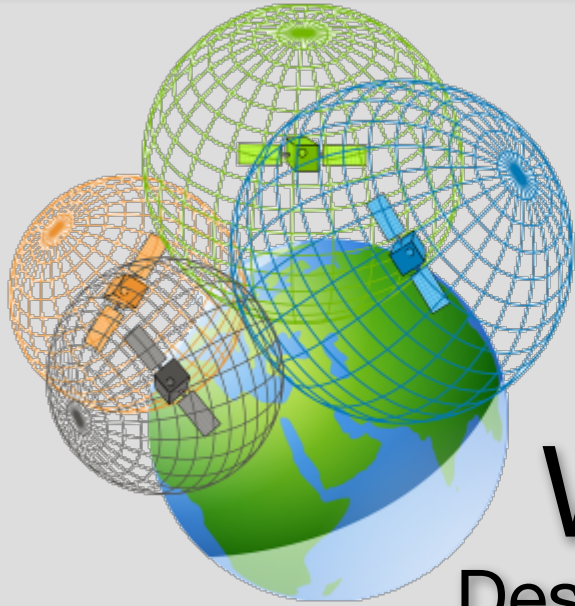Happy New Year!

Welcome Tigers.

Welcome to EECS 16A!
Designing Information Devices and Systems I

Ana Arias and Miki Lustig
Sp 2022

Lecture 2A
Span, Proofs
Linear (in)dependance

Announcements

- Quest: Tuesday 02/08/22 in class

- Last time:
  - Continue vectors
  - Matrix-Matrix and Matrix-vector Multiplications
  - Matrix-Vector Multiplications as linear set of equations
- Today:
  - Span
  - Proofs
  - Linear (in)dependance

Row vs Column Perspective

- Row / Measurement Perspective of $A\vec{x} = \vec{b}$

Row vs Column Perspective

- Row / Measurement Perspective of $A\vec{x} = \vec{b}$

Q: What does a row mean?

A: How each variable affect a particular measurement

Row vs Column Perspective

- Column Perspective of $A\vec{x} = \vec{b}$

Row vs Column Perspective

- Column Perspective of $A\overrightarrow{x} = \overrightarrow{b}$

Row vs Column Perspective

- Column Perspective of $A \vec{x} = \vec{b}$

Row vs Column Perspective

- Column Perspective of $A \vec{x} = \vec{b}$

Q: What does a column mean?

A: How a particular variable affects all measurements.

Linear combination of vectors

- Given set of vectors $\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_M\} \in \mathbb{R}^N$, and coefficients $\{\alpha_1, \alpha_2, \cdots, \alpha_M\} \in \mathbb{R}$

- A linear combination of vectors is defined as: $\vec{b} \triangleq \alpha_1 \vec{a}_1 + \alpha_2 \vec{a}_2 + \cdots + \alpha_M \vec{a}_M$
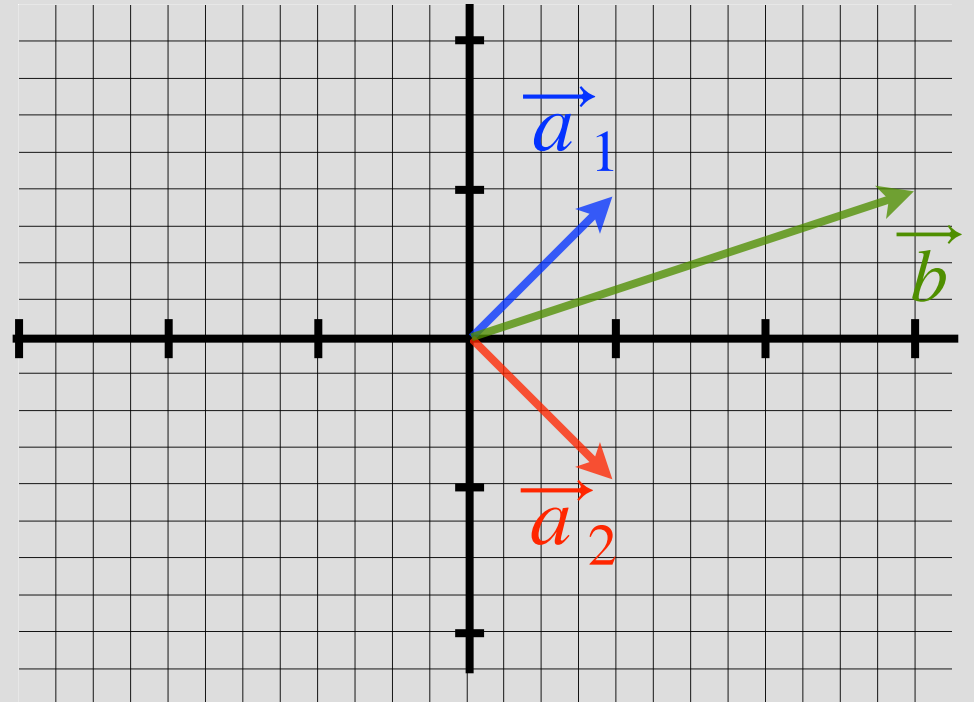
Recall: $A\vec{x}$:

Matrix-vector multiplication is a linear combination of the columns of A!

