

# Welcome to EECS 16A!

Designing Information Devices and Systems I

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Sp 2022

Lecture 1B

Vectors, Matrices, Multiplications,  
start span



# Announcements

- Quest: Tuesday 02/8/22 in Class (first 30min)
  - Covers everything including today
- Last time:
  - Gaussian Elimination
- Today:
  - vectors
  - Matrix-Matrix and Matrix-vector Multiplications
  - Matrix-Vector Multiplications as linear set of equations
  - Intro to span (most likely next Tuesday)

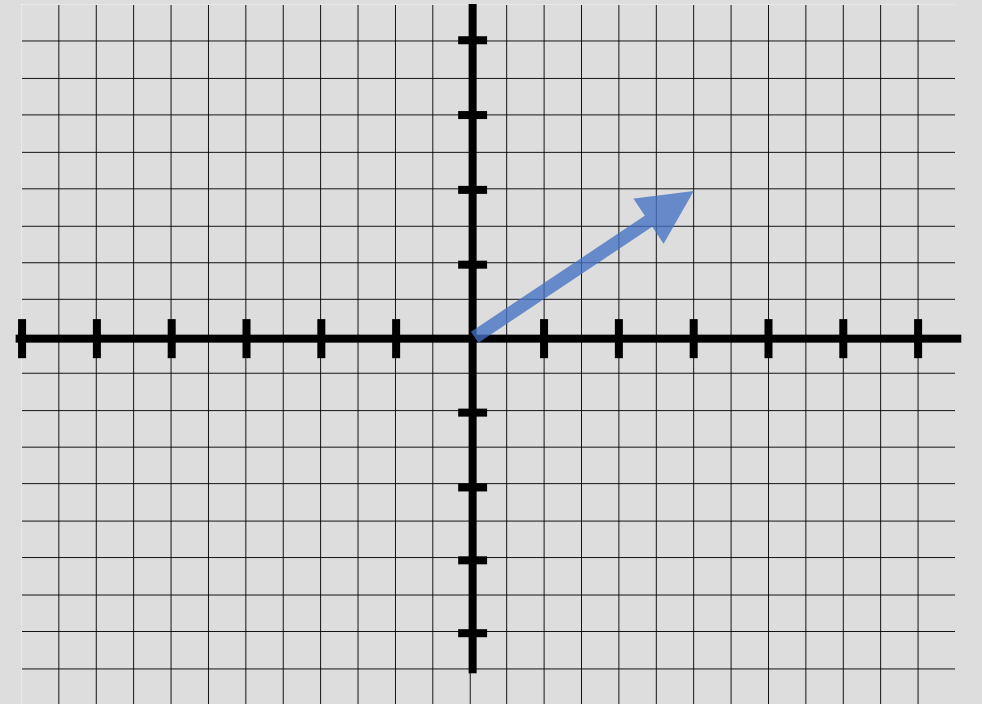
# Vectors

- An array of N numbers
  - Represents coordinates in an N-dimensional space

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}, \quad \vec{x} \in \mathbb{R}^N$$

- For example:

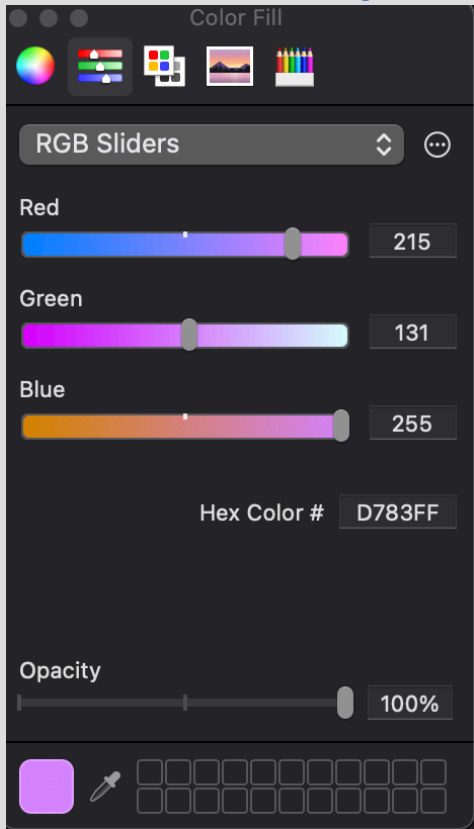
$$\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \vec{x} \in \mathbb{R}^2$$



# Vectors

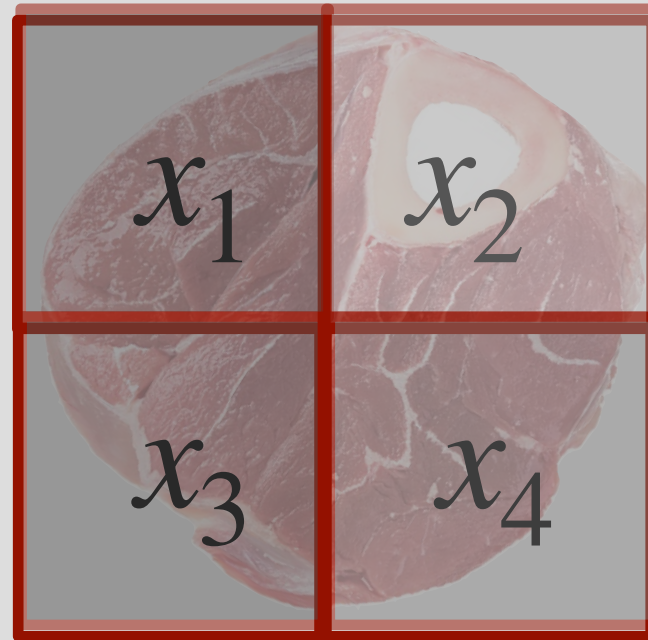
- Since it's an array of numbers, it can represent other things....

pixel color



$$\vec{x} = \begin{bmatrix} 215 \\ 131 \\ 25 \end{bmatrix}$$

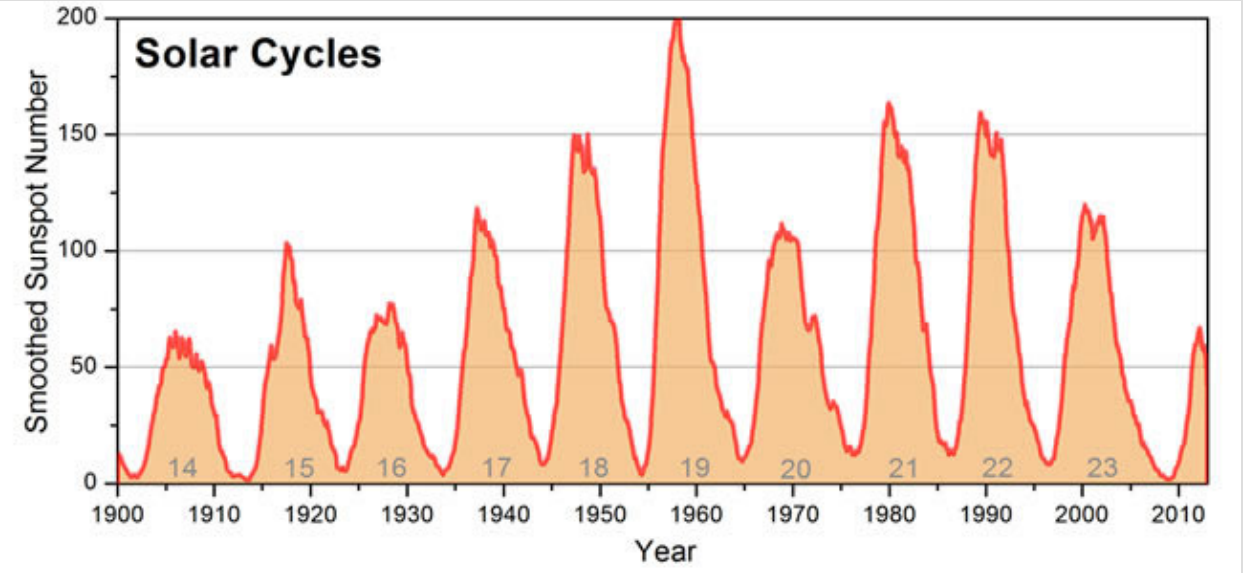
pixel values in an image



$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

# Vectors

## Data



$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{120} \end{bmatrix}$$

# Special Vectors

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\vec{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

$$\vec{e}_N = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

# Matrices

- A collection of numbers in a rectangular form

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1M} \\ x_{21} & x_{22} & \cdots & x_{2M} \\ \vdots & \vdots & \cdots & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{NM} \end{bmatrix}, \quad X \in \mathbb{R}^{N \times M}$$

- Or a collection of M, N-length vectors

$$X = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_M \end{bmatrix}, \quad X \in \mathbb{R}^{N \times M}$$

# Vectors as Matrices

- A vector is a degenerate matrix

$$\vec{x} = \begin{bmatrix} x_1 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad \vec{x} \in \mathbb{R}^{N \times 1}$$

