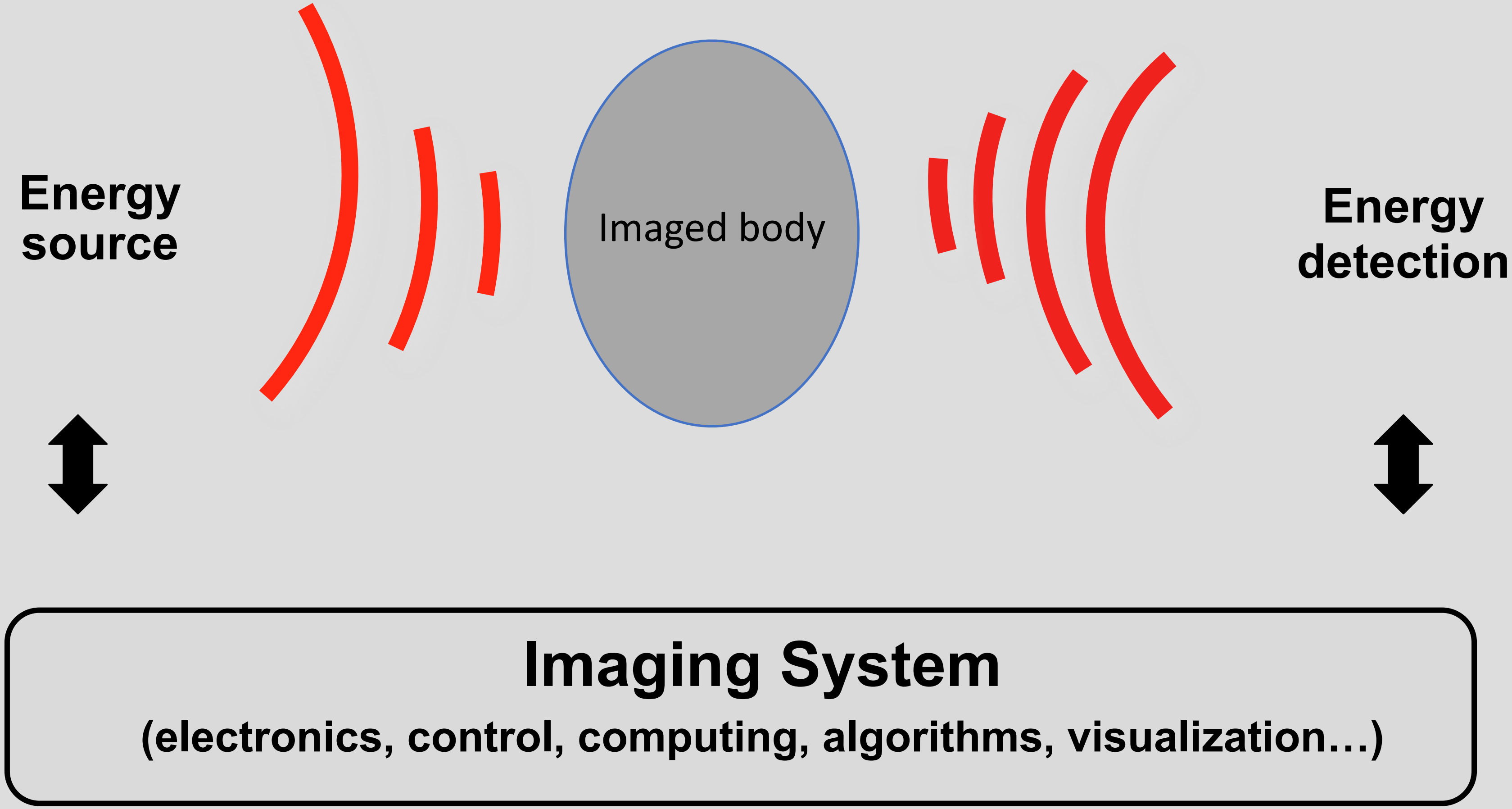






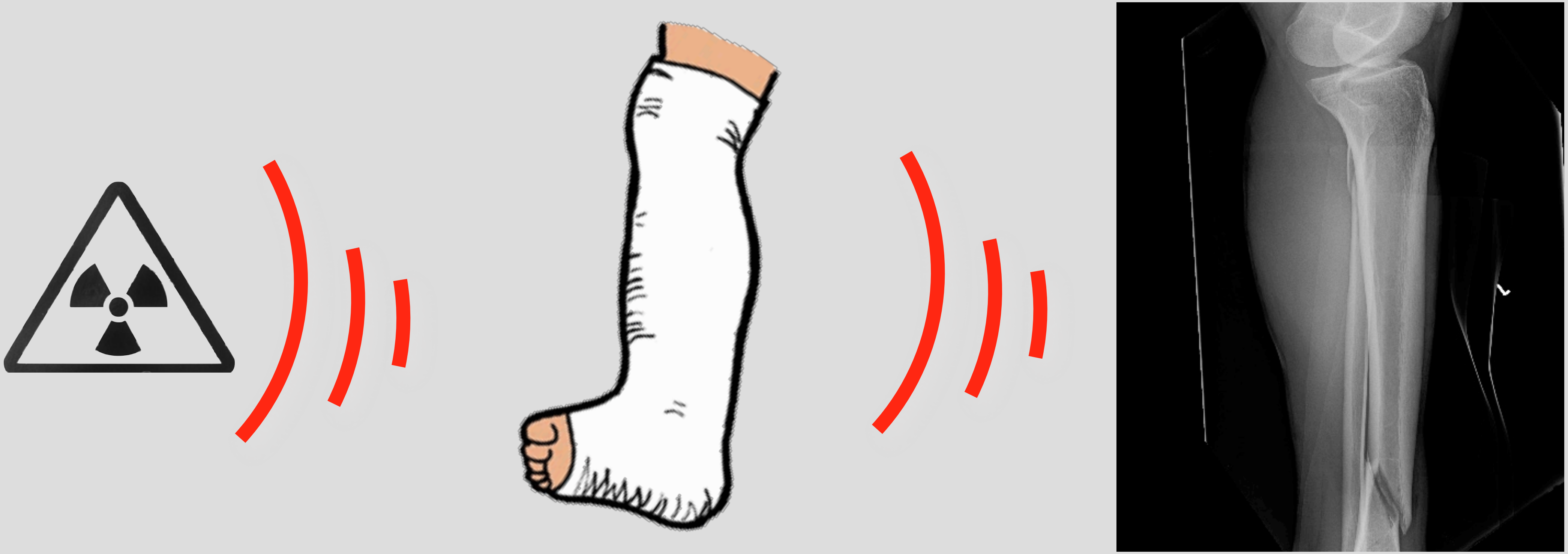


Imaging Systems in General

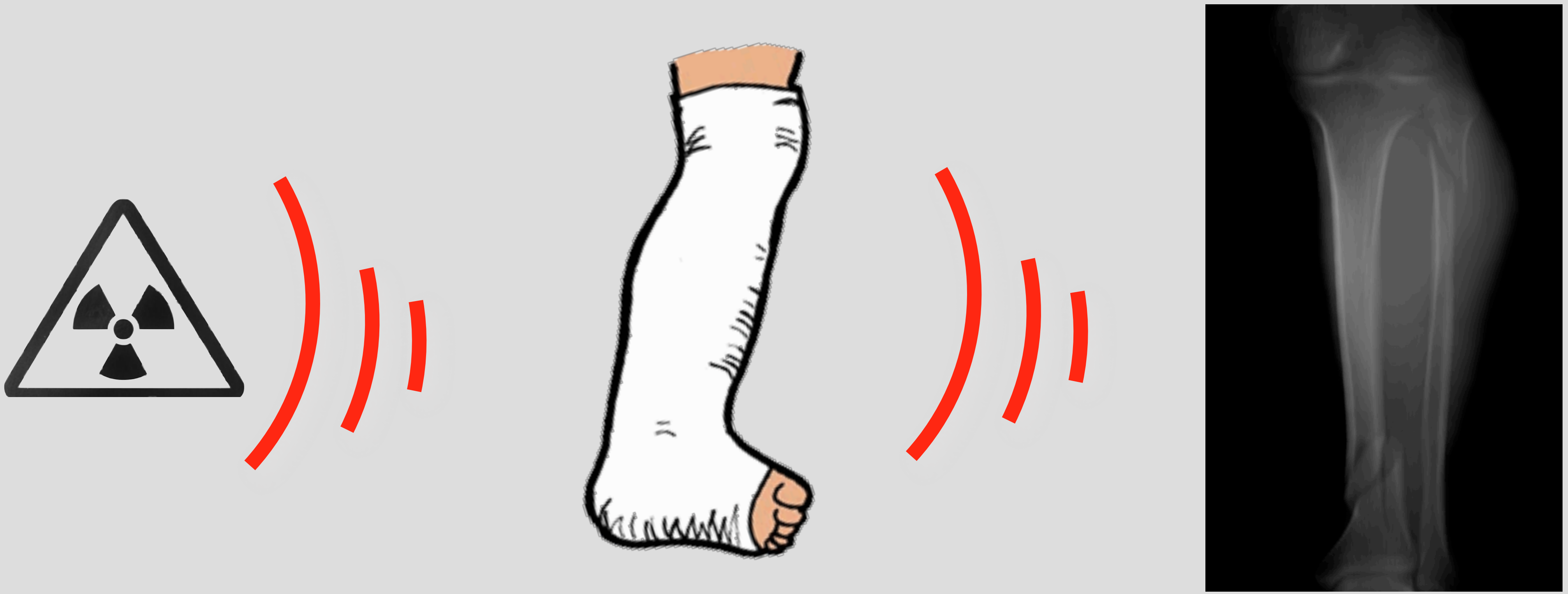


“Medical imaging” circa 1632
“The Anatomy Lesson of Dr. Nicolaes Tulp”, Rembrandt
Mauritshuis, The Hague

Projection Xray



Projection Xray



Tomography

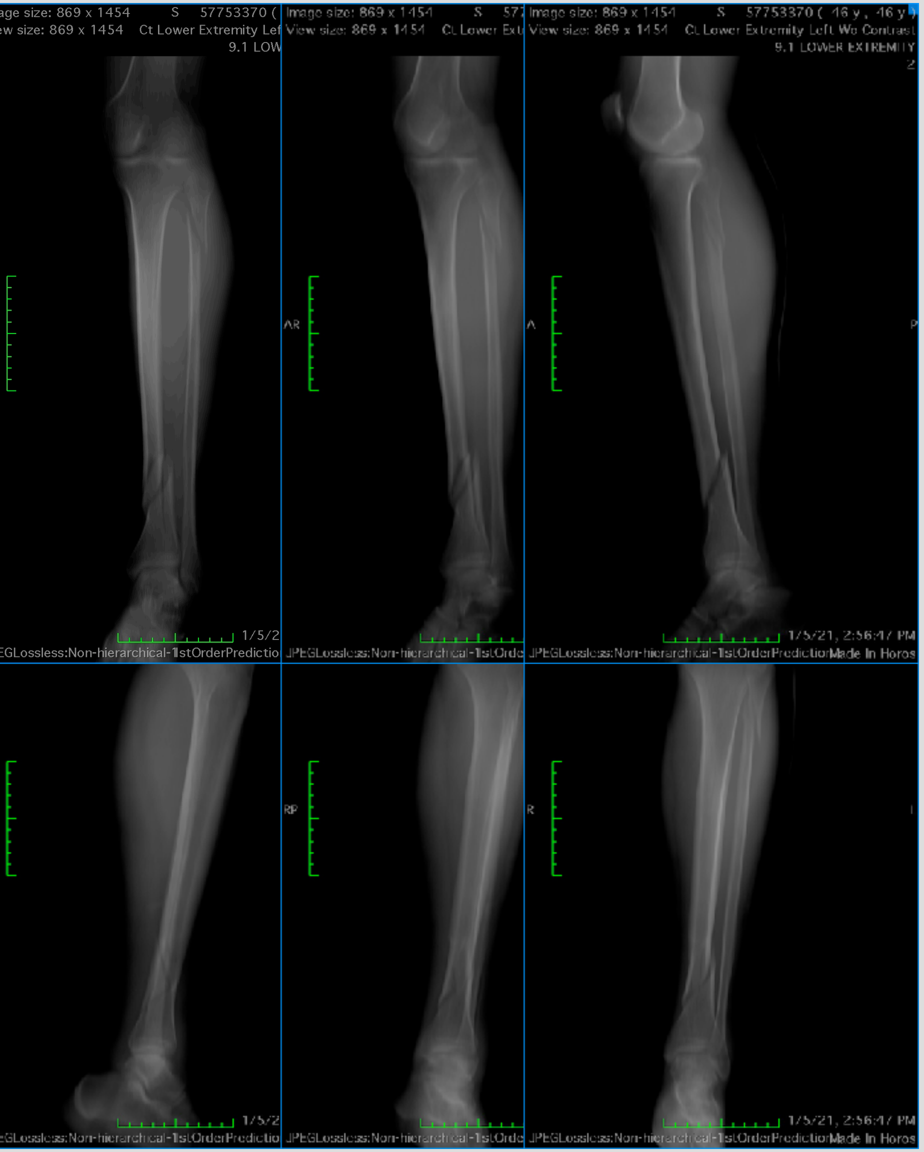
‘tomo’ – slice

‘graphy’ – to write

Assume it is not desirable to slice open leg. How does tomography visualizes cross-sectional slices?