Linear Set of Equations as a Linear Combination

- Consider the problem: $A\vec{x} = \vec{b}$:

$\vec{a}_1$ $\vec{a}_2$ $\vec{b}$

Linear Set of Equations as a Linear Combination

- Consider the problem: $A\vec{x} = \vec{b}$:

$\vec{a}_1$ $\vec{a}_2$ $\vec{b}$

Q: What linear combination of $\vec{a}_1$, $\vec{a}_2$ will give $\vec{b}$?
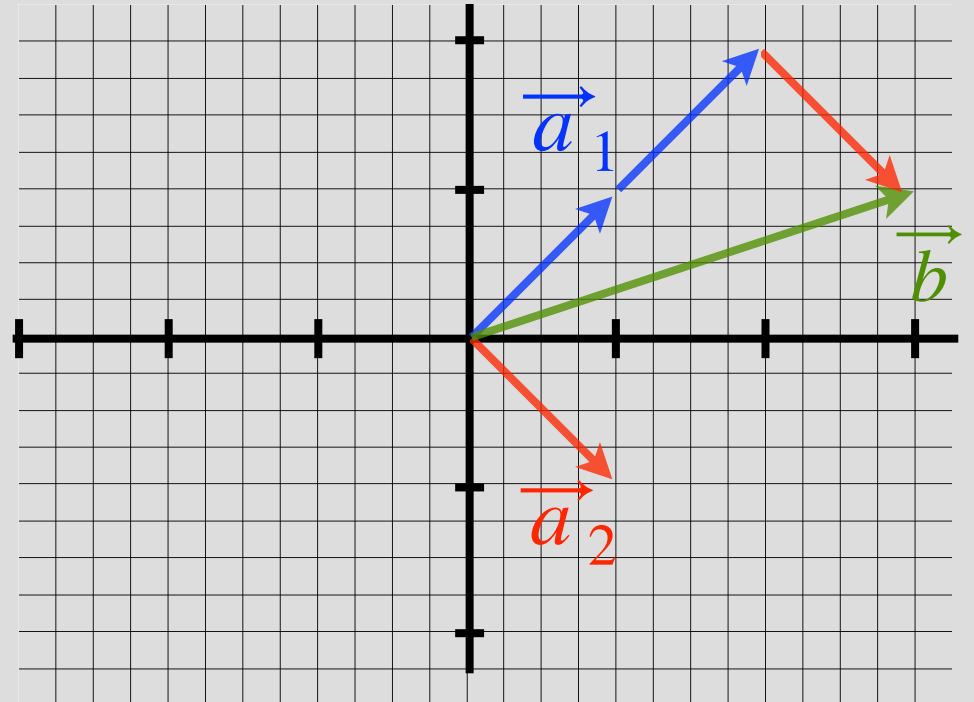
Linear Set of Equations as a Linear Combination

- Consider the problem: $A\vec{x} = \vec{b}$:

$\vec{a}_1$ $\vec{a}_2$ $\vec{b}$

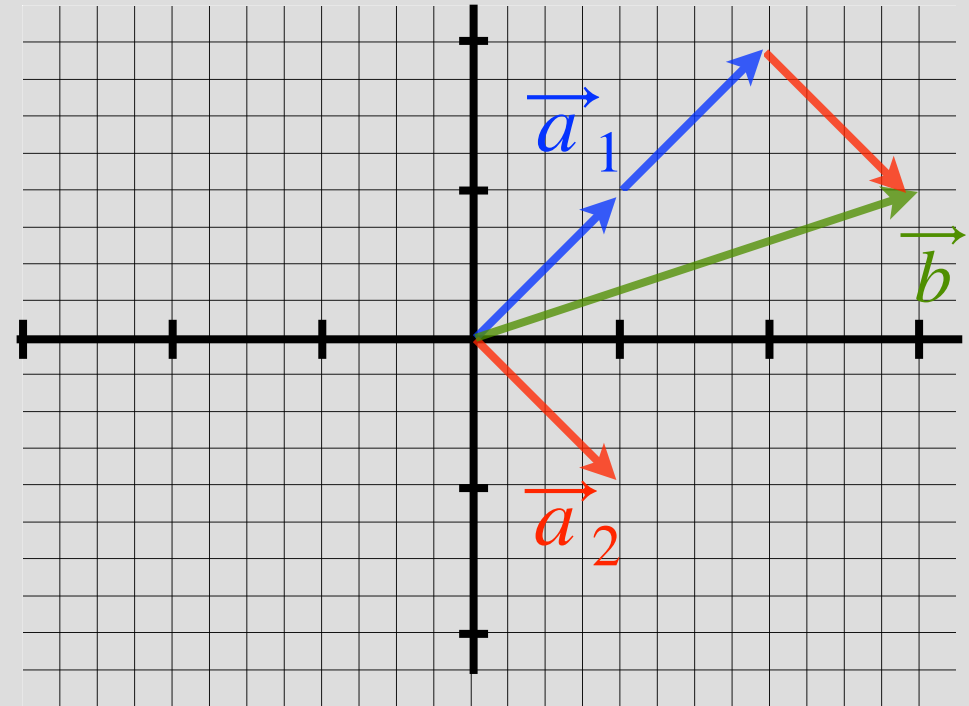Q: What linear combination of $\vec{a}_1$, $\vec{a}_2$ will give $\vec{b}$?