- A scalar is a degenerate vector or matrix

$$a \in \mathbb{R}^{1 \times 1}$$

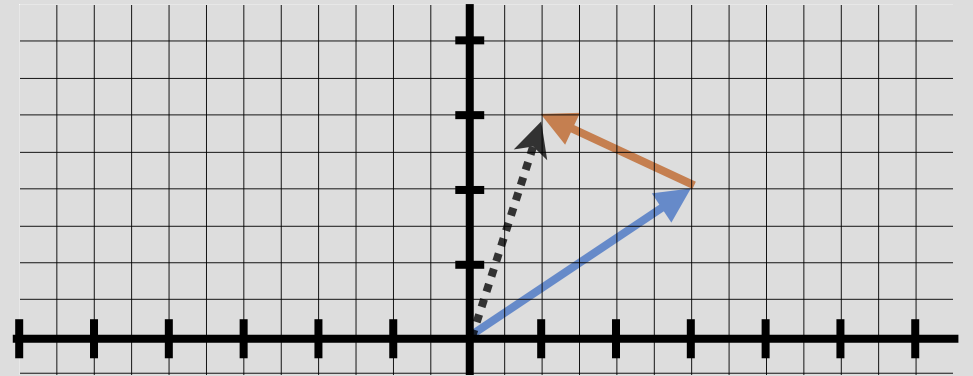


# Vector addition

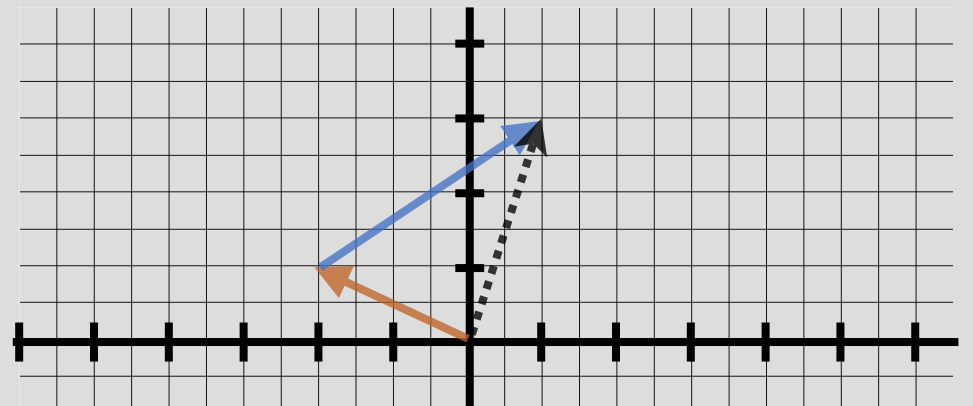
- Two vectors of the same length can be added
  - Addition is element-wise

$$\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\vec{x} + \vec{y} =$$



$$\vec{y} + \vec{x} =$$

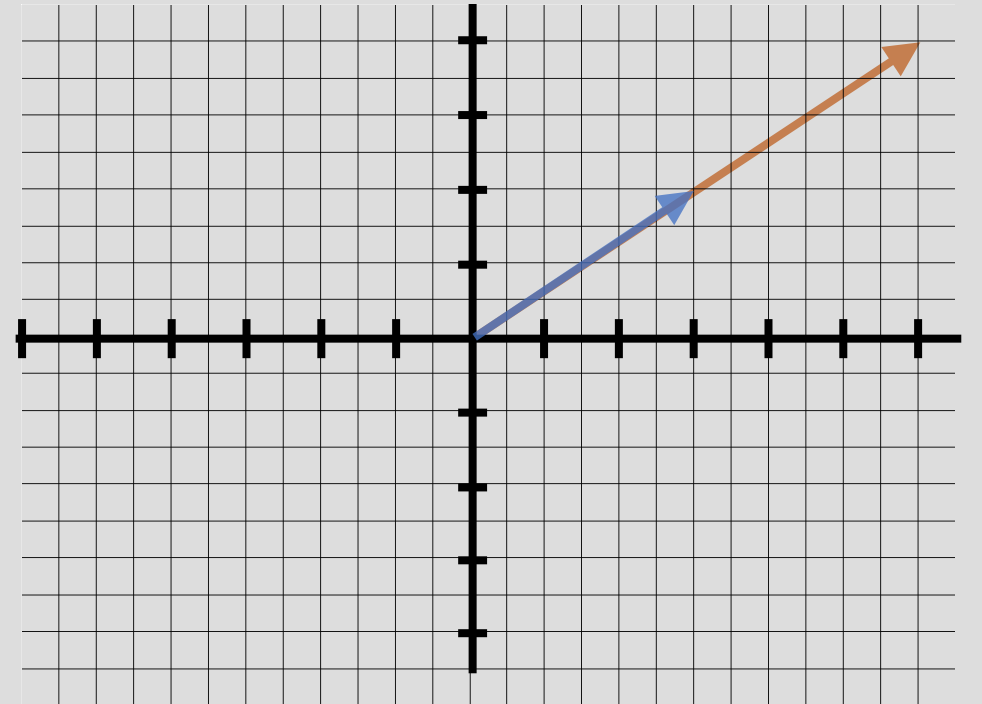


# Scalar Vector Multiplication

- Multiplying with a scalar result in multiplying each element.

$$a\vec{x} = \begin{bmatrix} ax_1 \\ ax_1 \\ \vdots \\ ax_{N \times 1} \end{bmatrix}$$

$$2 \cdot \vec{x} = 2 \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$



# Properties of vector addition

- Commutativity:  $\vec{x} + \vec{y} = \vec{y} + \vec{x}$
- Associativity:  $(\vec{x} + \vec{y}) + \vec{z} = \vec{x} + (\vec{y} + \vec{z})$
- Additive negative:  $\vec{x} + (-\vec{x}) = \vec{0}$
- Additive identity:  $\vec{x} + \vec{0} = \vec{x}$

# Matrix Addition

- When matrices are the same size, they can be added element-wise

$$X + Y = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

- Scalar multiplication — by all elements

$$2X = 2 \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

# Matrix Addition

- When matrices are the same size, they can be added element-wise

$$X + Y = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- Scalar multiplication — by all elements

$$2X = 2 \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -2 & -4 \end{bmatrix}$$

*diagonal matrix!*

# Vector Transpose

- $\vec{x}^T$  is the transpose of  $\vec{x}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_1 \\ \vdots \\ x_N \end{bmatrix}, \quad \vec{x} \in \mathbb{R}^{N \times 1}$$

$$\vec{x}^T = [ x_1 \quad x_1 \quad \cdots \quad x_N ], \quad \vec{x}^T \in \mathbb{R}^{1 \times N}$$

- $\vec{x}$  is always a column vector
- To represent a row vector, write:  $\vec{x}^T$

# Matrix Transpose

If the elements of the matrix  $A \in \mathbb{R}^{N \times M}$  are  $a_{ij}$

The elements of  $A^T \in \mathbb{R}^{M \times N}$  are  $a_{ji}$

Matrix transpose is not (generally) an inverse!

$$A \in \mathbb{R}^{N \times M} \left[ \begin{array}{c} \vec{a}_1 \quad \vec{a}_2 \quad \dots \quad \vec{a}_m \end{array} \right] \quad \left[ \begin{array}{c} \vec{a}_1^T \\ \vec{a}_2^T \\ \vdots \\ \vec{a}_m^T \end{array} \right] \quad A^T \in \mathbb{R}^{M \times N}$$

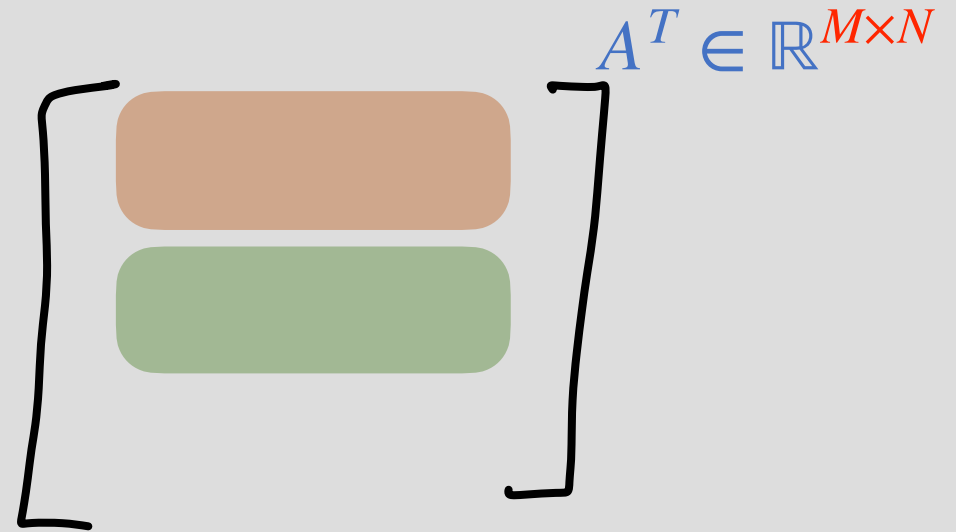
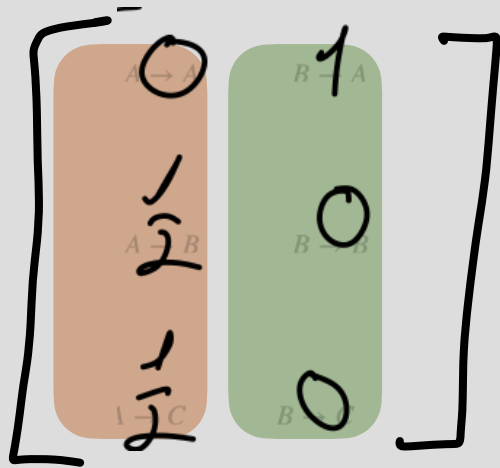
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$A \in \mathbb{R}^{N \times M}$





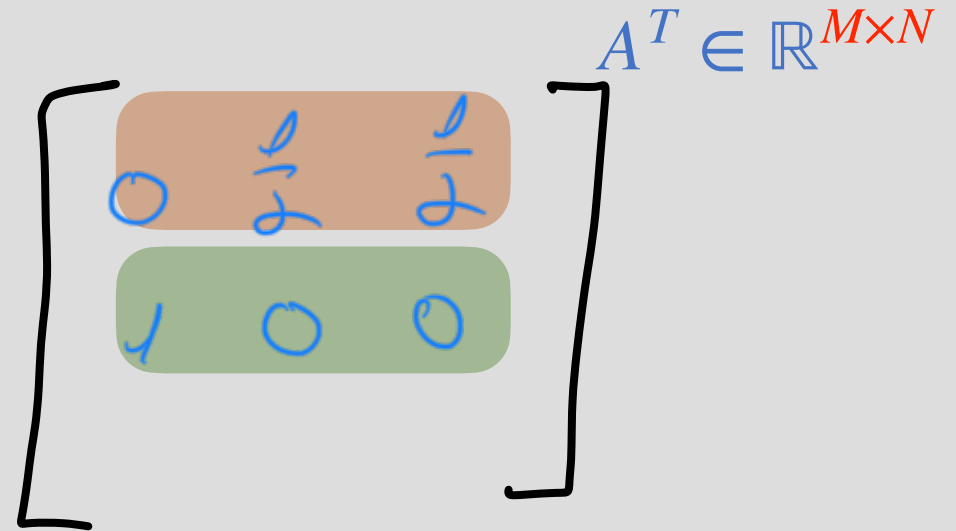
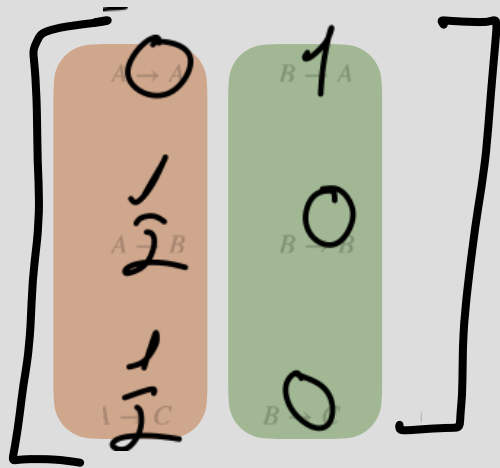
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Matrix transpose is not (generally) an inverse!