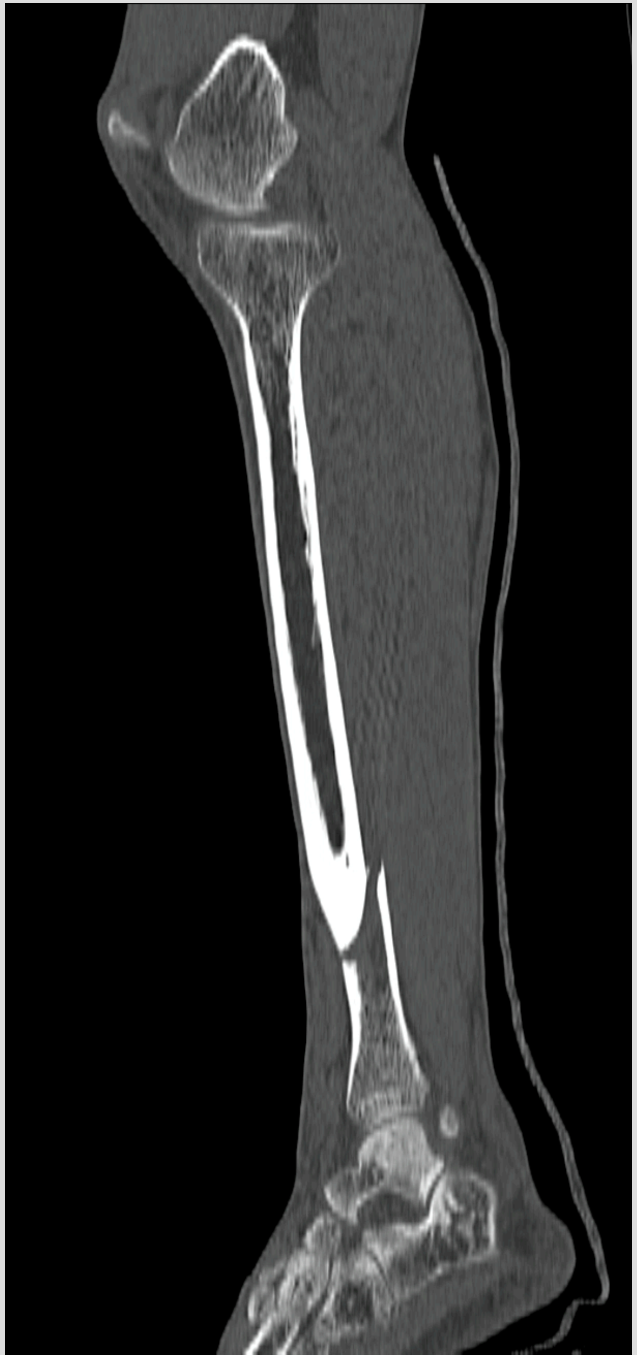
From Projections

Projections

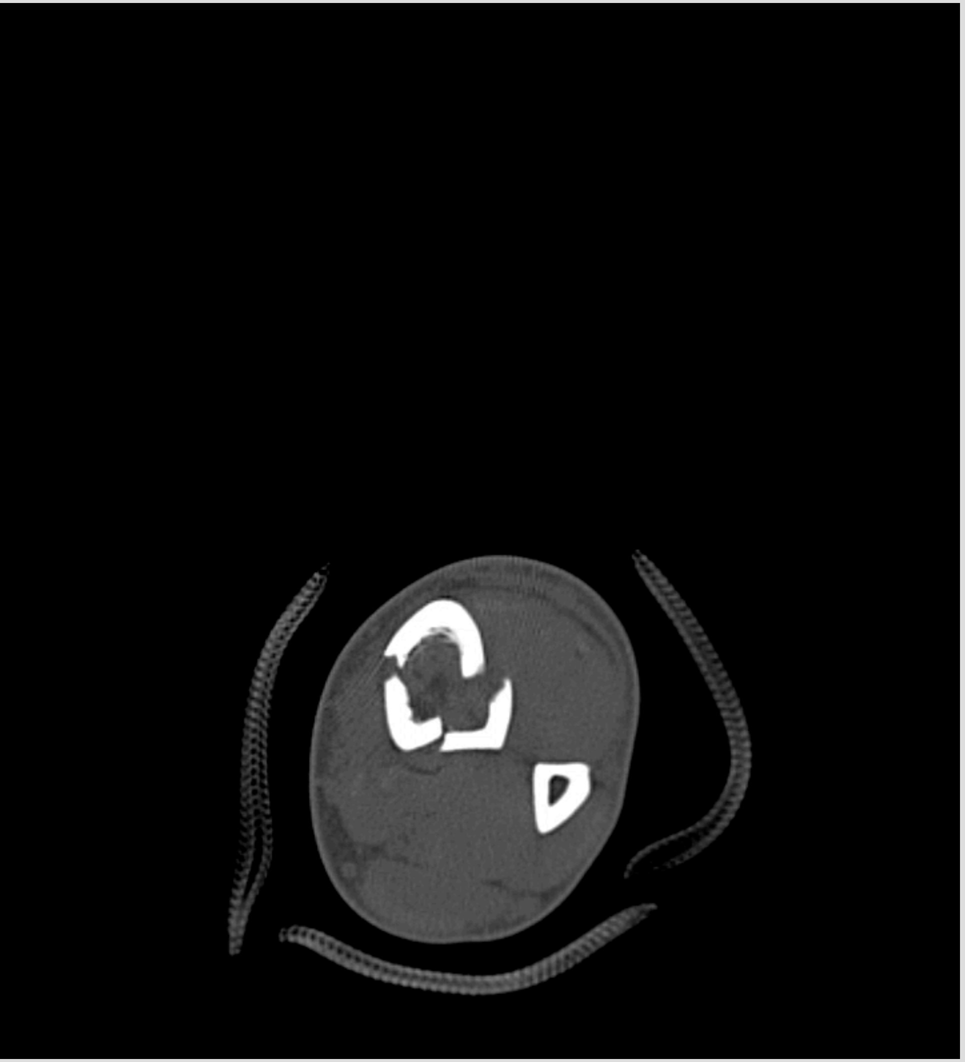


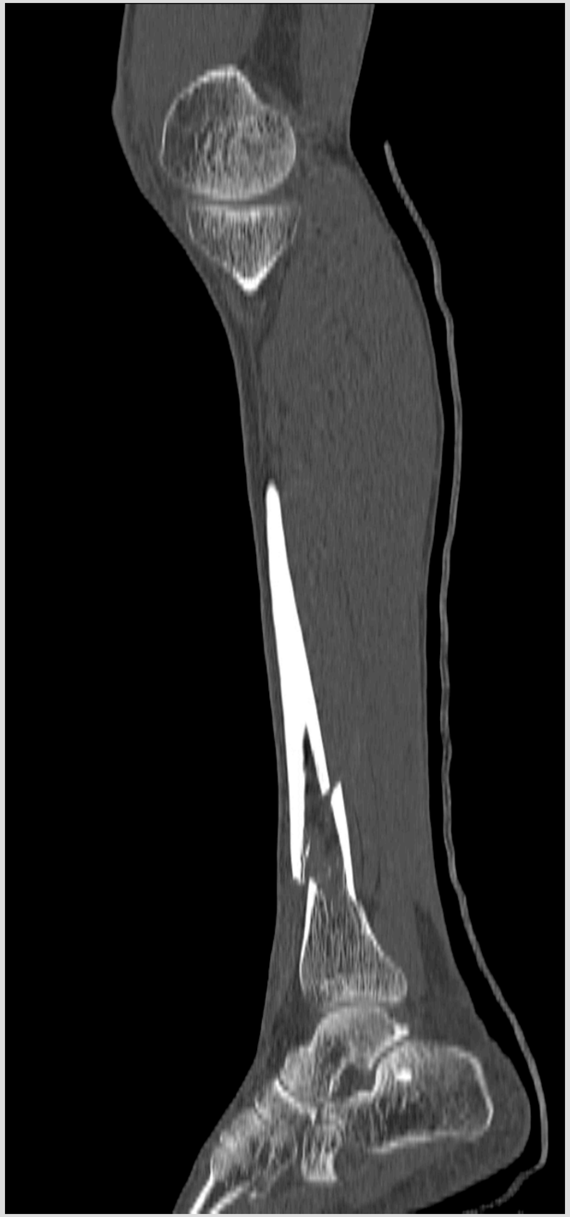
Axial Slices

Sagittal Slices

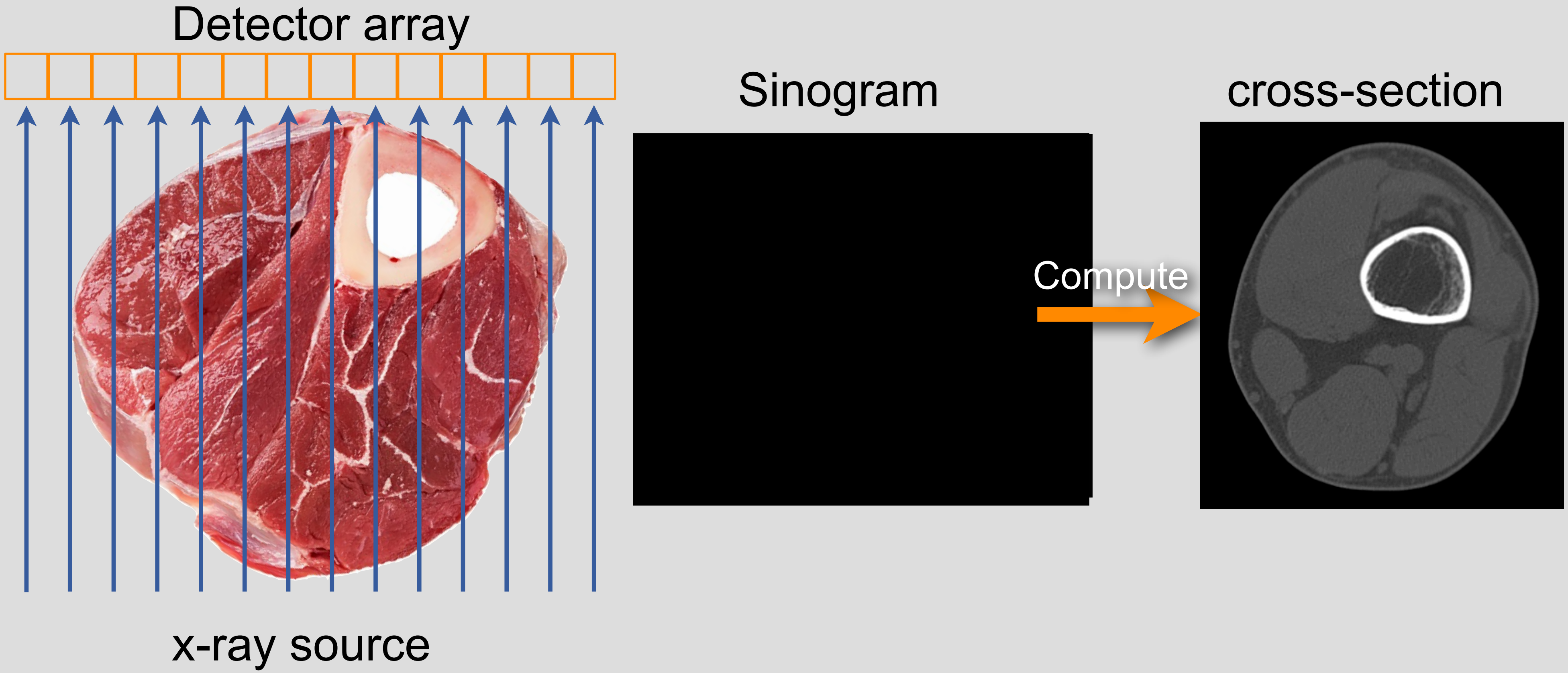


3D Rendering from Slices





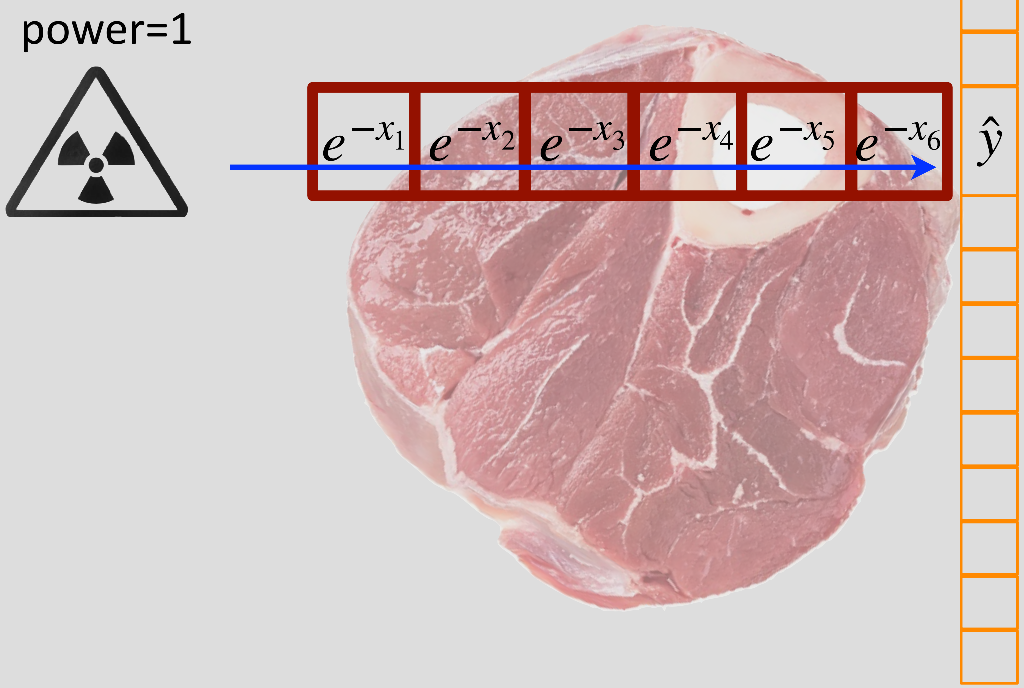
Computed Tomography



Computed Tomography

http://www.youtube.com/watch?v=4gklQHM19aY&feature=related

Modeling Tomography



$$1 \cdot e^{-(x_1+x_2+x_3+x_4+x_5+x_6)} = \hat{y}$$

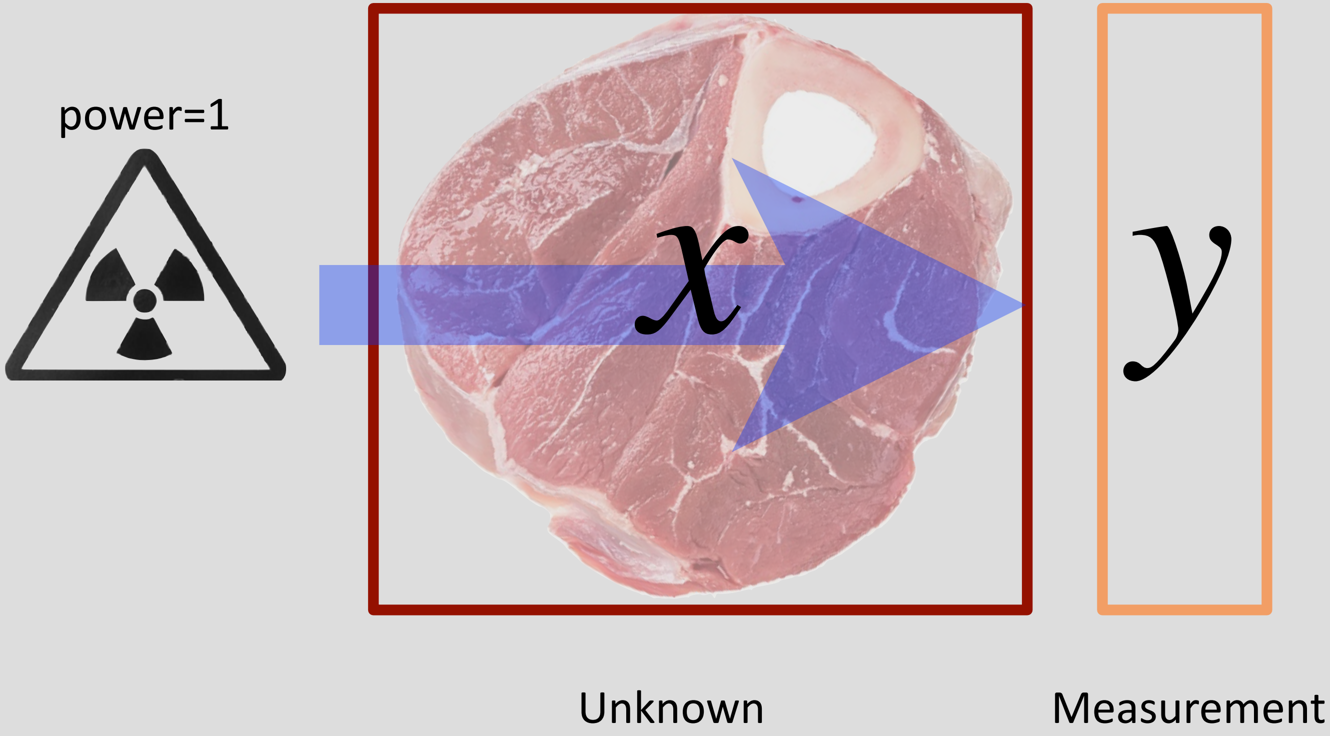