A: $2\vec{a}_1 + 1\vec{a}_2$

Linear Set of Equations as a Linear Combination

- Consider the problem: $A\vec{x} = \vec{b}$:

$\downarrow$
$\vec{a}_1$ $\quad$ $\downarrow$
$\vec{a}_2$ $\qquad\qquad\qquad$ $\downarrow$
$\vec{b}$
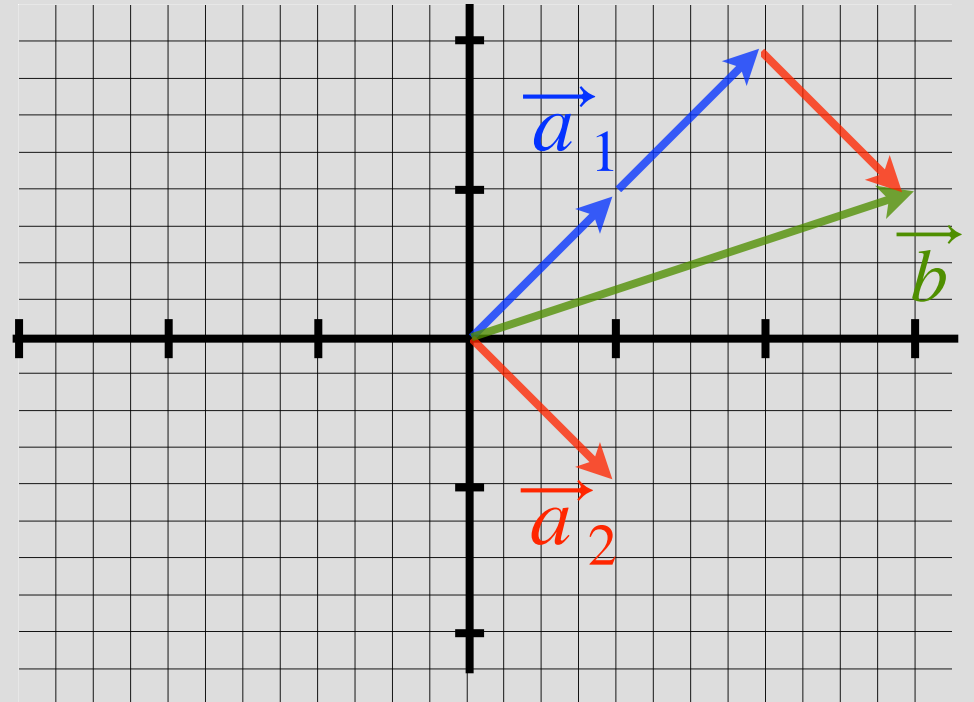
Q: What linear combination of $\vec{a}_1$, $\vec{a}_2$ will give $\vec{b}$?

A: $2\vec{a}_1 + 1\vec{a}_2$

Gaussian Elimination:

$\begin{bmatrix} 1 & 1 & | & 3 \\ 1 & -1 & | & 1 \end{bmatrix}$ $\quad$ $\begin{bmatrix} 1 & 1 & | & 3 \\ 0 & 1 & | & 1 \end{bmatrix}$ $\qquad$ $x_1 = 2$ $\quad \Rightarrow$ $\quad$ some as $\quad \vec{b} = 2\vec{a}_1 + 1 \cdot \vec{a}_2$

$\begin{bmatrix} 1 & 1 & | & 3 \\ 0 & -2 & | & -2 \end{bmatrix}$ $\quad$ $\begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & 1 \end{bmatrix}$ $\qquad$ $x_2 = 1$

Linear Set of Equations as a Linear Combination

- Consider the problem: $A\vec{x} = \vec{b}$:

$\vec{a}_1$ $\vec{a}_2$

Q: Can linear combination of $\vec{a}_1$, $\vec{a}_2$ give any $\vec{b}$?