$$A \in \mathbb{R}^{N \times M}$$

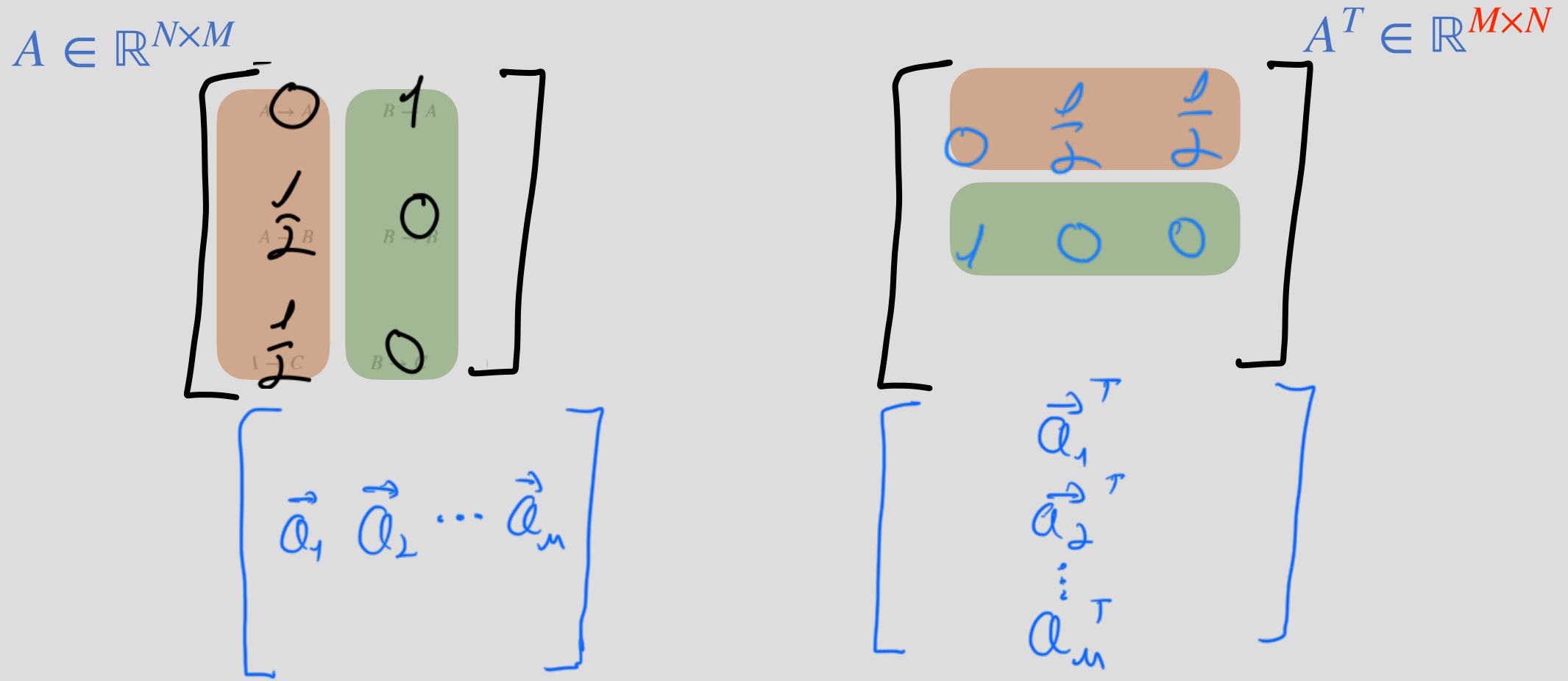


# Matrix Transpose

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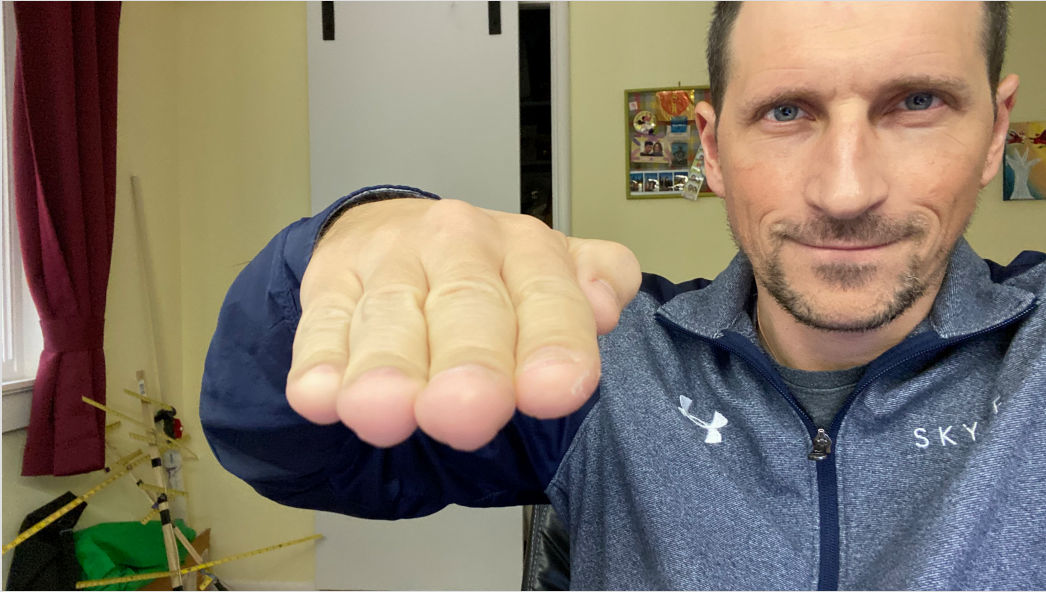
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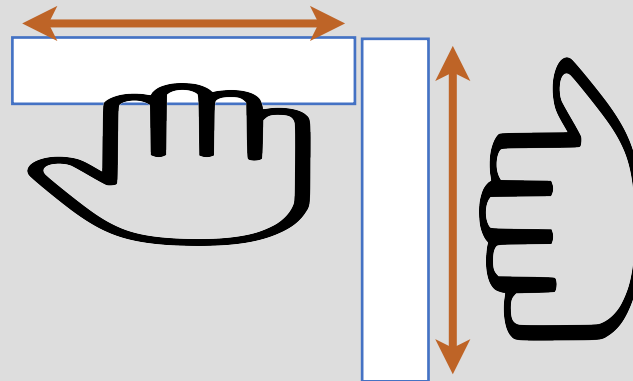
# Vector-Vector Multiplication

- Multiplication is valid only for specific matching dimensions!
  - Width of the 1st, matches length of the second

Like this....



and like that!



# Vector Vector Multiplication

$$\vec{x}, \vec{y} \in \mathbb{R}^{N \times 1}$$

$1 \times N$

$$\vec{y}^T \vec{x} =$$

$y_1$	$y_2$	$\dots$	$y_N$
-------	-------	---------	-------

$x_1$
$x_2$
$\vdots$
$x_N$

$N \times 1$

$$= y_1x_1 + y_2x_2 + y_3x_3 + \dots + y_Nx_N =$$

--

$1 \times 1$

scalar  $1 \times 1$

Also known as "inner product"  
or "dot product"

Like this....



and like that!

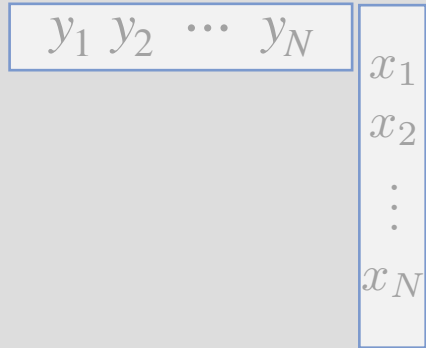


# Matrix-Vector Multiplication

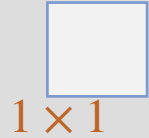
$$\vec{x}, \vec{y} \in \mathbb{R}^{N \times 1}$$

$1 \times N$

$$\vec{y}^T \vec{x} =$$



$$= y_1x_1 + y_2x_2 + y_3x_3 + \dots + y_Nx_N =$$



scalar  $1 \times 1$

Like this....



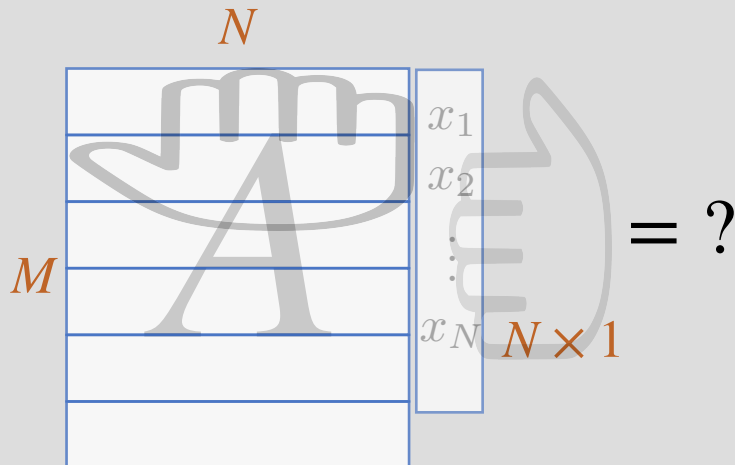
and like that!