$$\log\{e^{-(x_1+x_2+x_3+x_4+x_5+x_6)}\} = \log\{\hat{y}\}$$

$$y = -\,log\{\hat{y}\}$$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = y$$

…. or y is the sum of x-ray attenuation coefficients along a line

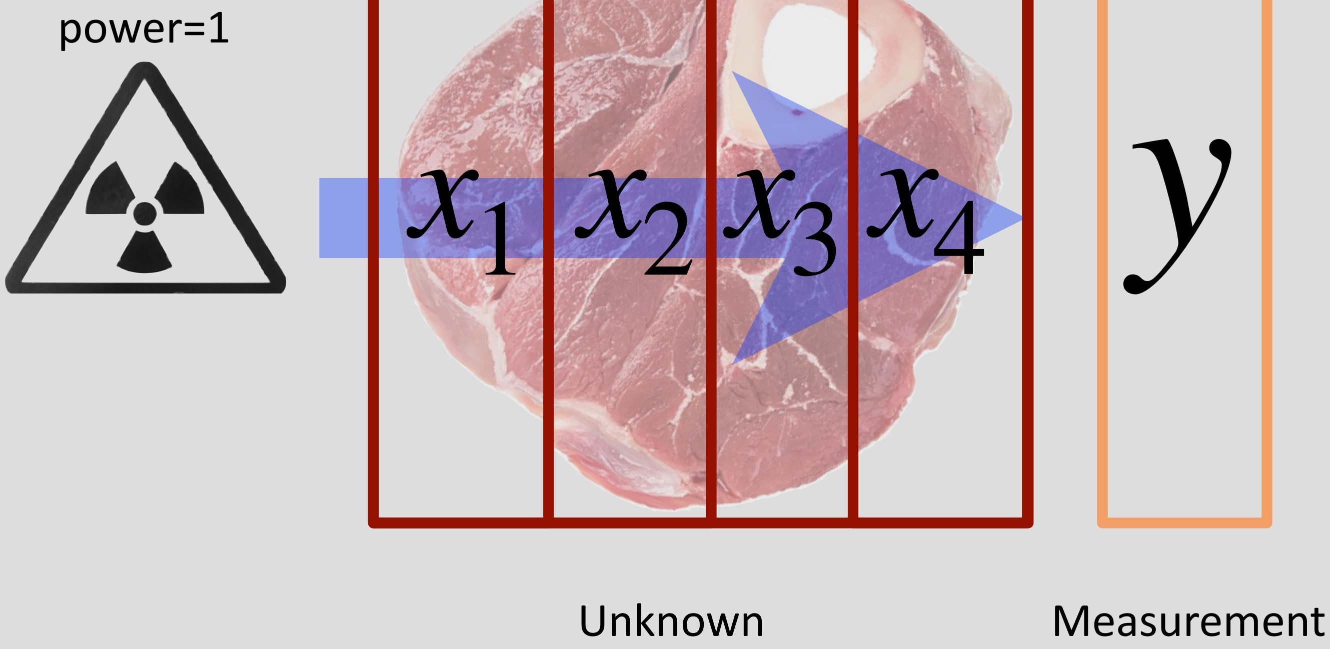
Modeling Tomography



$$y = 1 \cdot x$$

$$x = y$$

Modeling Tomography

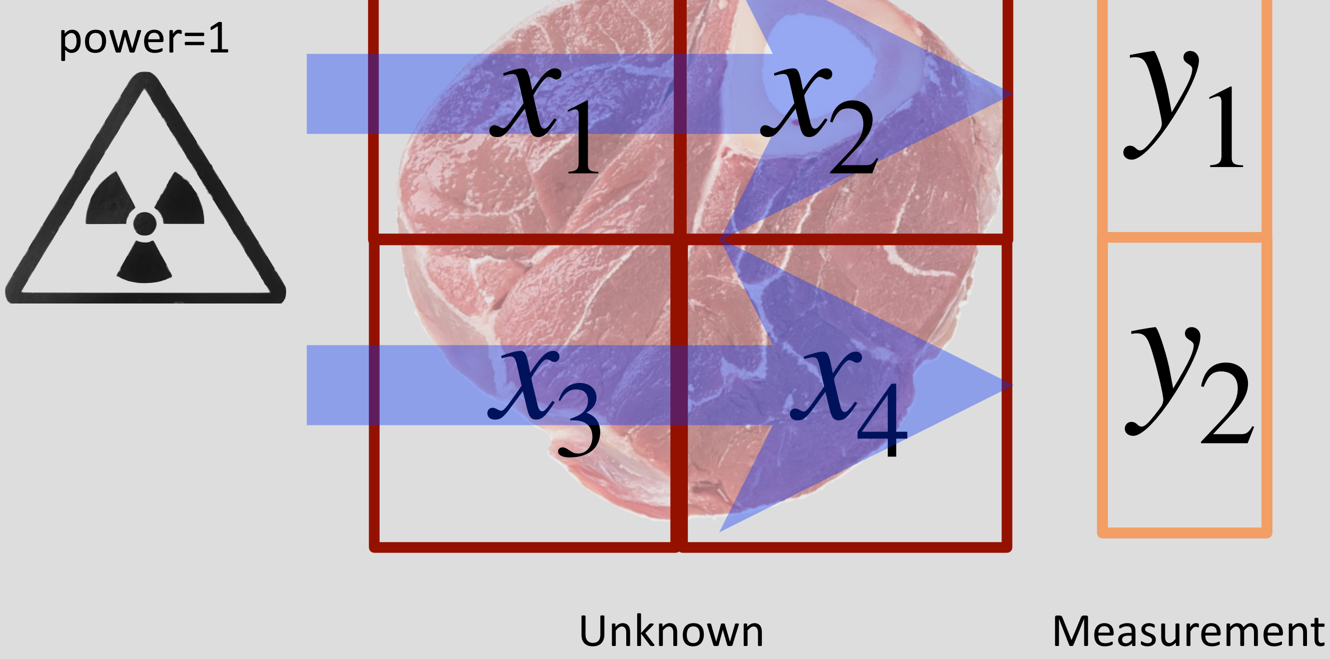


$$y = x_1 + x_2 + x_3 + x_4$$

1 equation 4 unknowns!

😭

Modeling Tomography



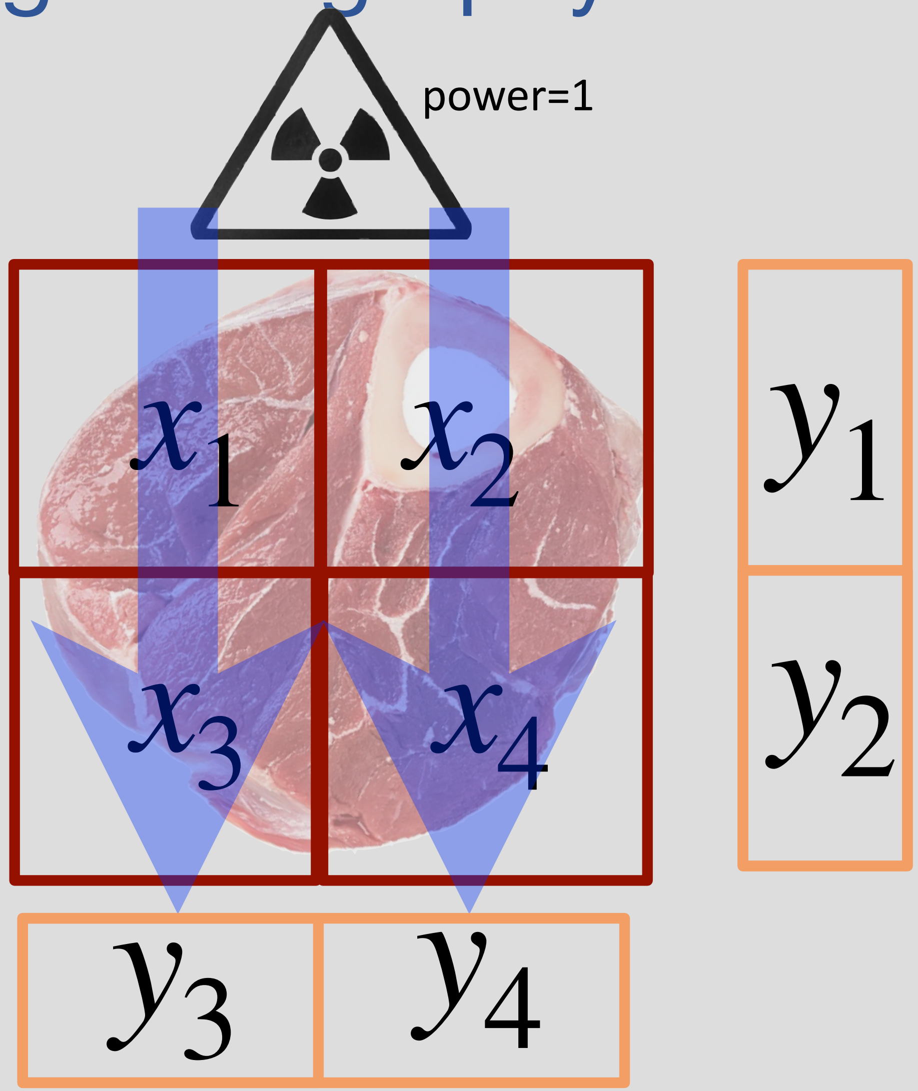
$$y_1 = x_1 + x_2$$
$$y_2 = x_3 + x_4$$

↓

2 equation 4 unknowns!