A: Hmmm….I think so….
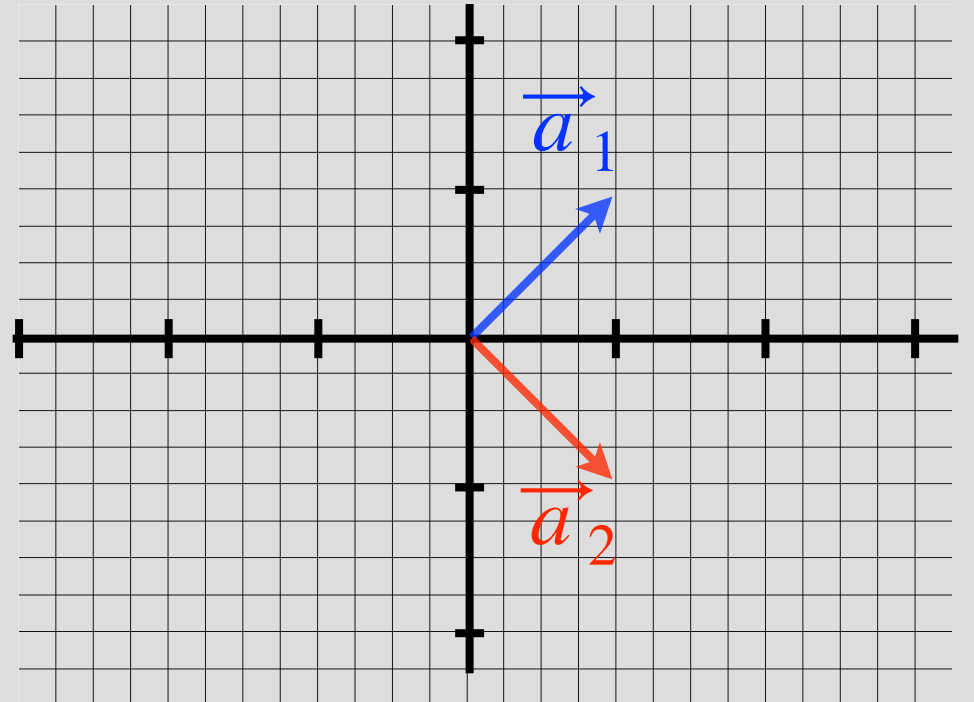
Linear Set of Equations as a Linear Combination

- Consider the problem: $A\vec{x} = \vec{b}$:

$\vec{a}_1$ $\vec{a}_2$

Q: Can linear combination of $\vec{a}_1$, $\vec{a}_2$ give any $\vec{b}$?

A: Hmmm….I don't think so…. Unless its along the line $\vec{a}_1$
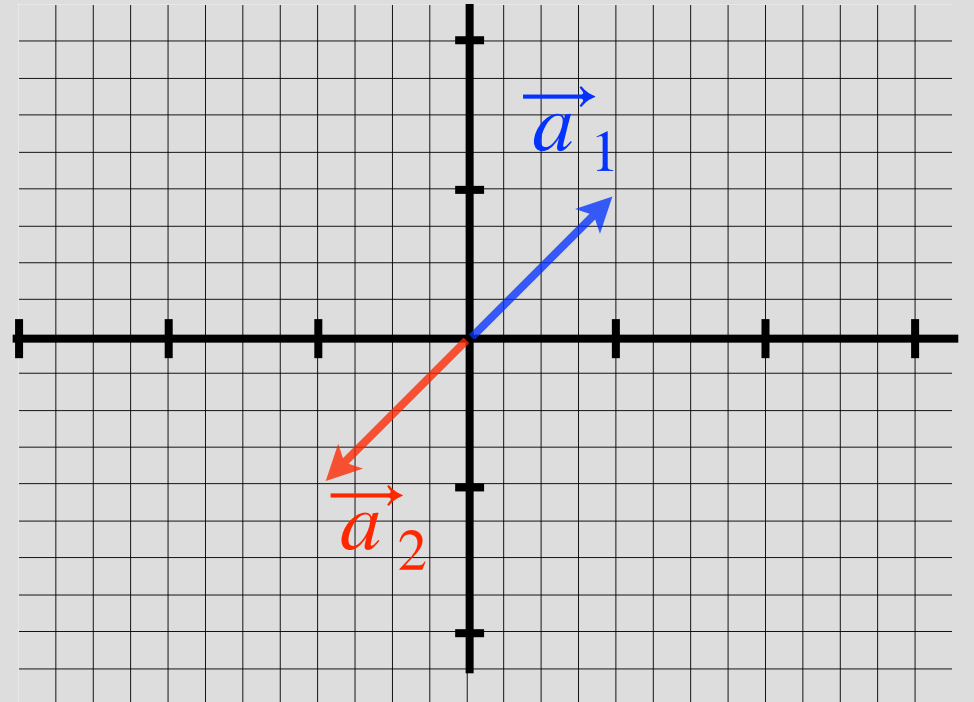
Linear Set of Equations as a Linear Combination

- Consider the problem: $A\vec{x} = \vec{b}$:

$$\vec{a}_1 \quad \vec{a}_2$$

Q: Can linear combination of $\vec{a}_1$, $\vec{a}_2$ give any $\vec{b}$?