What about this case....

$$A \in \mathbb{R}^{M \times N}, \vec{x} \in \mathbb{R}^{N \times 1}$$

$$A \vec{x} =$$



Also known as “inner product” or “dot product”

# Matrix-Vector Multiplication

$$A \in \mathbb{R}^{M \times N}, \vec{x} \in \mathbb{R}^{N \times 1}$$



Like this....



and like that!

$$A \vec{x} = \begin{matrix} & & & N \\ & & & a_{11} & a_{12} & \cdots & a_{1N} \\ & & & a_{21} & a_{22} & \cdots & a_{2N} \\ & & & \vdots & \vdots & \ddots & \vdots \\ & & & \vdots & \vdots & \vdots & \vdots \\ & & & a_{M1} & a_{M2} & \cdots & a_{MN} \\ M & & & & & & \end{matrix} \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_N \\ N \times 1 \end{matrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N \end{bmatrix}$$

$$[a_{11} \ a_{12} \ \cdots \ a_{1N}] \vec{x} = \vec{y}_1^T \vec{x}$$



# Matrix-Vector Multiplication

$$A \in \mathbb{R}^{M \times N}, \vec{x} \in \mathbb{R}^{N \times 1}$$

Like this....



and like that!



$$A \vec{x} = \begin{matrix} & \begin{matrix} N \\ a_{11} & a_{12} & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} \end{matrix} \\ \begin{matrix} M \\ \vdots \\ \vdots \\ \vdots \end{matrix} & \begin{matrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{matrix} \end{matrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N \\ \vdots \\ a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N \end{bmatrix} = \begin{matrix} 1 \\ \vdots \\ \vdots \\ \vdots \end{matrix}$$

$$\begin{aligned} [a_{11} \ a_{12} \ \cdots \ a_{1N}] \vec{x} &= \vec{y}_1^T \vec{x} \\ [a_{21} \ a_{22} \ \cdots \ a_{2N}] \vec{x} &= \vec{y}_2^T \vec{x} \\ &\vdots \\ [a_{M1} \ a_{M2} \ \cdots \ a_{MN}] \vec{x} &= \vec{y}_M^T \vec{x} \end{aligned}$$





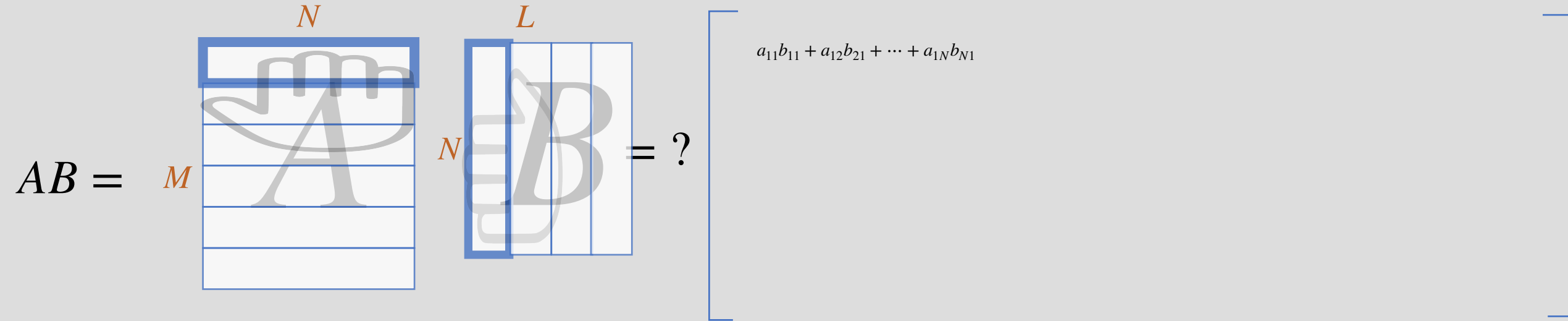
# Matrix-Matrix Multiplication

$$A \in \mathbb{R}^{M \times N}, B \in \mathbb{R}^{N \times L}$$

Like this....



and like that!



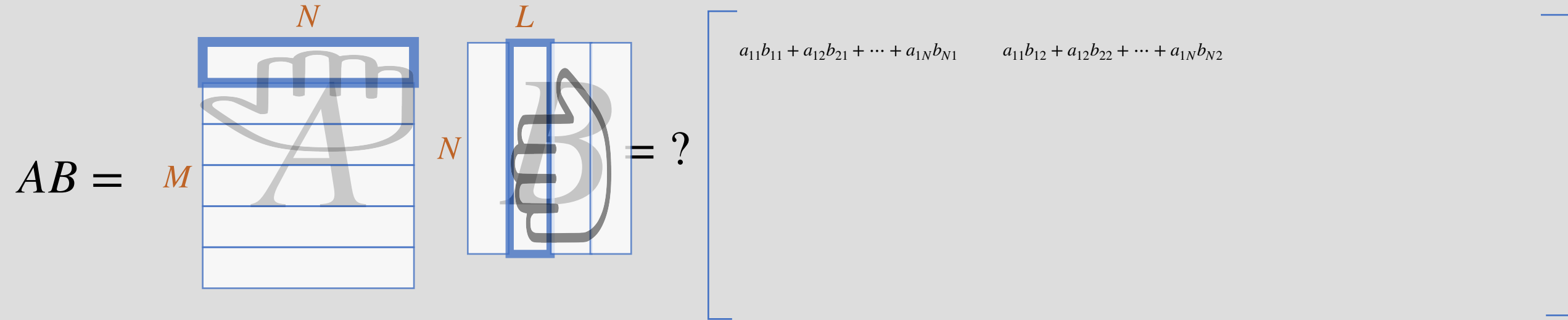
# Matrix-Matrix Multiplication

$$A \in \mathbb{R}^{M \times N}, B \in \mathbb{R}^{N \times L}$$

Like this....



and like that!



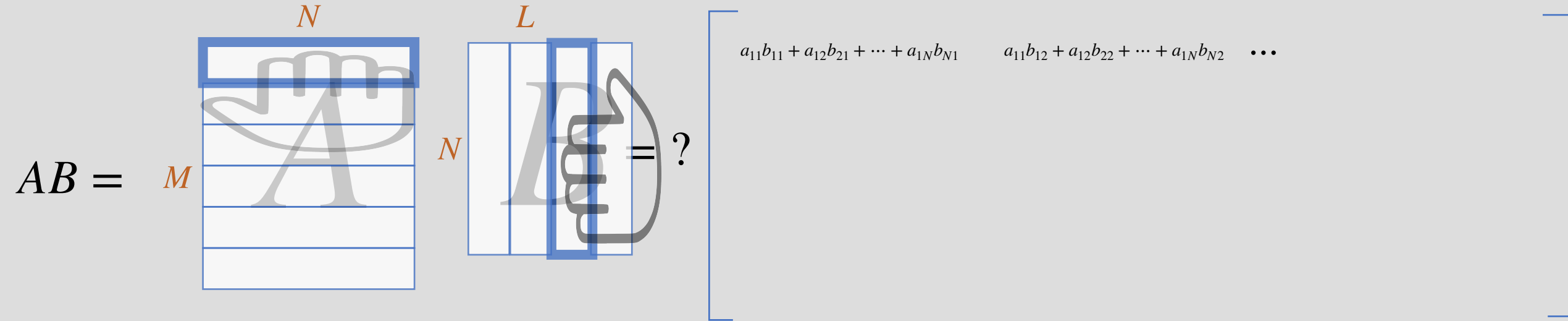
# Matrix-Matrix Multiplication

$$A \in \mathbb{R}^{M \times N}, B \in \mathbb{R}^{N \times L}$$

Like this....



and like that!



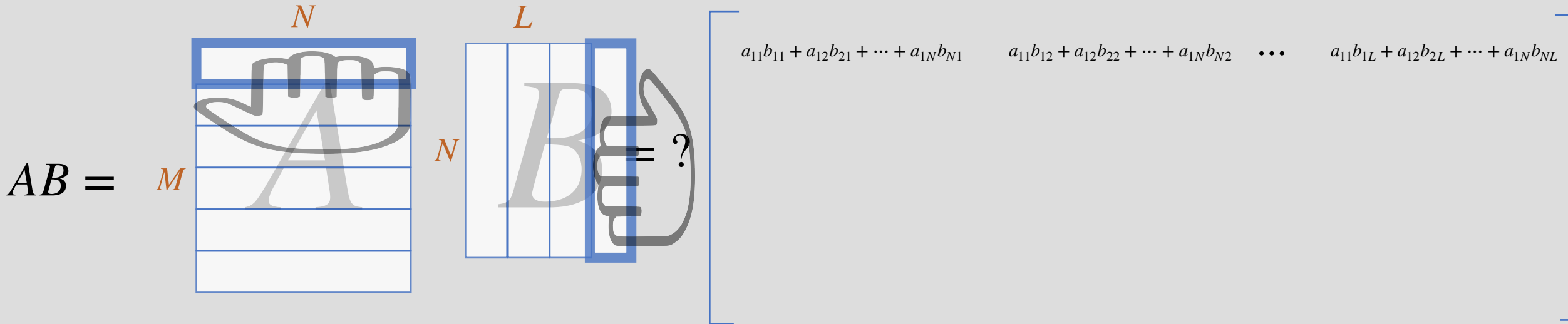
# Matrix-Matrix Multiplication

$$A \in \mathbb{R}^{M \times N}, B \in \mathbb{R}^{N \times L}$$

Like this....



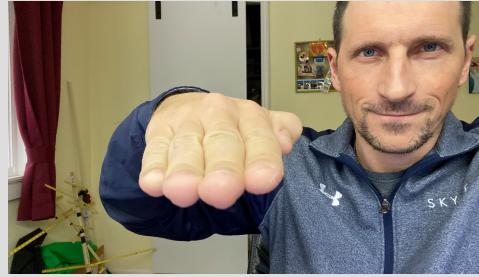
and like that!