😭

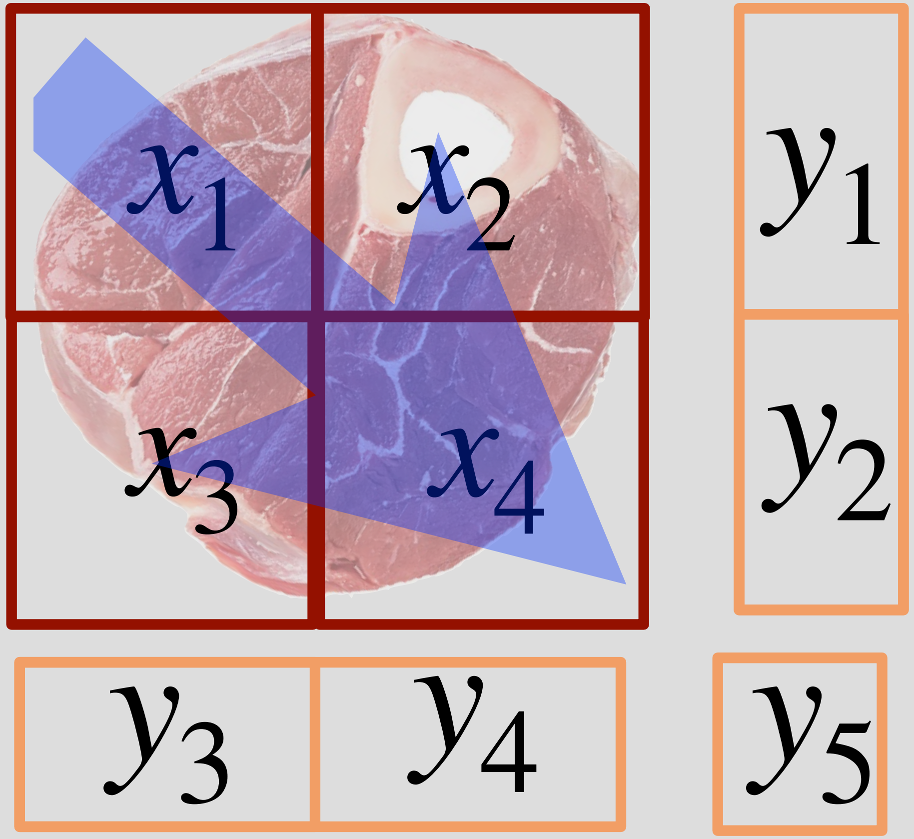
Modeling Tomography



$$
\begin{aligned}
y_1 &= x_1 + x_2 \\
y_2 &= x_3 + x_4 \\
y_3 &= x_1 + x_3 \\
y_4 &= x_2 + x_4
\end{aligned}
$$

Can we solve this?

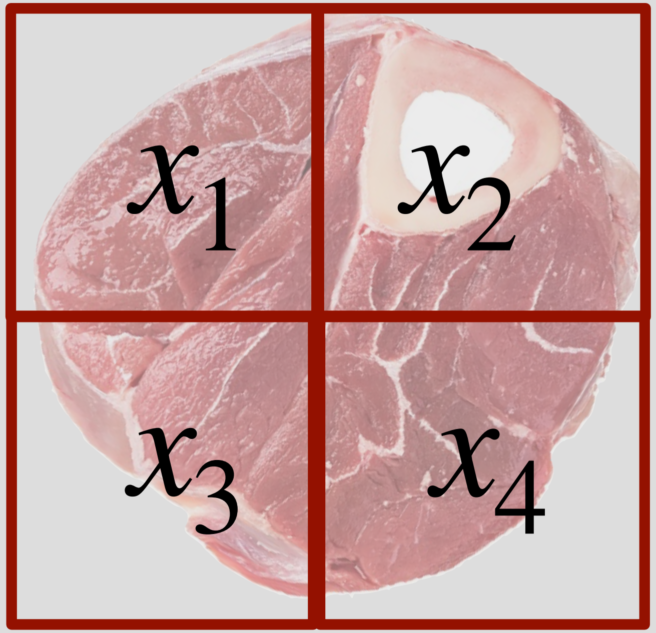
Modeling Tomography



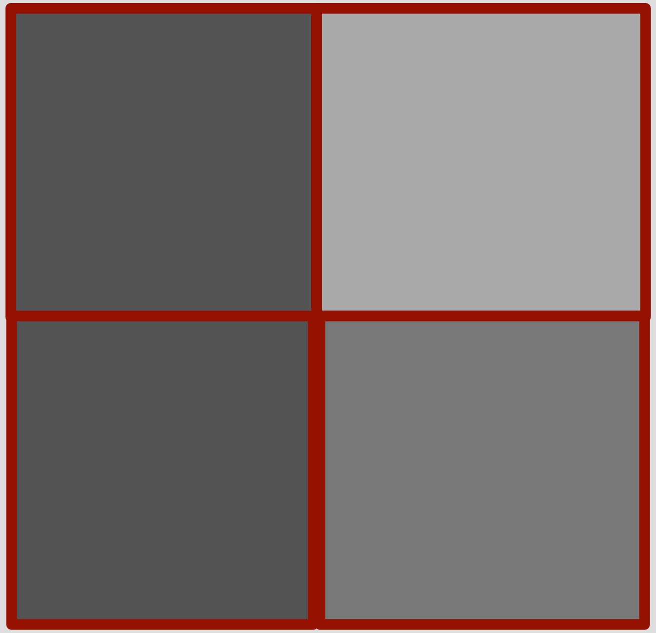
$$y_1 = x_1 + x_2$$
$$y_2 = x_3 + x_4$$
$$y_3 = x_1 + x_3$$
$$y_4 = x_2 + x_4$$
$$y_5 \approx \sqrt{2}x_1 + \sqrt{2}x_4$$

May be able to solve this!

Modeling Tomography



Possible reconstruction



Blurred version of :

All our measurements are (converted to) *linear*

What does that mean?

Each variable (x) is multiplied by a scalar to contribute to the measurement

$$
\begin{aligned}
y_1 &= x_1 + x_2 \\
y_2 &= x_3 + x_4 \\
y_3 &= x_1 + x_3 \\
y_4 &= + x_2 + x_4 \\
y_5 &= \sqrt{2}x_1 + \sqrt{2}x_4
\end{aligned}
$$

This is called a
system of linear equations

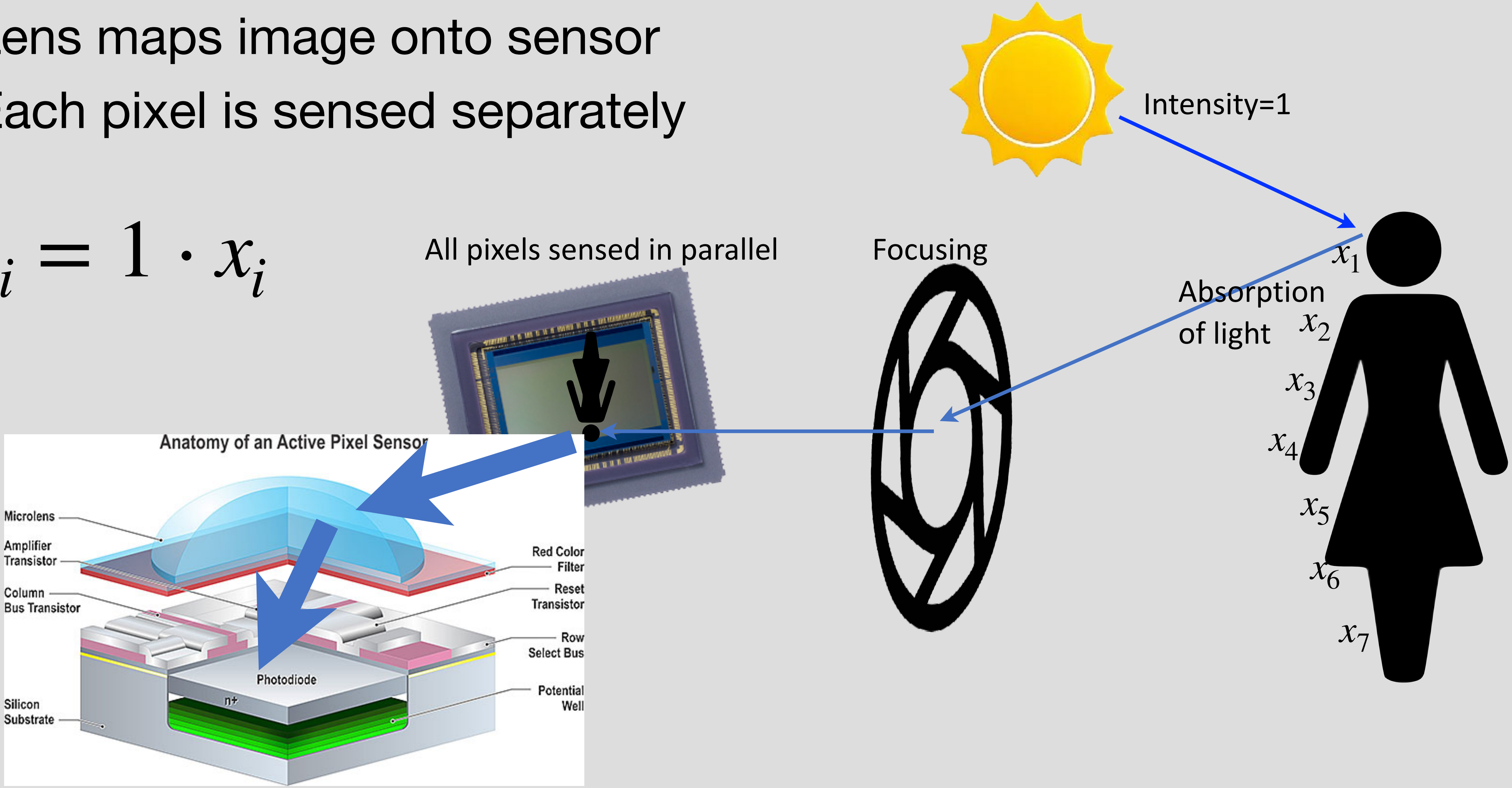
Linear Algebra **is what we need to solve it!**