A: Hmmm….yes!
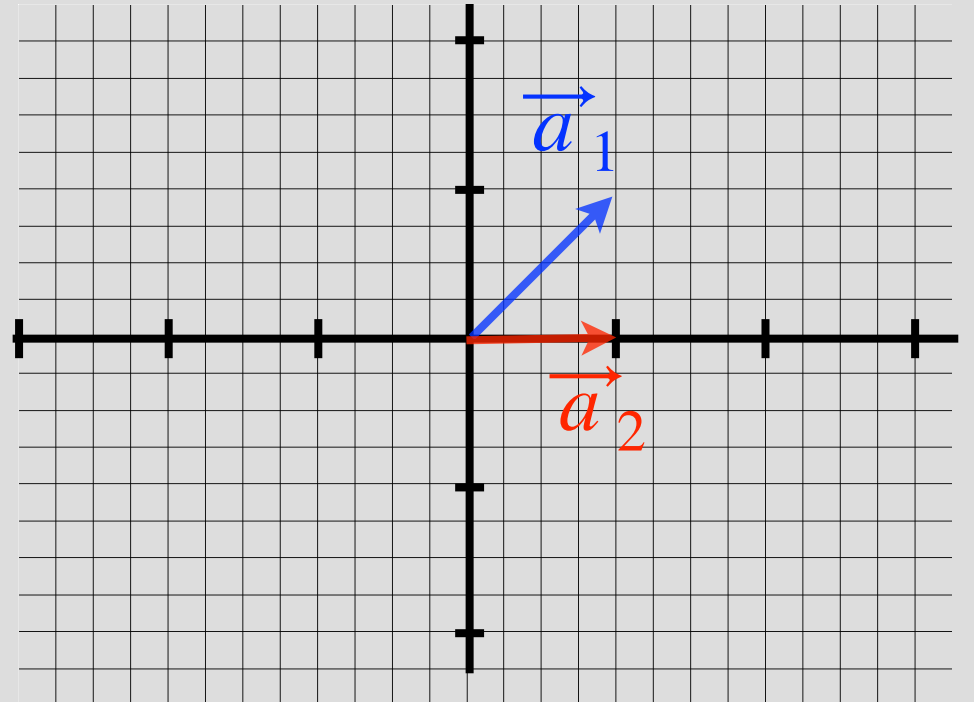
Span / Column Space / Range

- Span of the columns of A is the set of all vectors $\vec{b}$ such that $A\vec{x} = \vec{b}$ has a solution

  - i.e. the set of all vectors that can be reached by all possible linear combinations of the columns of A
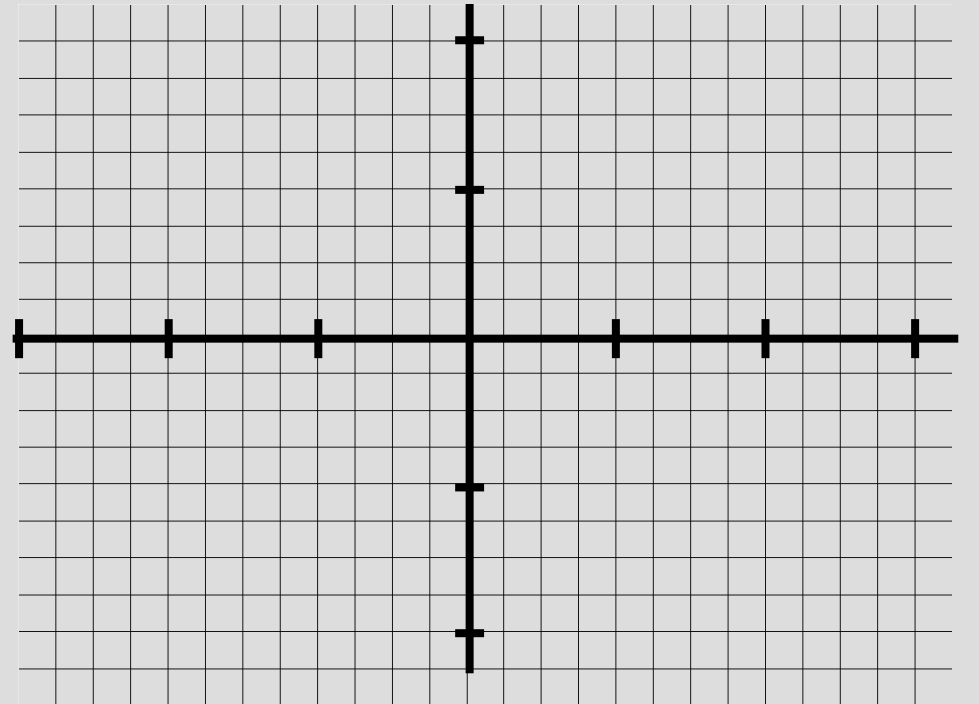
Span / Column Space / Range

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Example: What is the span of the cols of A?