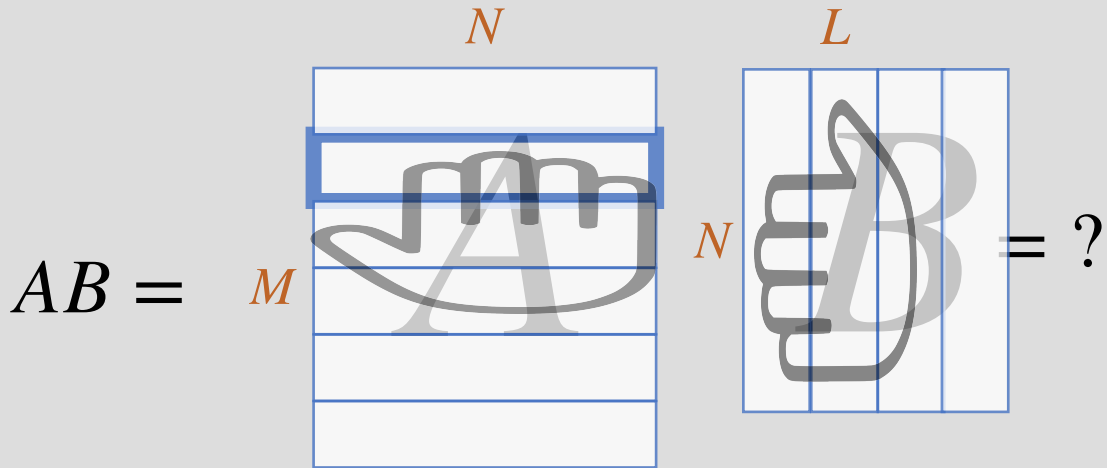
# Matrix-Matrix Multiplication

$$A \in \mathbb{R}^{M \times N}, B \in \mathbb{R}^{N \times L}$$

Like this....



and like that!



$$\begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1N}b_{N1} & a_{11}b_{12} + a_{12}b_{22} + \dots + a_{1N}b_{N2} & \dots & a_{11}b_{1L} + a_{12}b_{2L} + \dots + a_{1N}b_{NL} \\ a_{21}b_{11} + a_{22}b_{21} + \dots + a_{2N}b_{N1} & a_{21}b_{12} + a_{22}b_{22} + \dots + a_{2N}b_{N2} & \dots & a_{21}b_{1L} + a_{22}b_{2L} + \dots + a_{2N}b_{NL} \end{bmatrix}$$

# Matrix-Matrix Multiplication

$$A \in \mathbb{R}^{M \times N}, B \in \mathbb{R}^{N \times L}$$

Like this....



and like that!



$AB =$

$a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1N}b_{N1}$	$a_{11}b_{12} + a_{12}b_{22} + \dots + a_{1N}b_{N2}$	$\dots$	$a_{11}b_{1L} + a_{12}b_{2L} + \dots + a_{1N}b_{NL}$
$a_{21}b_{11} + a_{22}b_{21} + \dots + a_{2N}b_{N1}$	$a_{21}b_{12} + a_{22}b_{22} + \dots + a_{2N}b_{N2}$	$\dots$	$a_{21}b_{1L} + a_{22}b_{2L} + \dots + a_{2N}b_{NL}$
$\vdots$	$\vdots$		$\vdots$
$a_{M1}b_{11} + a_{M2}b_{21} + \dots + a_{MN}b_{N1}$	$a_{M1}b_{12} + a_{M2}b_{22} + \dots + a_{MN}b_{N2}$		$a_{M1}b_{1L} + a_{M2}b_{2L} + \dots + a_{MN}b_{NL}$

# Matrix-Matrix Multiplication

Like this....

and like that!



$$A \in \mathbb{R}^{M \times N}, B \in \mathbb{R}^{N \times L}$$

$AB =$

$M$

$N$

$N$

$L$

= ?

$a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1N}b_{N1}$	$a_{11}b_{12} + a_{12}b_{22} + \dots + a_{1N}b_{N2}$	$\dots$	$a_{11}b_{1L} + a_{12}b_{2L} + \dots + a_{1N}b_{NL}$
$a_{21}b_{11} + a_{22}b_{21} + \dots + a_{2N}b_{N1}$	$a_{21}b_{12} + a_{22}b_{22} + \dots + a_{2N}b_{N2}$	$\dots$	$a_{21}b_{1L} + a_{22}b_{2L} + \dots + a_{2N}b_{NL}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a_{M1}b_{11} + a_{M2}b_{21} + \dots + a_{MN}b_{N1}$	$a_{M1}b_{12} + a_{M2}b_{22} + \dots + a_{MN}b_{N2}$	$\dots$	$a_{M1}b_{1L} + a_{M2}b_{2L} + \dots + a_{MN}b_{NL}$

Result at location 2x2 =  $a_{21}b_{12} + a_{22}b_{22} + \dots + a_{2N}b_{N2}$



# Matrix-Vector Multiplication

Like this....



and like that!



$M \times N$



$N \times L$

=



$M \times L$



# Vector Vector Multiplication

$$\vec{x}, \vec{y} \in \mathbb{R}^{N \times 1}$$

Like this....



and like that!



$$\vec{y}^T \vec{x} =$$

$$= y_1x_1 + y_2x_2 + y_3x_3 + \dots + y_Nx_N$$

scalar  $1 \times 1$

Also known as “inner product”  
or “dot product”

$$\vec{x} \vec{y}^T =$$

$N \times 1$   $1 \times N$

$$= \begin{bmatrix} x_1y_1 & x_1y_2 & \dots & x_1y_N \\ x_2y_1 & x_2y_2 & \dots & x_2y_N \\ \vdots & \vdots & \dots & \vdots \\ x_Ny_1 & x_Ny_2 & \dots & x_Ny_N \end{bmatrix}$$

$N \times N$

Do not commute!

Also known as “outer product”

# Matrix Matrix Multiplication

$$\begin{matrix} \boxed{A} \\ N \times M \end{matrix} \begin{matrix} \boxed{B} \\ M \times N \end{matrix} = \begin{matrix} \boxed{N \times N} \\ \text{😍} \end{matrix}$$

$$\begin{matrix} \boxed{M \times 1} \\ 1 \times 1 \end{matrix} \begin{matrix} \boxed{1 \times N} \\ 1 \times N \end{matrix} = \begin{matrix} \boxed{M \times N} \\ \text{😍} \end{matrix}$$

$$\begin{matrix} \boxed{M \times N} \\ M \times N \end{matrix} \begin{matrix} \boxed{A} \\ N \times M \end{matrix} = \begin{matrix} \boxed{M \times M} \\ \text{😍} \end{matrix}$$

$$\begin{matrix} \boxed{1 \times N} \\ 1 \times N \end{matrix} \begin{matrix} \boxed{N \times M} \\ N \times M \end{matrix} = \begin{matrix} \boxed{1 \times M} \\ \text{😍} \end{matrix}$$

Matrix multiplication does not commute!

# Matrix-Vector Form of Systems of Linear Equations

- Consider the matrix equation:  $A\vec{x} = \vec{b}$

$$A\vec{x} = \begin{array}{|cccc|} \hline a_{11} & a_{12} & \cdots & a_{1N} \\ \hline a_{21} & a_{22} & \cdots & a_{2N} \\ \hline & \bullet & & \\ \hline & \bullet & & \\ \hline & \bullet & & \\ \hline a_{M1} & a_{M2} & \cdots & a_{MN} \\ \hline \end{array} \begin{array}{|c|} \hline x_1 \\ \hline x_2 \\ \hline \vdots \\ \hline x_N \\ \hline \end{array} = \begin{array}{|c|} \hline a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N \\ \hline a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N \\ \hline \bullet \\ \hline \bullet \\ \hline a_{M1}x_1 + a_{M2}x_2 + \cdots + a_{MN}x_N \\ \hline \end{array} = \begin{array}{|c|} \hline b_1 \\ \hline b_2 \\ \hline \vdots \\ \hline b_M \\ \hline \end{array}$$

$M \times N$      $N \times 1$ 
 $M \times 1$

Same as the Augmented Matrix!

$A\vec{x} = \vec{b}$  is another way to write  
A linear set of equations!

$$\left[ \begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1N} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2N} & b_2 \\ \vdots & & & & \vdots \\ a_{M1} & a_{M2} & \cdots & a_{MN} & b_M \end{array} \right]$$

# Row vs Column Perspective

- Row / Measurement Perspective of  $A \vec{x} = \vec{b}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} \phantom{a_{11}} & \phantom{a_{12}} & \phantom{a_{13}} \\ \phantom{a_{21}} & \phantom{a_{22}} & \phantom{a_{23}} \end{bmatrix} = \begin{bmatrix} \phantom{b_1} \\ \phantom{b_2} \end{bmatrix}$$

# Row vs Column Perspective

- Row / Measurement Perspective of  $A \vec{x} = \vec{b}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Q: What does a row mean?

A: How each variable affect a particular measurement

# Row vs Column Perspective

- Column Perspective of  $A\vec{x} = \vec{b}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} \phantom{a_{11}} \\ \phantom{a_{21}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} =$$



# Row vs Column Perspective

- Column Perspective of  $A \vec{x} = \vec{b}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{b}$$

# Row vs Column Perspective

- Column Perspective of  $A\vec{x} = \vec{b}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3 =$$

# Row vs Column Perspective

- Column Perspective of  $A \vec{x} = \vec{b}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3 =$$

$$= \begin{bmatrix} a_{11}x_1 \\ a_{21}x_1 \end{bmatrix} + \begin{bmatrix} a_{12}x_2 \\ a_{22}x_2 \end{bmatrix} + \begin{bmatrix} a_{13}x_3 \\ a_{23}x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Q: What does a column mean?

A: How a particular variable affects all measurements.