Camera Model

Lens maps image onto sensor

Each pixel is sensed separately

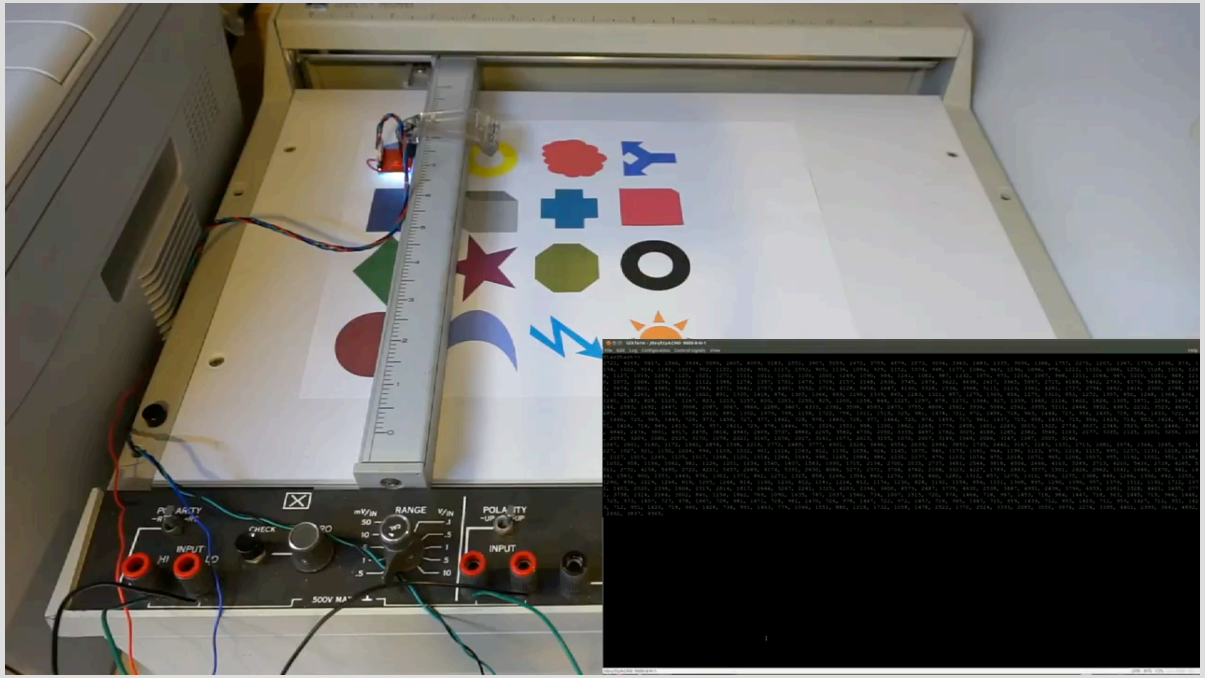
$$y_i = 1 \cdot x_i$$

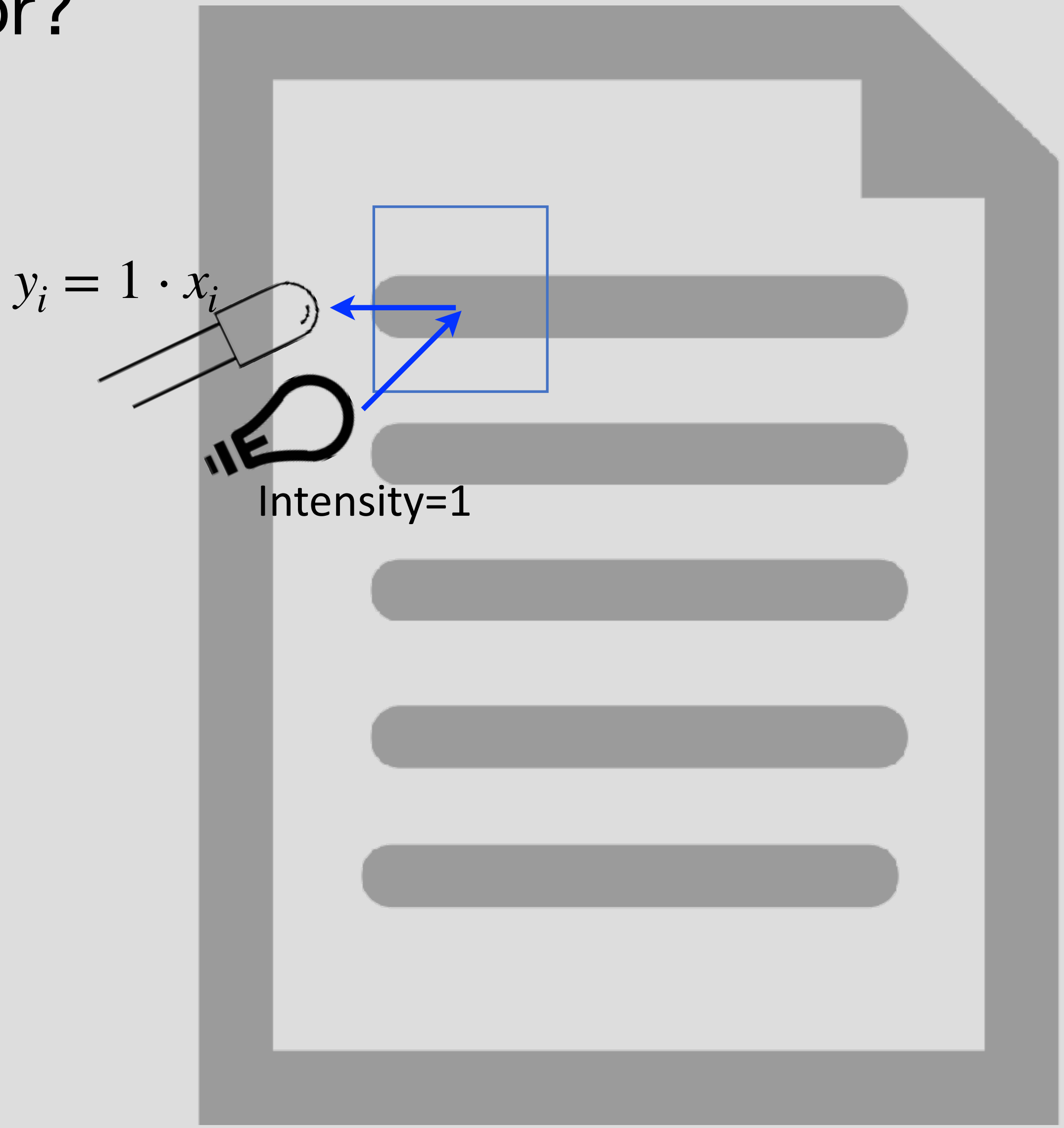


Single Pixel Scanner

- What if we had only a single sensor?
- How can we create an image?

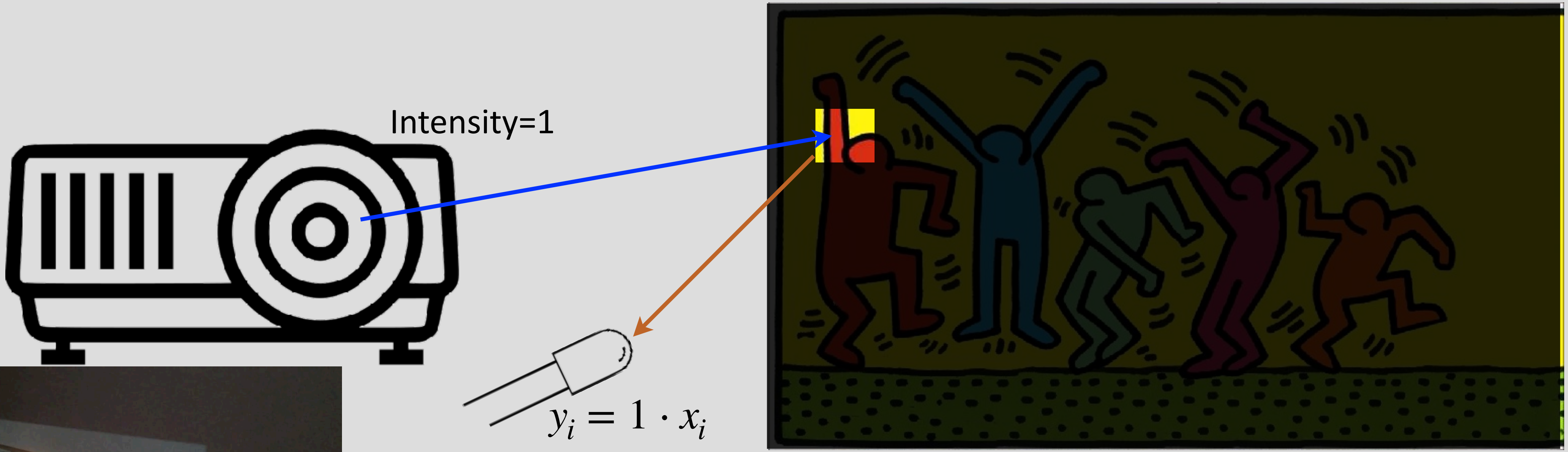
https://www.youtube.com/watch?v=U5PwsVqHT8Y





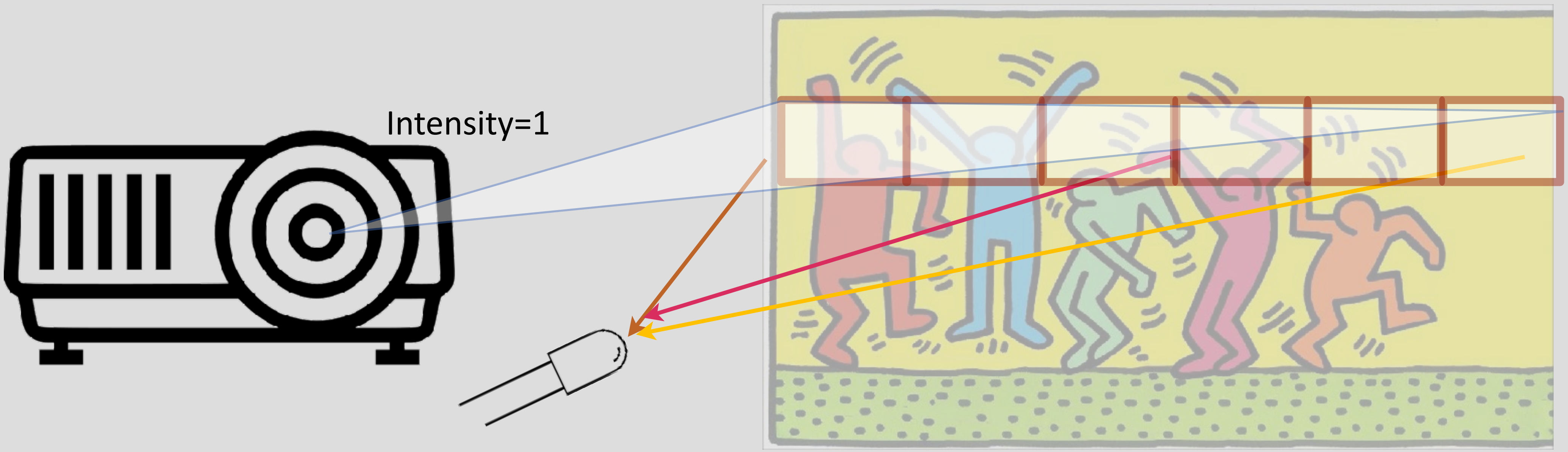
Non-moving Single Pixel Camera

- Use a projector to illuminate pixels
- Sense reflected light with a sensor



Non-moving Single Pixel Camera

- Use a projector to illuminate several pixels!
- Sense reflected light with a sensor
- Make many measurements and solve the equations!



$$y_1 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

Similar math as Tomography!

Non-moving Single Pixel Camera

- Use a projector to illuminate several pixels!
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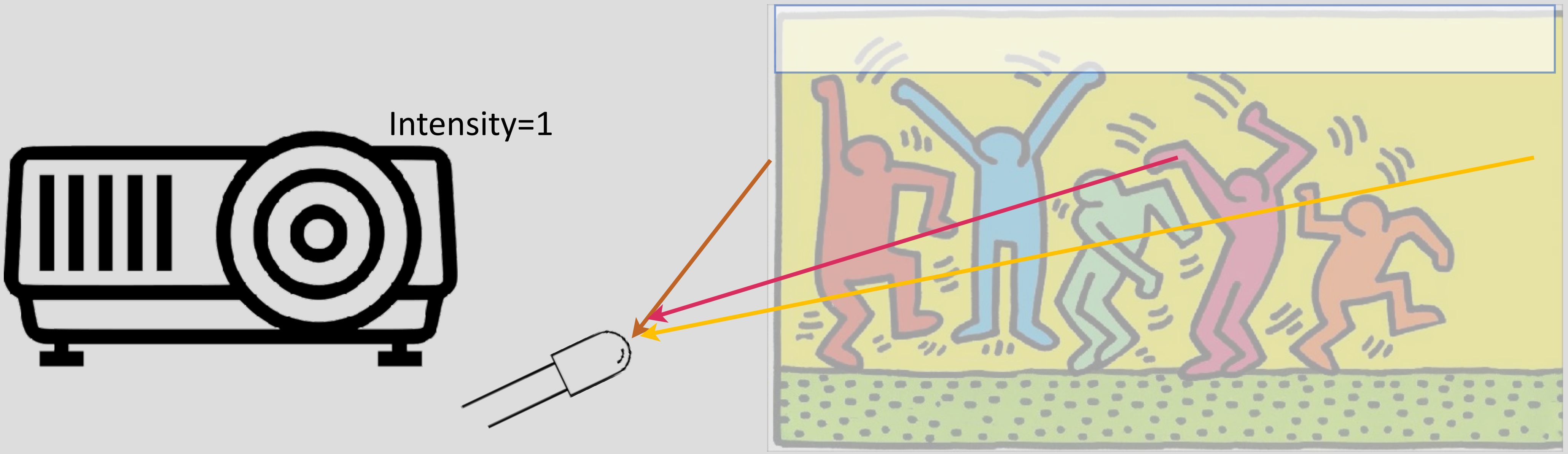