Span / Column Space / Range

- Span of the columns of A is the set of all vectors $\vec{b}$ such that $A\vec{x} = \vec{b}$ has a solution
  - i.e. the set of all vectors that can be reached by all possible linear combinations of the columns of A

Example: What is the span of the cols of A?

A: $\mathbb{R}^2$!
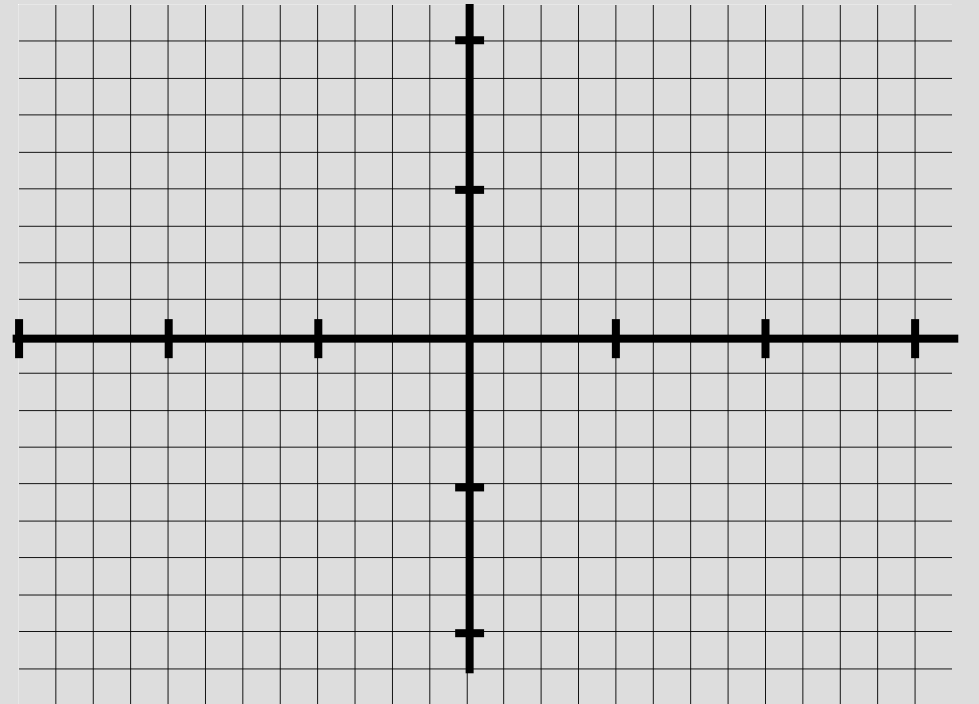
Span / Column Space / Range