# Linear combination of vectors

- Given set of vectors  $\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_M\} \in \mathbb{R}^N$ , and coefficients  $\{\alpha_1, \alpha_2, \dots, \alpha_M\} \in \mathbb{R}$
- A linear combination of vectors is defined as:  $\vec{b} \triangleq \alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 + \dots + \alpha_M \vec{a}_M$

Recall:  $A\vec{x}$ :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$= x_1 \vec{a}_1 + x_2 \vec{a}_2 + x_3 \vec{a}_3$$

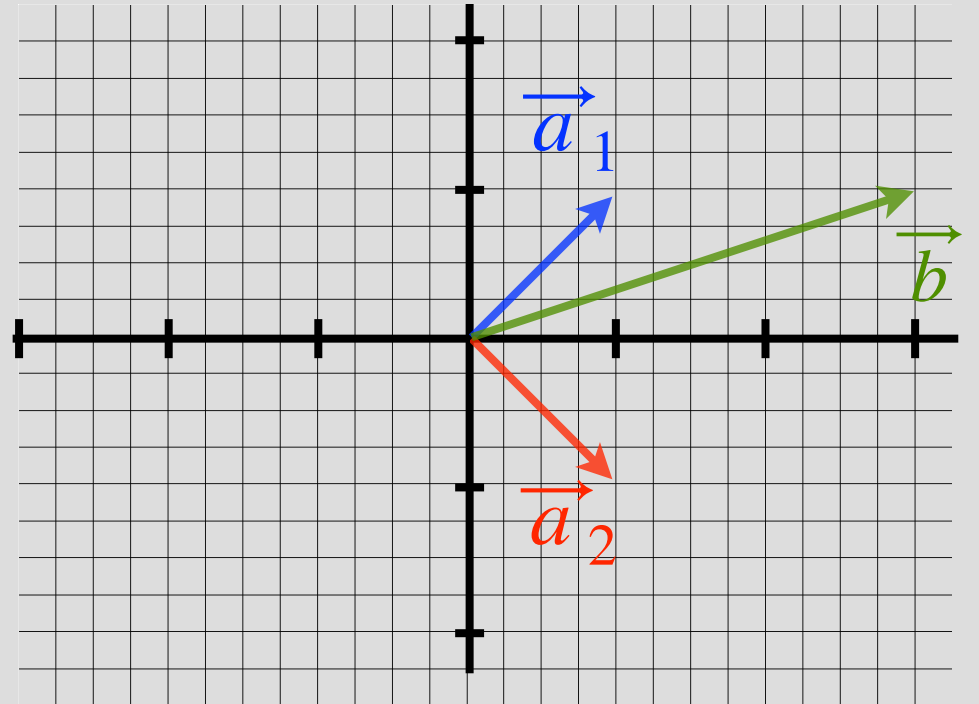
Matrix-vector multiplication is a linear combination of the columns of A!

# Linear Set of Equations as a Linear Combination

- Consider the problem:  $A\vec{x} = \vec{b}$ :

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$\downarrow$   $\downarrow$   $\downarrow$   
 $\vec{a}_1$   $\vec{a}_2$   $\vec{b}$



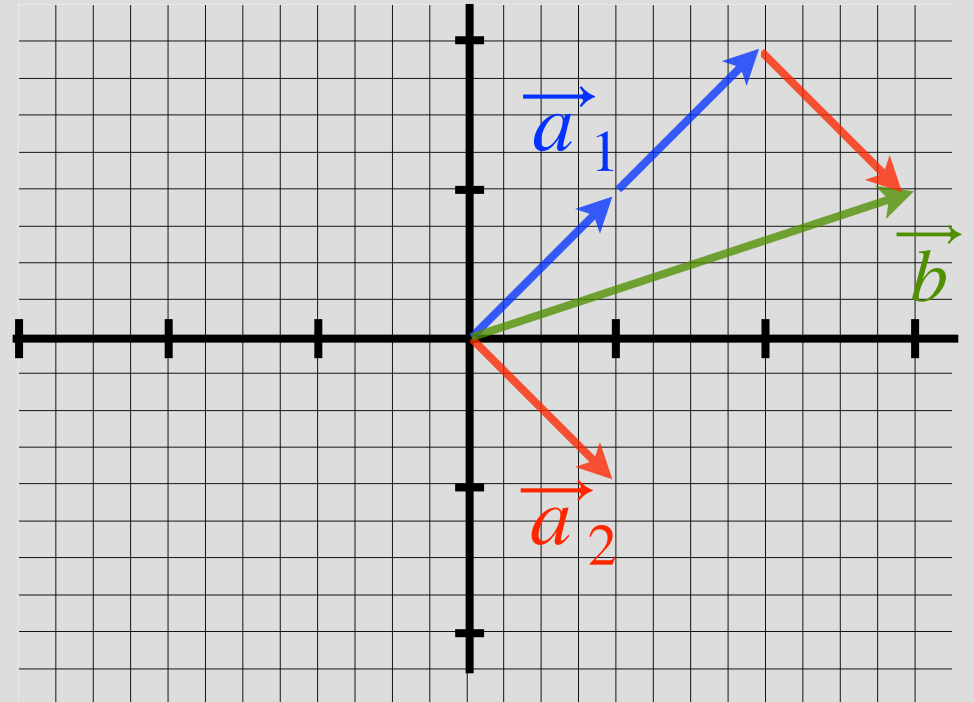
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$\downarrow$                        $\downarrow$                        $\downarrow$   
 $\vec{a}_1$                        $\vec{a}_2$                        $\vec{b}$

Q: What linear combination of  $\vec{a}_1, \vec{a}_2$  will give  $\vec{b}$ ?



# Linear Set of Equations as a Linear Combination

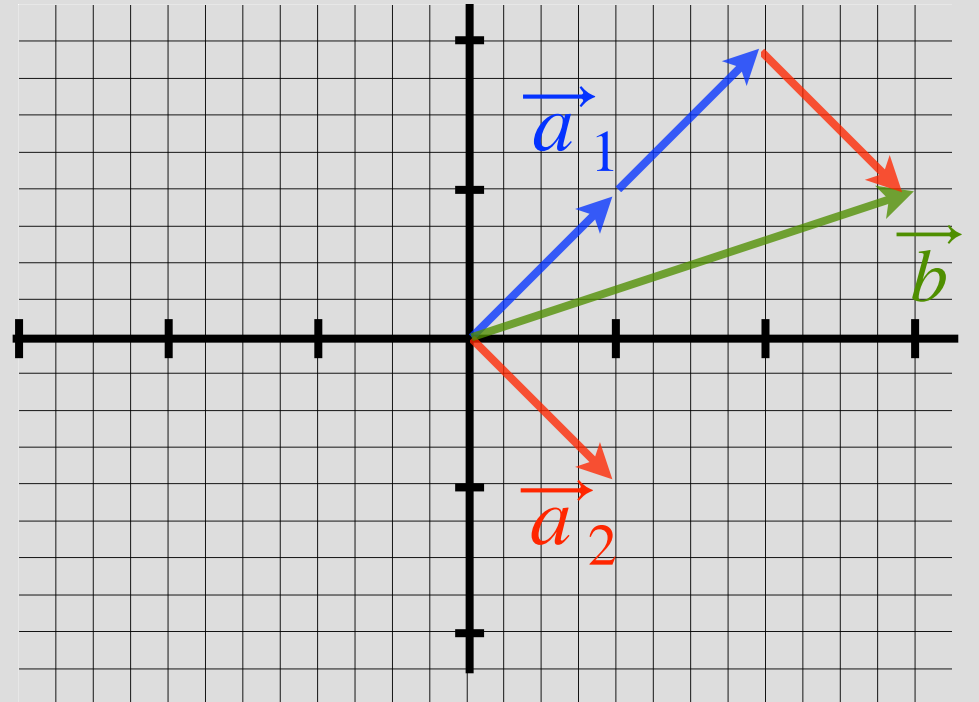
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$\downarrow$                        $\downarrow$                        $\downarrow$   
 $\vec{a}_1$                        $\vec{a}_2$                        $\vec{b}$

Q: What linear combination of  $\vec{a}_1, \vec{a}_2$  will give  $\vec{b}$ ?

A:  $2\vec{a}_1 + 1\vec{a}_2$

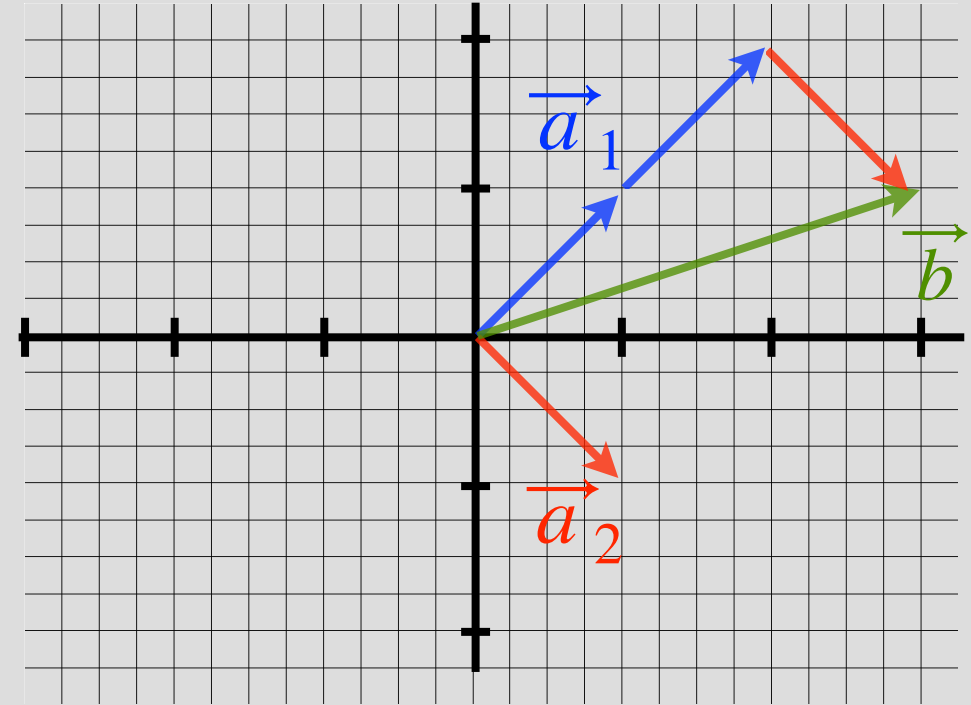


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$\downarrow$   $\downarrow$   $\downarrow$   
 $\vec{a}_1$   $\vec{a}_2$   $\vec{b}$



Q: What linear combination of  $\vec{a}_1, \vec{a}_2$  will give  $\vec{b}$ ?

A:  $2\vec{a}_1 + 1\vec{a}_2$

Gaussian Elimination:

$$\begin{bmatrix} 1 & 1 & | & 3 \\ 1 & -1 & | & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & | & 3 \\ 0 & 1 & | & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & | & 3 \\ 0 & -2 & | & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & 1 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 2 \\ x_2 &= 1 \end{aligned}$$

same as

$$\vec{b} = 2\vec{a}_1 + 1\vec{a}_2$$

!