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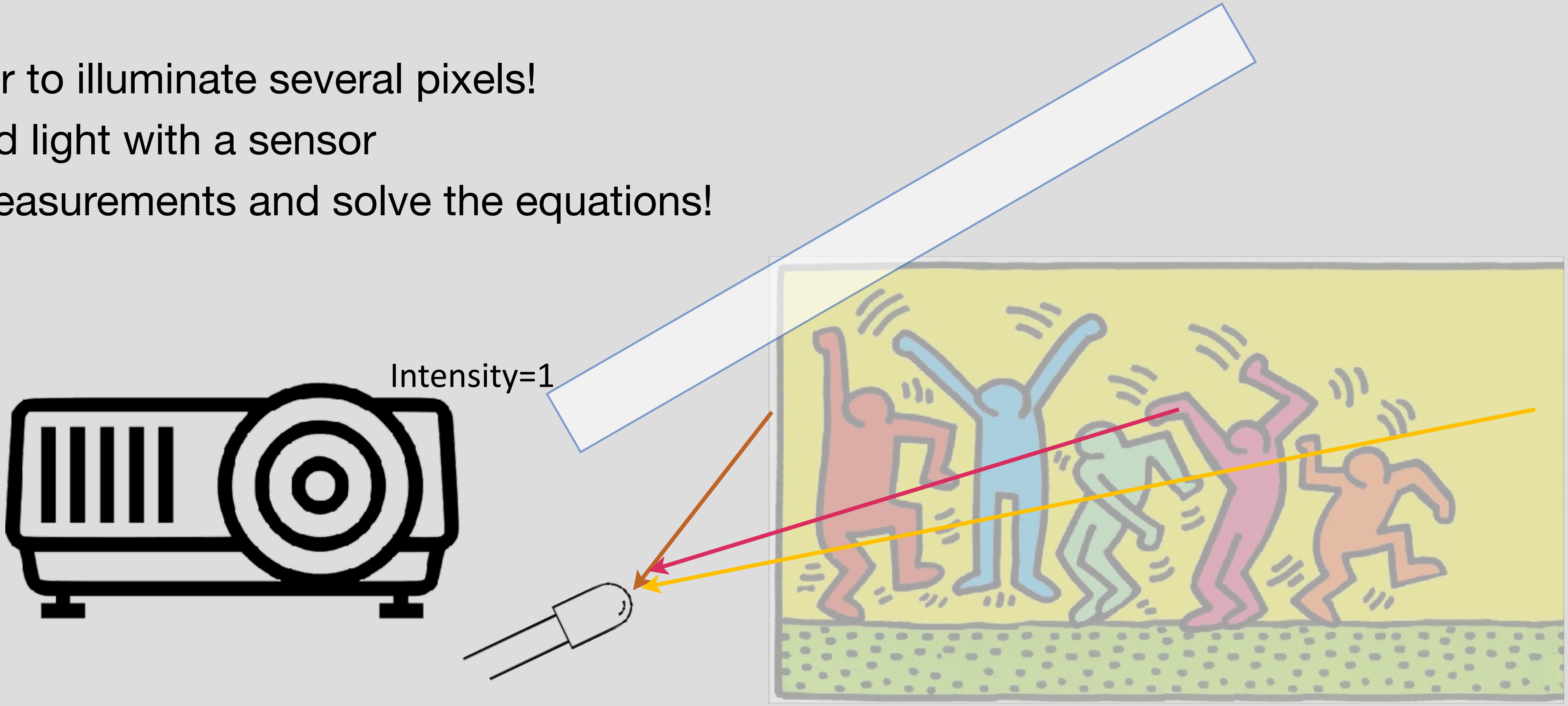

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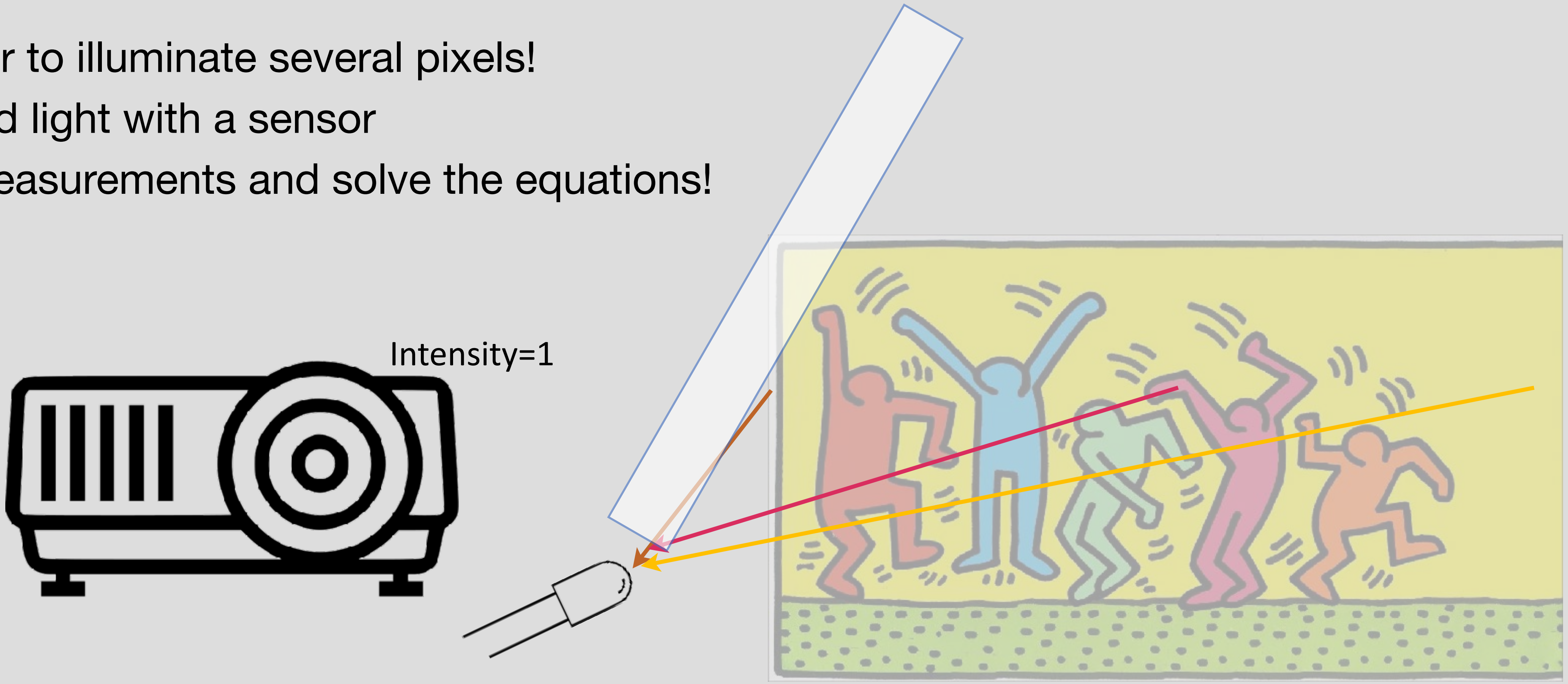

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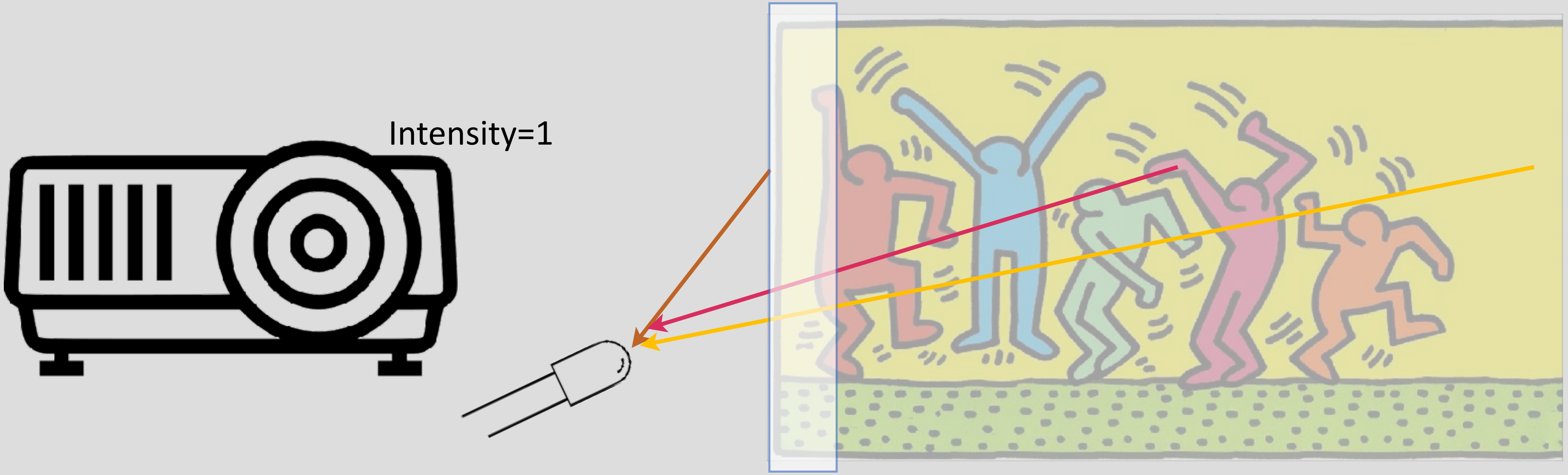

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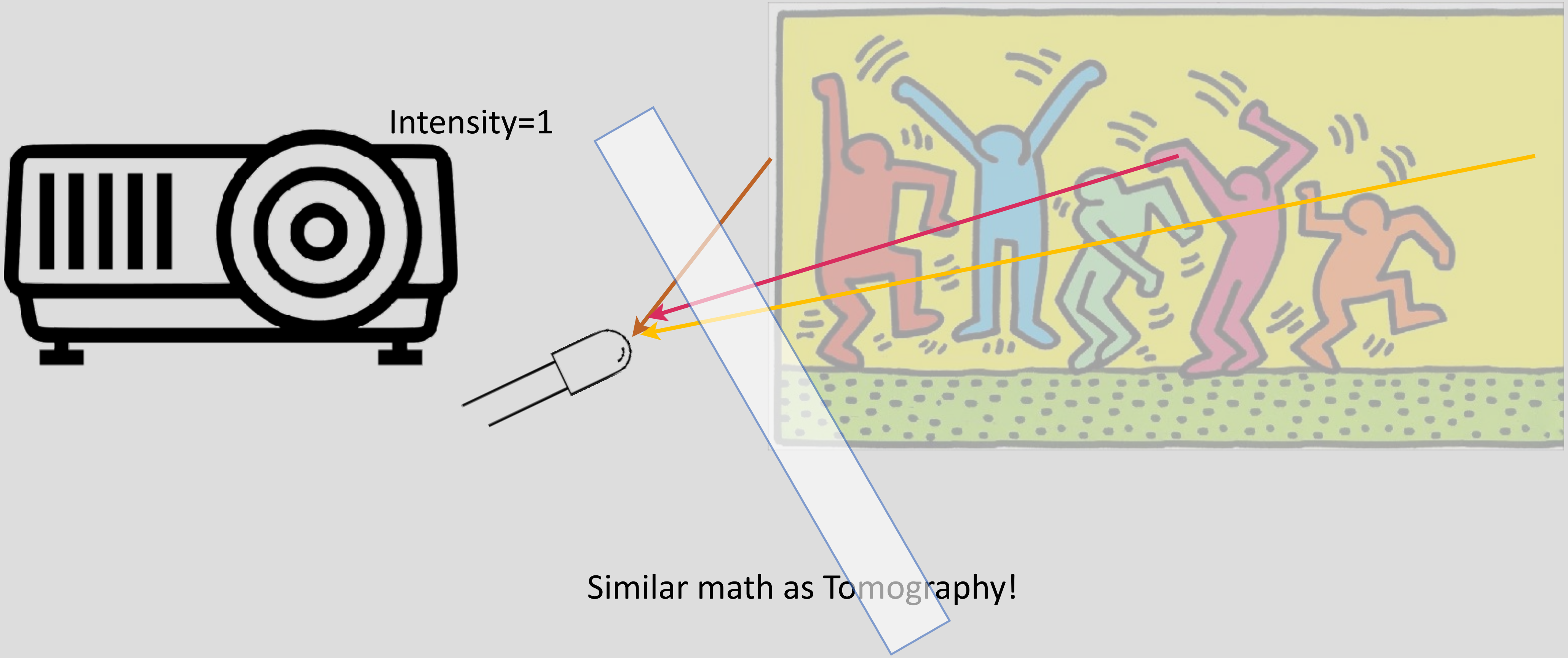