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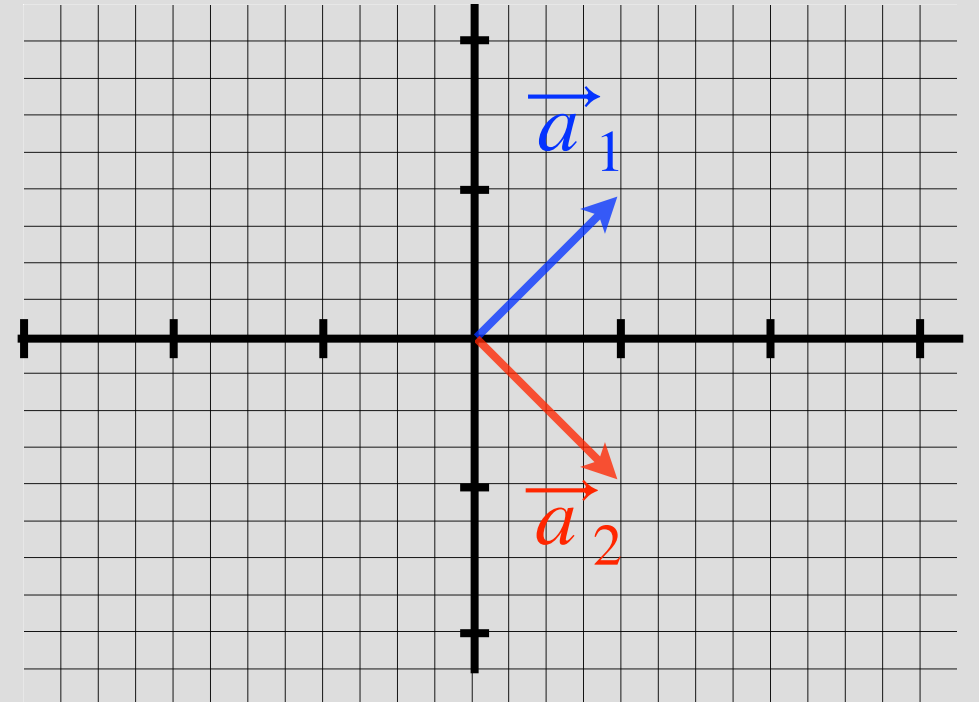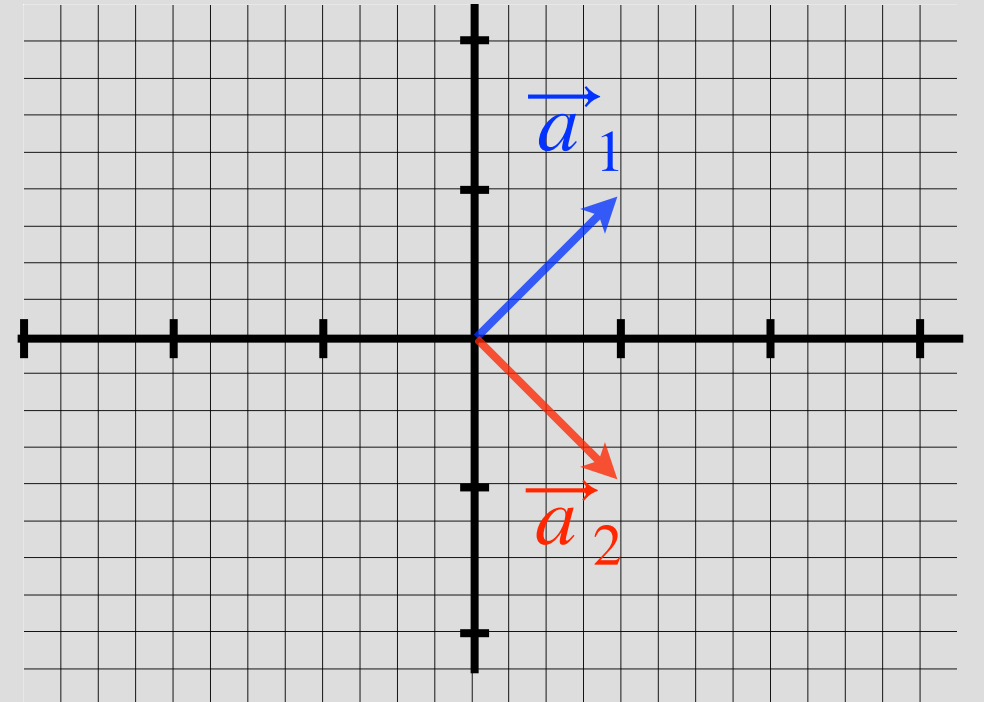
Span / Column Space / Range

Example 2: What is the span of the cols of A?

A: The line $x_1 = x_2$
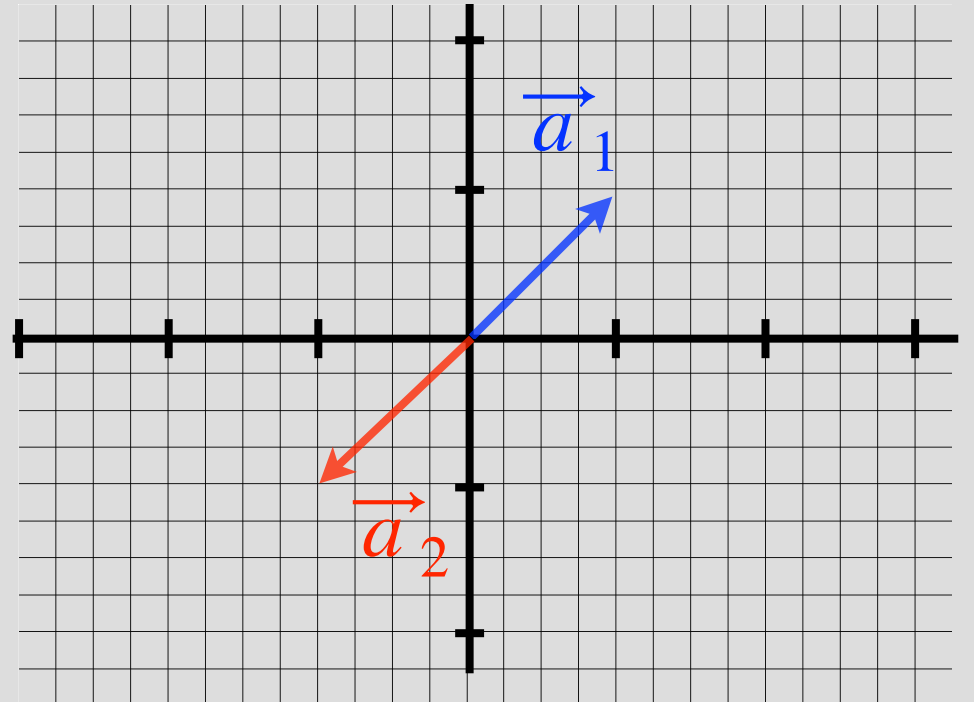
Span / Column Space / Range

- Definition:

  If $\exists\, \vec{x}$ s.t. $A\vec{x} = \vec{b}$ then $\vec{b} \in \mathrm{span}\{\mathrm{cols}(A)\}$

  Q: What if $\vec{b} \notin \mathrm{span}\{\mathrm{cols}(A)\}$?