# Linear Set of Equations as a Linear Combination

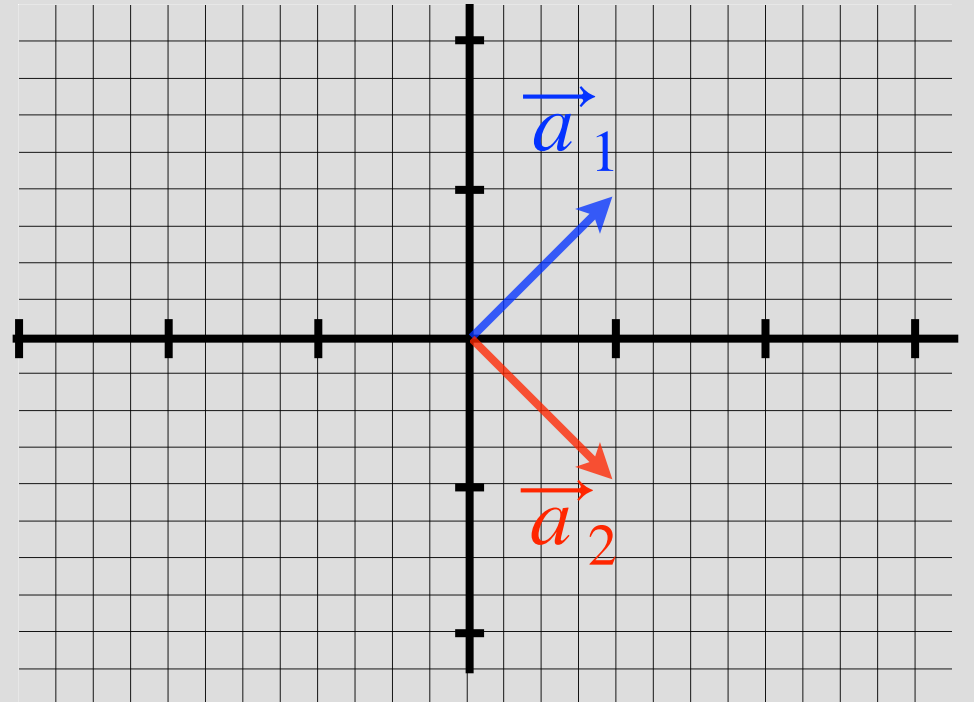
- Consider the problem:  $A\vec{x} = \vec{b}$ :

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{b}$$

$\downarrow$                        $\downarrow$   
 $\vec{a}_1$                        $\vec{a}_2$

Q: Can linear combination of  $\vec{a}_1, \vec{a}_2$  give any  $\vec{b}$ ?

A: Hmmm.....I think so.....



# Linear Set of Equations as a Linear Combination

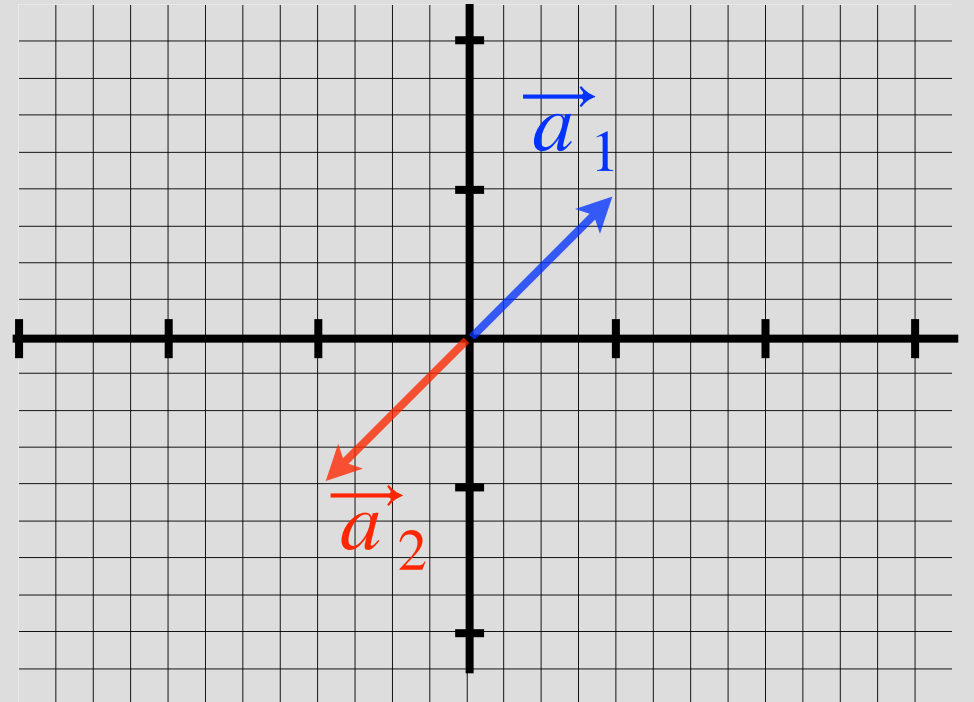
- Consider the problem:  $A\vec{x} = \vec{b}$ :

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{b}$$

$\downarrow$                        $\downarrow$   
 $\vec{a}_1$                        $\vec{a}_2$

Q: Can linear combination of  $\vec{a}_1, \vec{a}_2$  give any  $\vec{b}$ ?

A: Hmmmm....I don't think so.... Unless its along the line  $\vec{a}_1$



# Linear Set of Equations as a Linear Combination

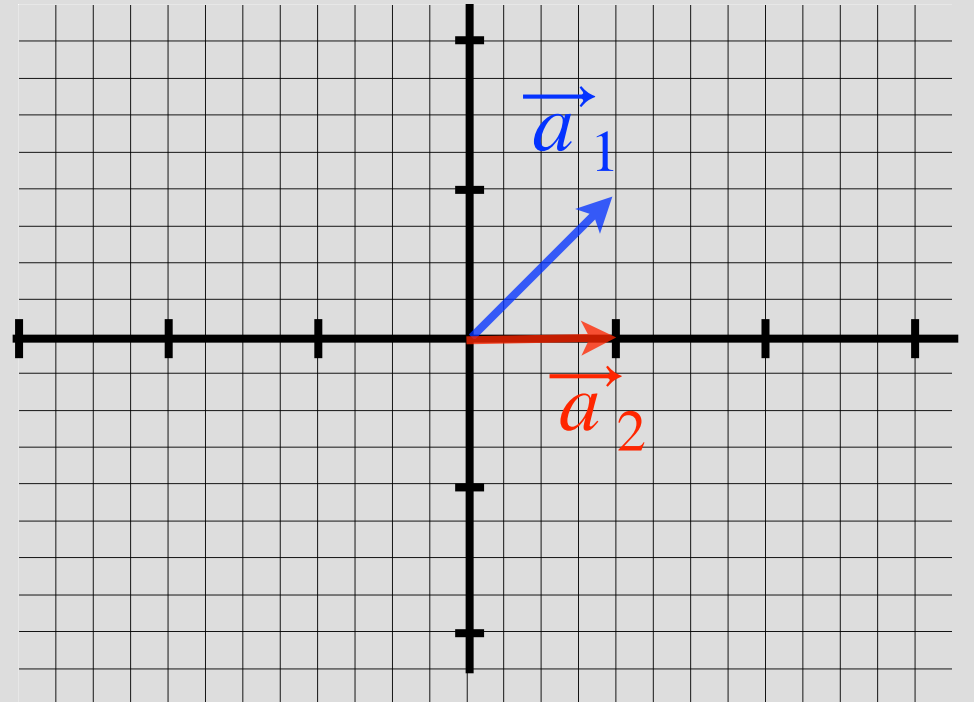
- Consider the problem:  $A\vec{x} = \vec{b}$ :

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{b}$$

$\downarrow$        $\downarrow$   
 $\vec{a}_1$     $\vec{a}_2$

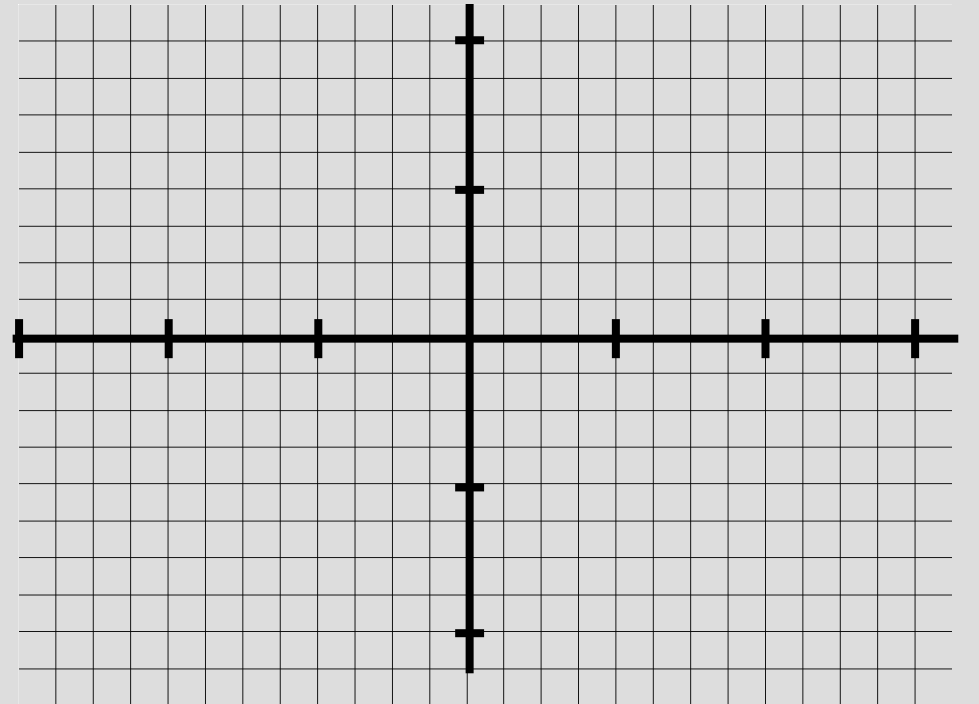
Q: Can linear combination of  $\vec{a}_1, \vec{a}_2$  give any  $\vec{b}$ ?

A: Hmmm....yes!