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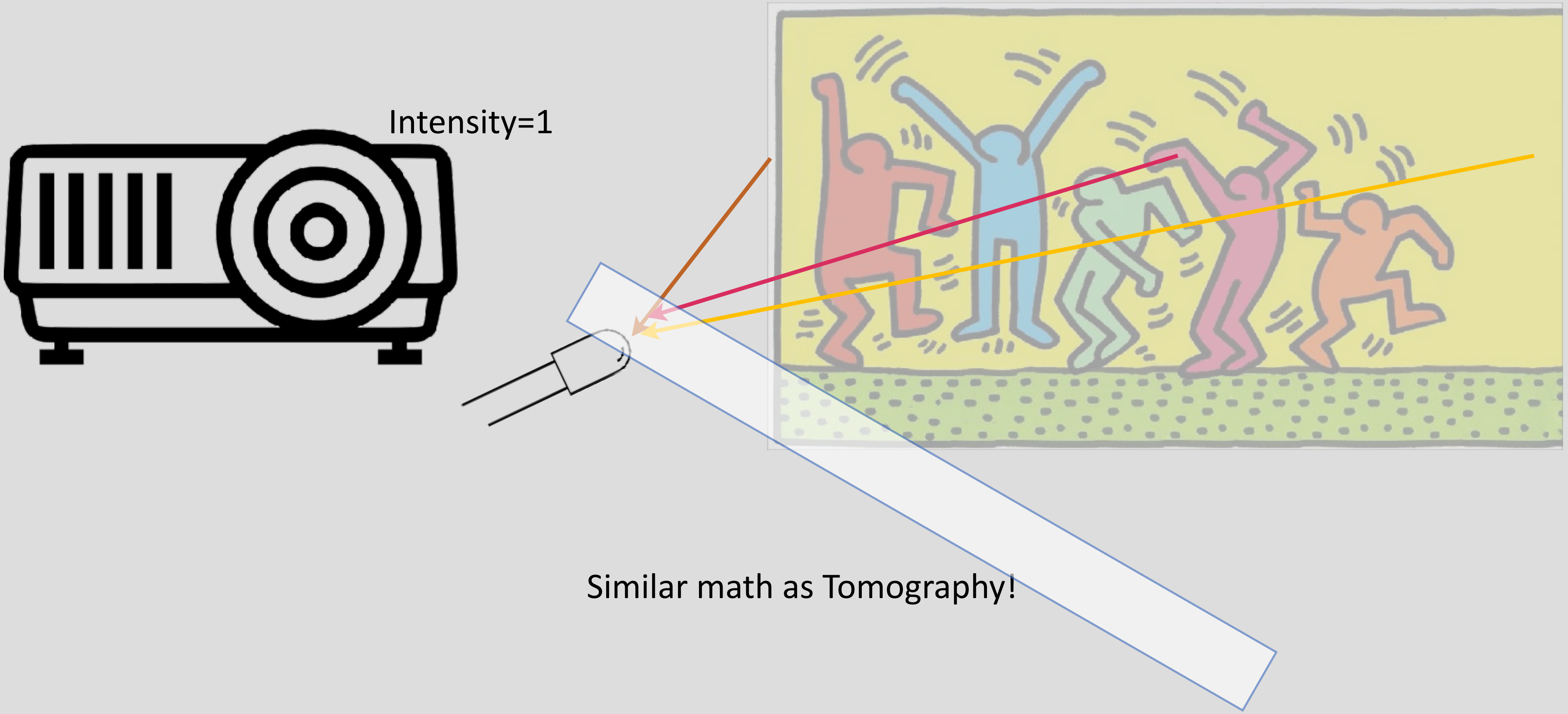
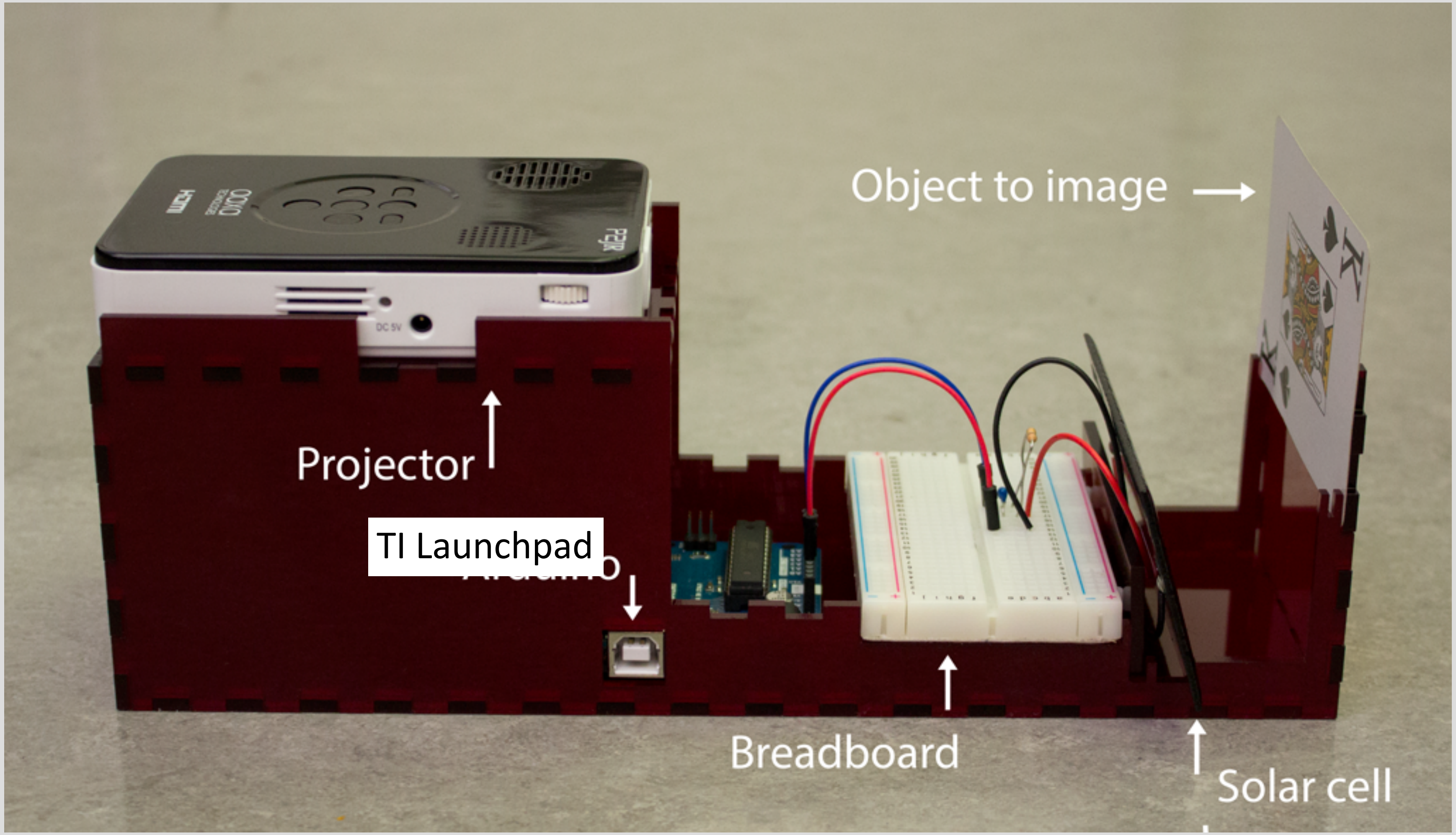
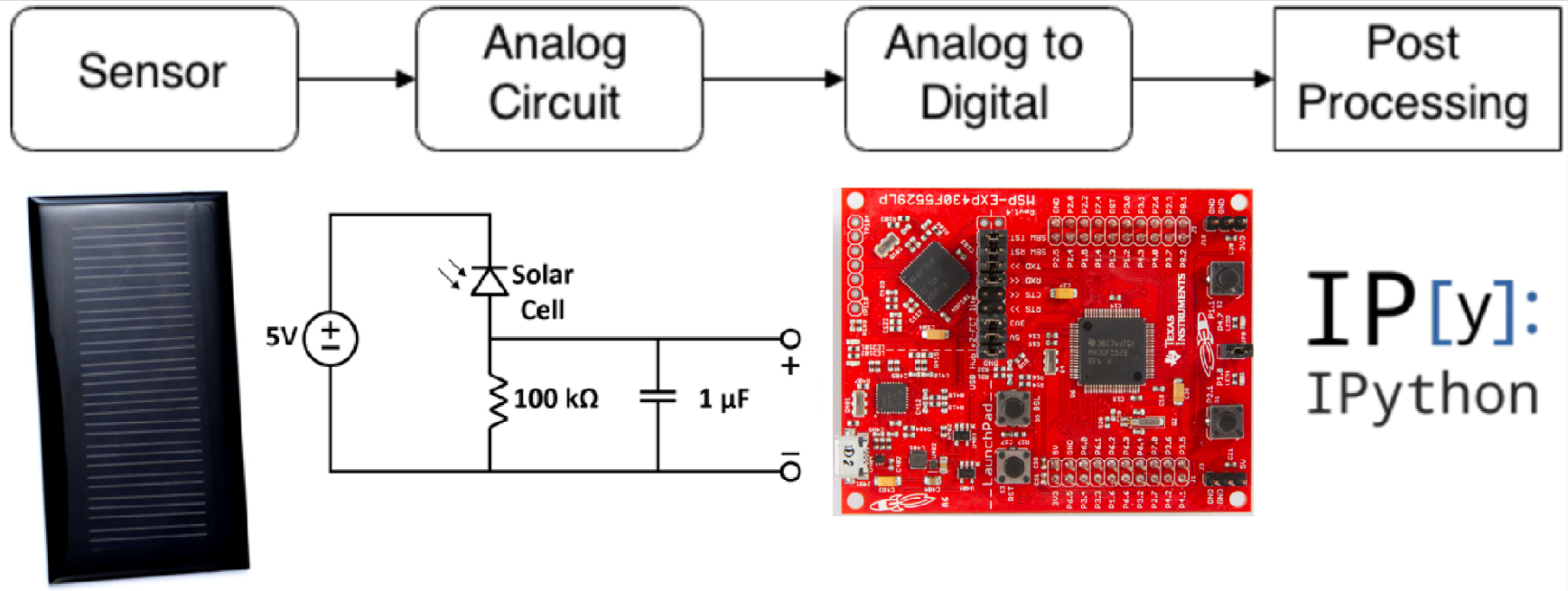


Imaging Lab #1 Setup

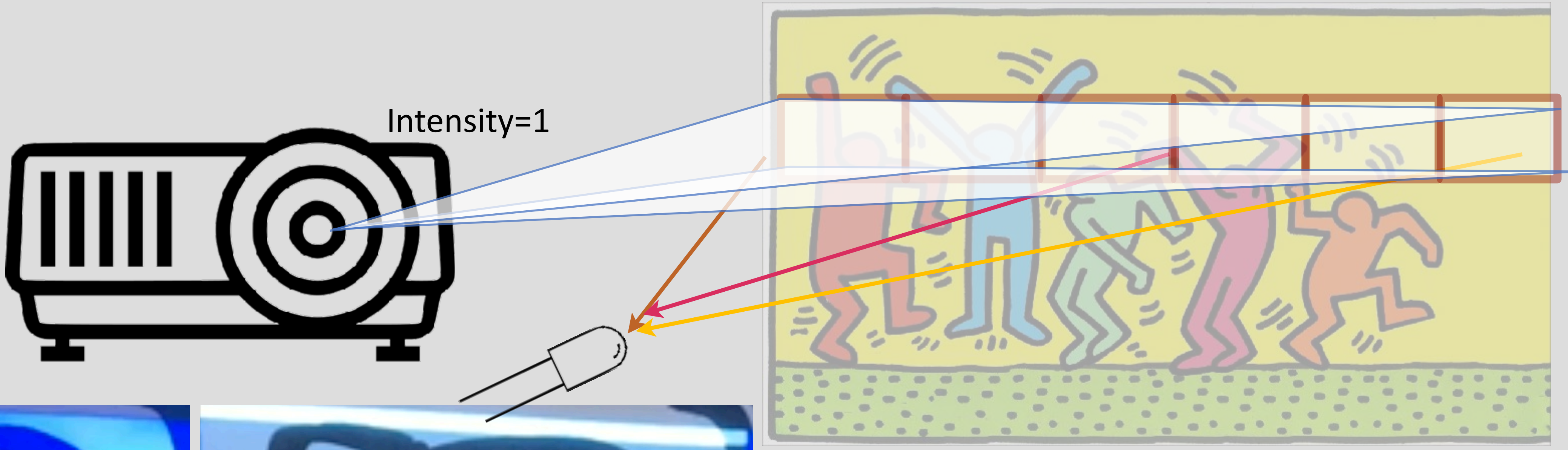


Imaging Lab #1



Non-moving Single Pixel Camera

- How many measurements do you need?
- What are the best patterns?



What is linear algebra?

- The study of linear functions and linear equations, typically using vectors and matrices
- Linearity is not always applicable, but can be a good first-order approximation
- There exist good fast algorithms to solve these problems

Linear Equations

- Definition:

  Consider: $f(x_1, x_2, \cdots, x_N) : \mathbb{R}^n \to \mathbb{R}$

  f is *linear if the following identity holds:*

  (1) Homogeneity:

$$f(\alpha x_1, \cdots, \alpha x_N) = \alpha f(x_1, \ldots, x_N)$$

  (2) Super Position (distributivity): if $x_i = y_i + z_i$, *then*

$$f(y_1 + z_1, \cdots, y_N + z_N) = f(y_1, \cdots, y_N) + f(z_1, \cdots, z_N)$$

  Claim: linear functions can always be expressed as:

$$f(x_1, x_2, \cdots, x_N) = c_1 x_1 + c_2 x_2 + \cdots + c_N x_N$$

Proof for $\mathbb{R}^2$

- $f(x_1, x_2) : \mathbb{R}^2 \Rightarrow \mathbb{R}$ is linear. Need to prove: $f(x_1, x_2) = c_1x_1 + c_2x_2$

Trick:

$$x_1 = 1 \cdot x_1 + 0 \cdot x_2 \quad (1 = y_1,\ 0 = z_1) \qquad \Rightarrow x_1 = x_1y_1 + x_2z_1$$

$$x_2 = 0 \cdot x_1 + 1 \cdot x_2 \quad (0 = y_2,\ 1 = z_2) \qquad \Rightarrow x_2 = x_1y_2 + x_2z_2$$

So,

$$\begin{aligned} f(x_1, x_2) &= f(x_1y_1 + x_2z_1,\ x_1y_2 + x_2z_2) \\ &= f(x_1y_1, x_1y_2) + f(x_2z_1, x_2z_2) \\ &= x_1f(y_1, y_2) + x_2f(z_1, z_2) \\ &= x_1f(1,0) + x_2f(0,1) \quad (f(1,0) = c_1,\ f(0,1) = c_2) \\ &= c_1x_1 + c_2x_2 \end{aligned}$$

Linear Set of Equations

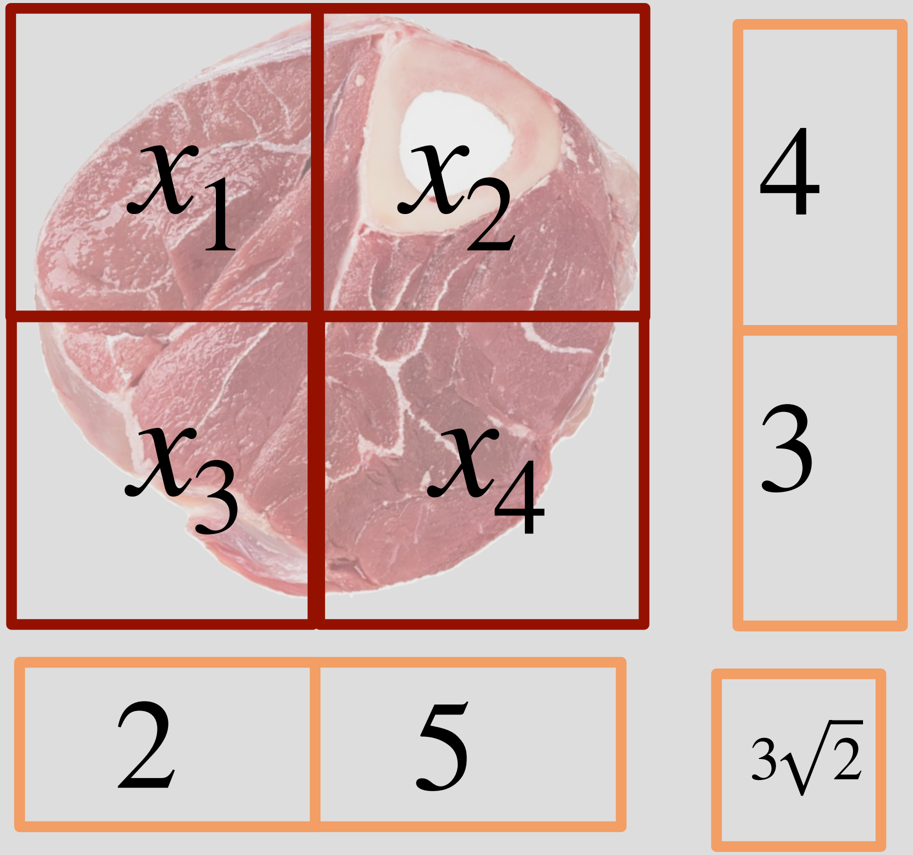
- Consider the set of M linear equations with N variables:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N &= b_2 \\
\vdots \qquad\qquad\qquad & \vdots \\
a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N &= b_M
\end{aligned}
$$

- Can be written compactly using augmented matrix:

$$
\left[\begin{array}{cccc|c}
a_{11} & a_{12} & \cdots & a_{1N} & b_1 \\
a_{21} & a_{22} & \cdots & a_{2N} & b_2 \\
\vdots & & \vdots & & \vdots \\
a_{M1} & a_{M2} & \cdots & a_{MN} & b_M
\end{array}\right]
$$

Back to Tomography



$$1 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 4$$

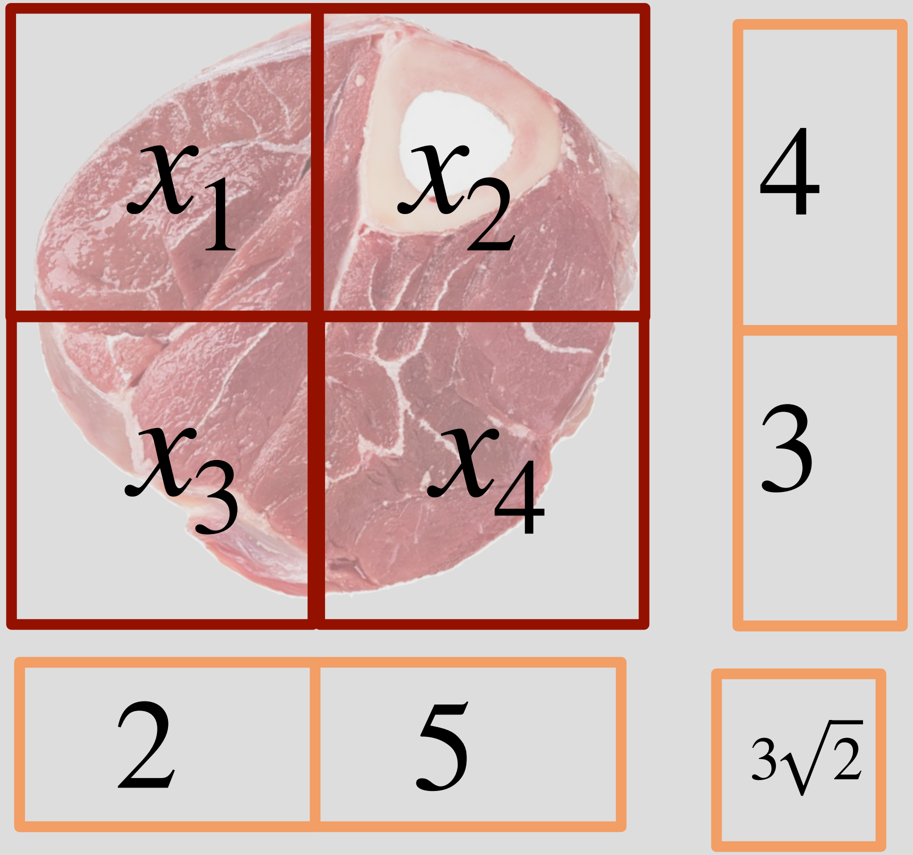
$$0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 1 \cdot x_4 = 3$$

$$1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 = 2$$

$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 5$$

$$\sqrt{2}x_1 + 0 \cdot x_2 + 0 \cdot x_3 + \sqrt{2}x_4 = 3\sqrt{2}$$

Back to Tomography



$$1 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 4$$
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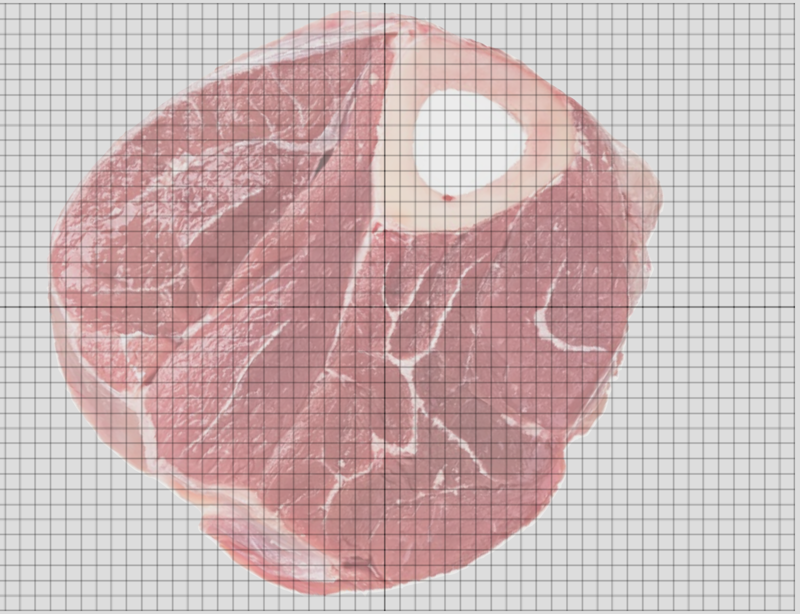
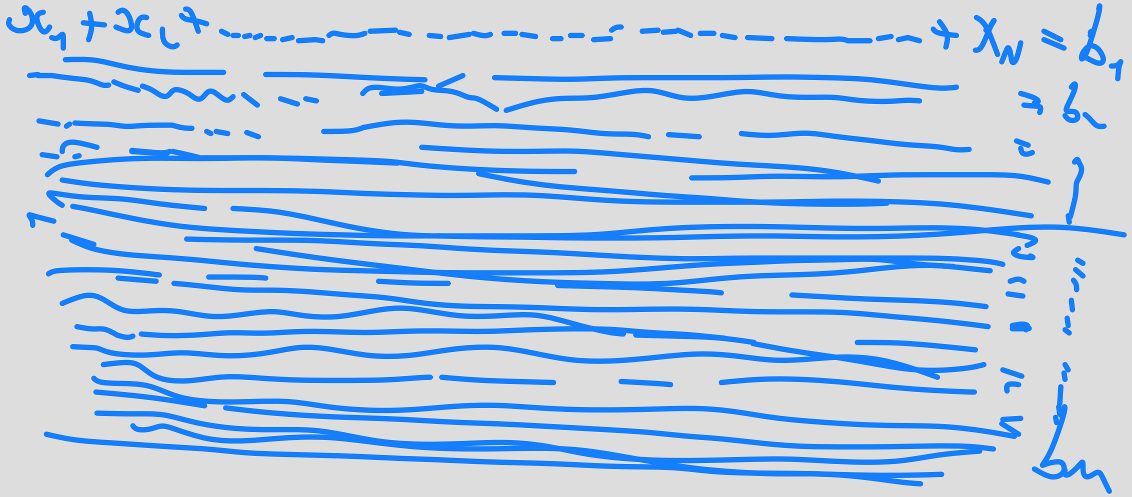
$$1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 = 2$$
$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 5$$
$$\sqrt{2}x_1 + 0 \cdot x_2 + 0 \cdot x_3 + \sqrt{2}x_4 = 3\sqrt{2}$$

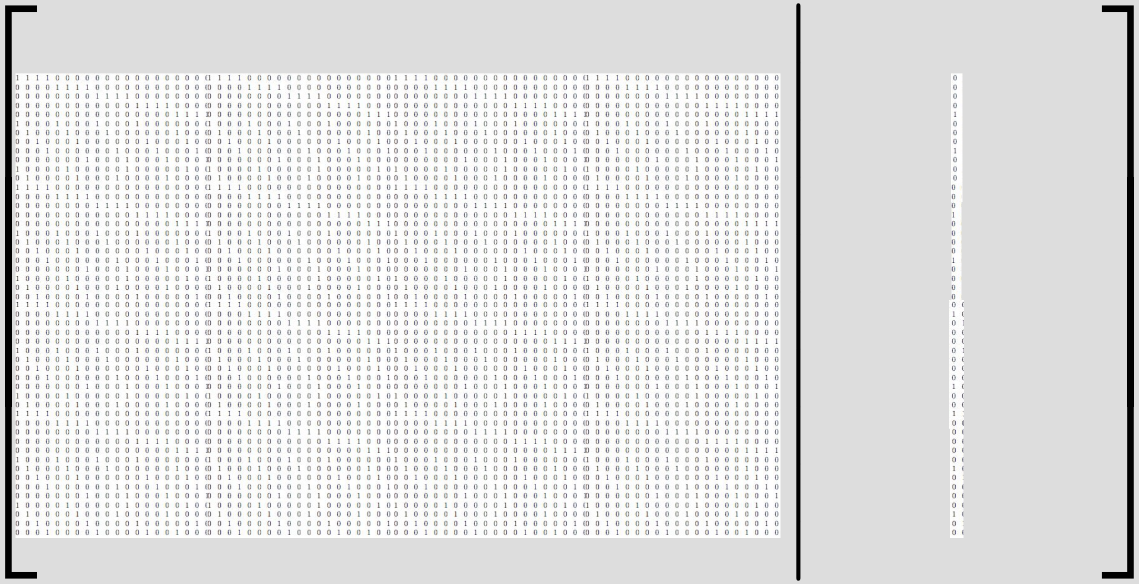
$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \\ \sqrt{2} & 0 & 0 & \sqrt{2} & 3\sqrt{2} \end{array}\right]$$

How do we solve it?

Back to Tomography







How do we systematically solve it?

Algorithm for solving linear equations

- Three basic operations that don’t change a solution:
  1. Multiply an equation with *nonzero* scalar
  2. Adding a scalar constant multiple of one equation to another
  3. Swapping equations

Algorithm for solving linear equations

- Three basic operations that don't change a solution:
  1. Multiply an equation with *nonzero* scalar
  2. Adding a scalar constant multiple of one equation to another
  3. Swapping equations

$$\begin{array}{rrcrcr} (1) & x & + & y & = & 2 \\ (2) & 3x & + & 2y & = & 5 \end{array}$$

and

$$\begin{array}{rrcrcr} (1) & 3x & + & 2y & = & 5 \\ (2) & x & + & y & = & 2 \end{array}$$

Have the same solution

Proof: Pretty obvious!

Algorithm for solving linear equations

- Three basic operations that don't change a solution:
  1. Multiply an equation with *nonzero* scalar

$2x + 3y = 4$ has the same solution as: $4x + 6y = 8$

Proof for N=2:

Let $ax + by = c$, with solution x_0, y_0
$\Rightarrow ax_0 + by_0 = c$

Show that $\beta ax + \beta by = \beta c$,
has the same solution.

Substitute x_0, y_0 for x, y:

$$\beta ax_0 + \beta by_0 = \beta c$$

$$\beta(ax_0 + by_0) = \beta c$$

$$\beta c = \beta c$$

But is it the only solution?

$\beta ax + \beta by = \beta c$, with solution: x_1, y_1
$\Rightarrow \beta ax_1 + \beta by_1 = \beta c$

Show that $ax + by = c$,
has the same solution.....

Since $\beta \neq 0$....

$$\beta ax_1 + \beta by_1 = \beta c \Rightarrow ax_1 + by_1 = c$$

SOLUTION OF ONE, IMPLIES THE OTHER
AND VICE-VERSA!