# Span / Column Space / Range

- Span of the columns of  $A$  is the set of all vectors  $\vec{b}$  such that  $A\vec{x} = \vec{b}$  has a solution
  - i.e. the set of all vectors that can be reached by all possible linear combinations of the columns of  $A$

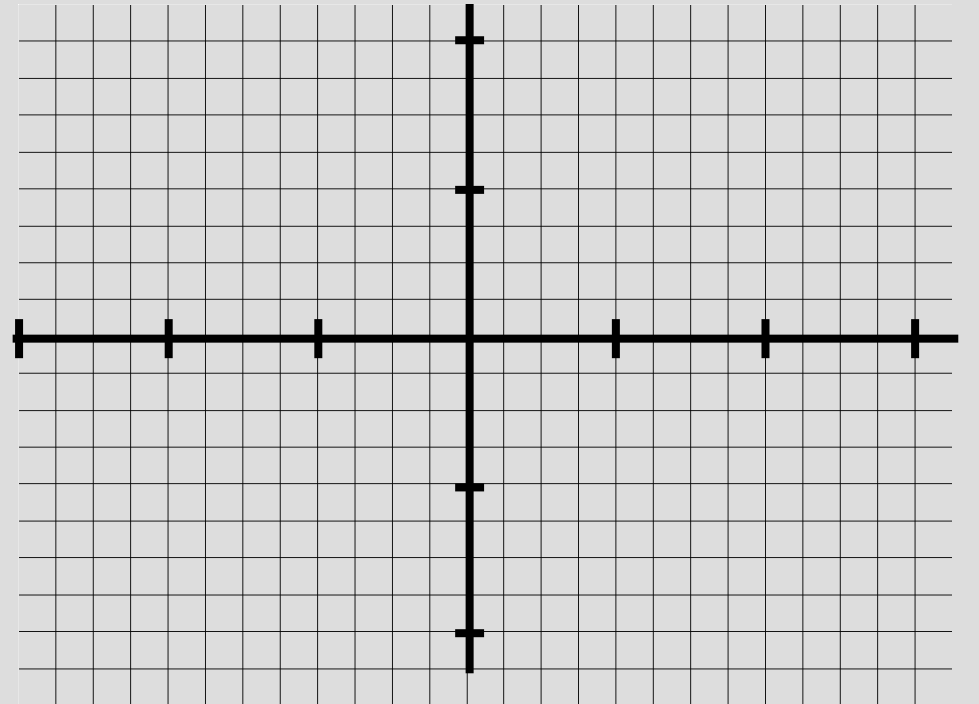


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Example: What is the span of the cols of  $A$ ?

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$



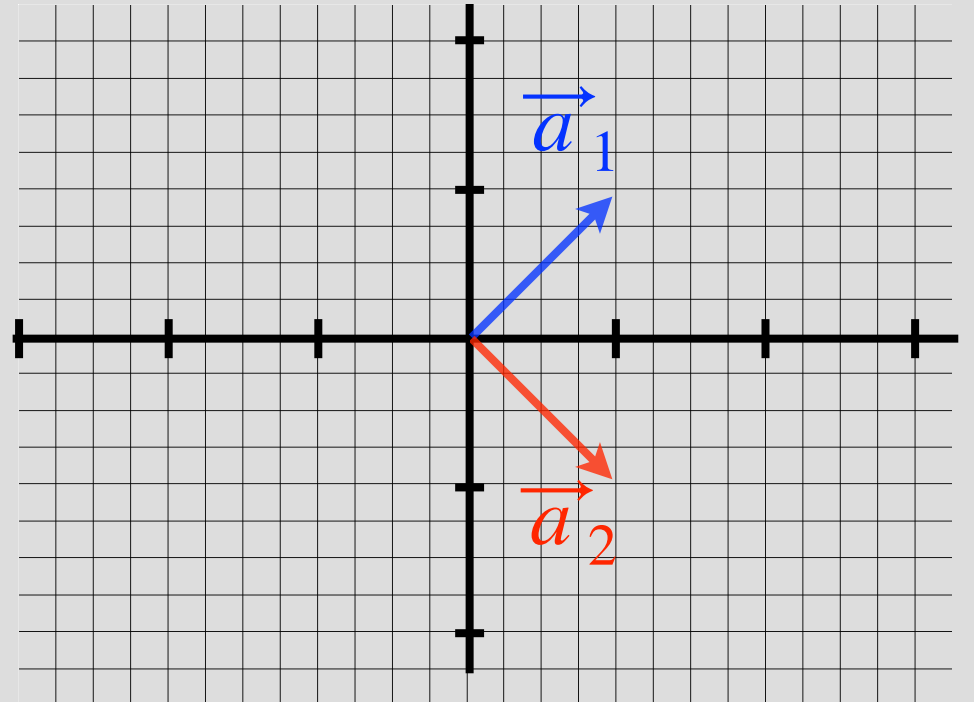
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$A: \mathbb{R}^2!$



# Span / Column Space / Range

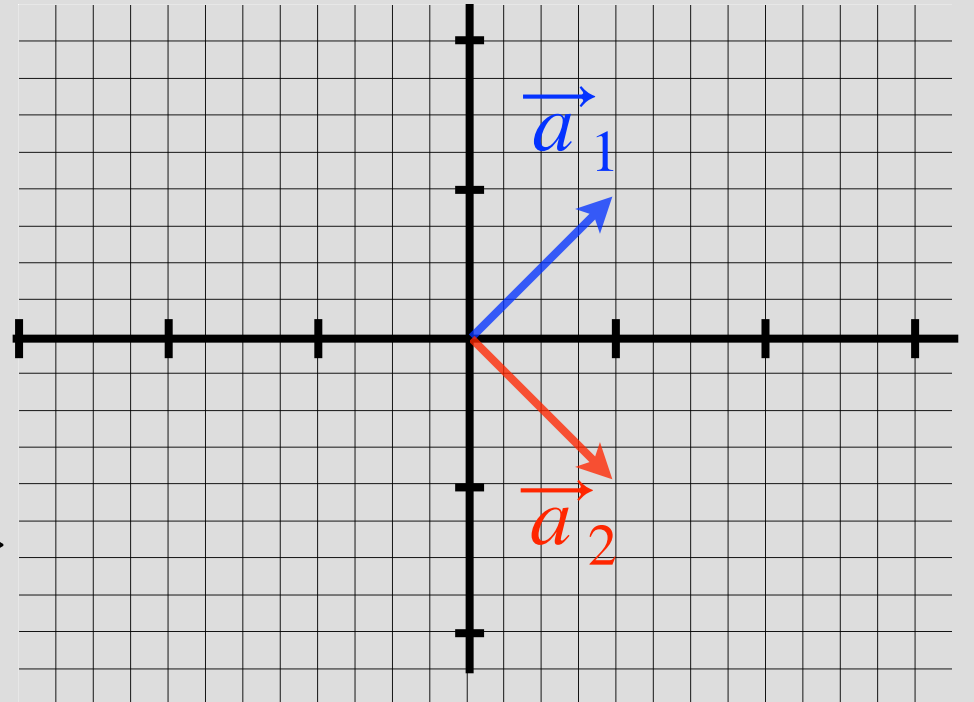
- Span of the columns of  $A$  is the set of all vectors  $\vec{b}$  such that  $A\vec{x} = \vec{b}$  has a solution
  - i.e. the set of all vectors that can be reached by all possible linear combinations of the columns of  $A$

Example: What is the span of the cols of  $A$ ?

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$A: \mathbb{R}^2!$

$$\text{span}(\text{cols of } A) = \left\{ \vec{v} \mid \vec{v} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \alpha, \beta \in \mathbb{R} \right\}$$



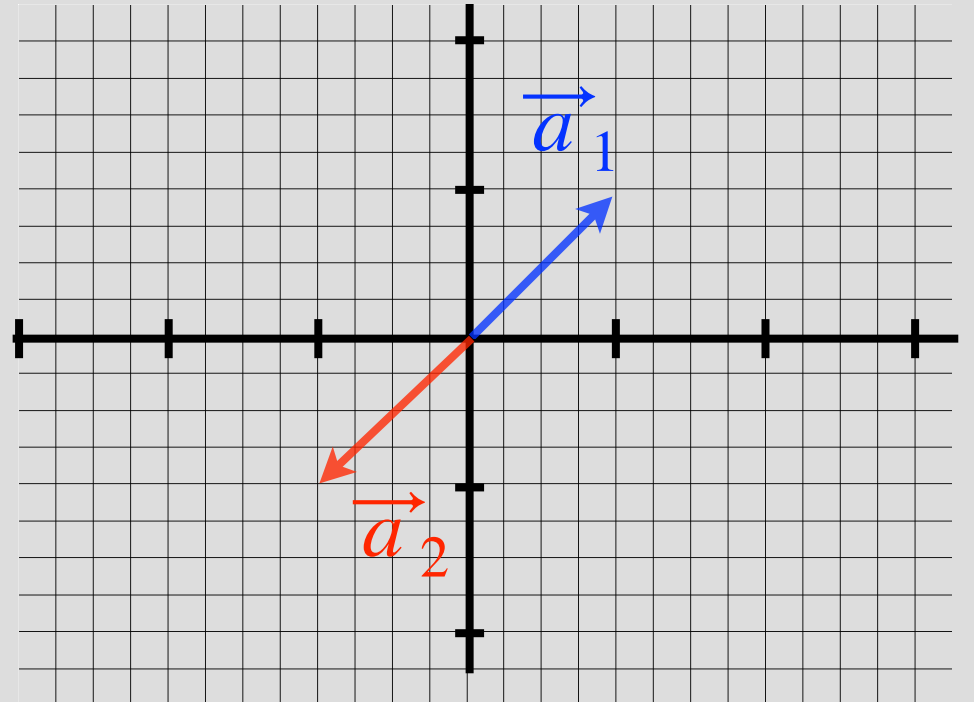
# Span / Column Space / Range

Example 2: What is the span of the cols of  $A$ ?

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

A: The line  $x_1 = x_2$

$$\text{span}(\text{cols of } A) = \left\{ \vec{v} \mid \vec{v} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \alpha \in \mathbb{R} \right\}$$





# Span / Column Space / Range

- Definition:

If  $\exists \vec{x}$  s.t.  $A\vec{x} = \vec{b}$  then  $\vec{b} \in \text{span}\{A\}$

Converse:  $\vec{b} \in \text{span}\{\text{cols}(A)\}$ , there is a solution for  $A\vec{x} = \vec{b}$

Q: What if  $\vec{b} \notin \text{span}\{\text{cols}(A)\}$ ?

A: There is no solution for  $A\vec{x} = \vec{b}$