  A: There is no solution for $A\vec{x} = \vec{b}$

Steps for a proof

- Write out the statement, note direction ("if" → "then")

- Try a simple example (to see a pattern)
  - Use what is known, definitions and other theorems

- Manipulate both sides of the arguments
  - Must justify each step

- Know the different styles of proofs to try
  - Constructive
  - Proof by contradiction
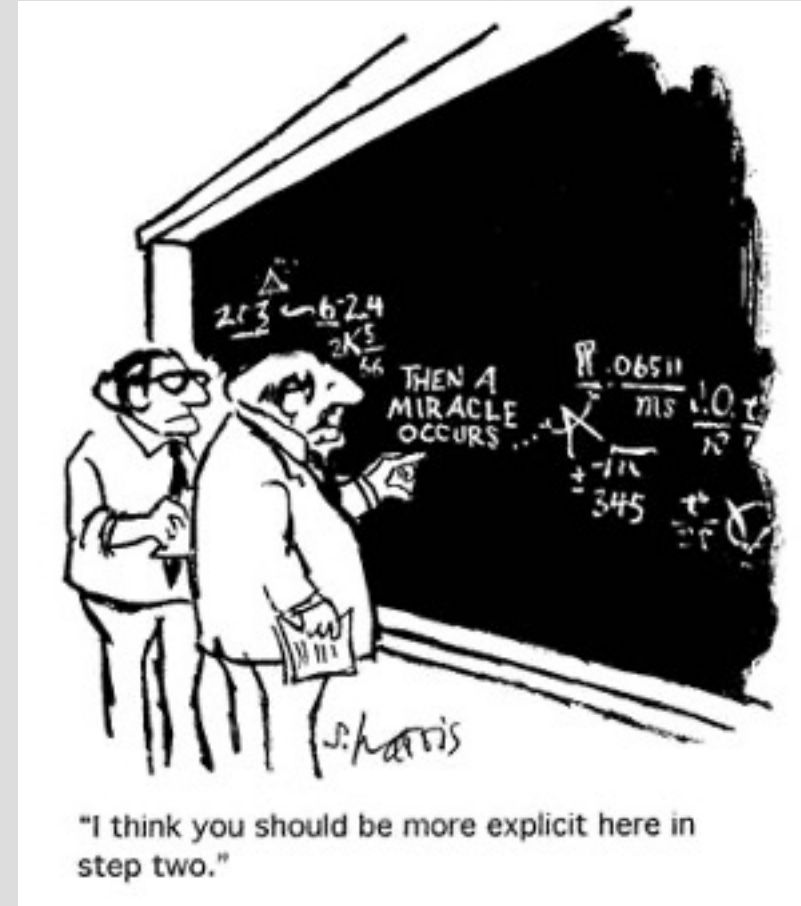
"I think you should be more explicit here in step two."

Algorithm for solving linear equations

- Three basic operations that don't change a solution:

  1. Multiply an equation with *nonzero* scalar

$$2x + 3y = 4 \quad \text{has the same solution as:} \quad 4x + 6y = 8$$

Proof for N=2:

Let $ax + by = c$, with solution x_0, y_0
$\Rightarrow ax_0 + by_0 = c$

Show that $\beta ax + \beta by = \beta c$, has the same solution.

Substitute x_0, y_0 for x, y:

$$\beta ax_0 + \beta by_0 = \beta c$$

$$\beta(ax_0 + by_0) = \beta c$$

$$\beta c = \beta c \quad \text{But is it the only solution?}$$

$\beta ax + \beta by = \beta c$, with solution: x_1, y_1
$\Rightarrow \beta ax_1 + \beta by_1 = \beta c$

Show that $ax + by = c$, has the same solution…..

Since $\beta \neq 0$….

$$\beta ax_1 + \beta by_1 = \beta c \Rightarrow ax_1 + by_1 = c$$

SOLUTION OF ONE, IMPLIES THE OTHER
AND VICE-VERSA!

Algorithm for solving linear equations

- Three basic operations that don't change a solution:

    1. Multiply an equation with *nonzero* scalar

    2. Adding a scalar constant multiple of one equation to another

    and

    Have the same solution

    Concept of proof: look at explicit solution, show they are the same

    Also show the reverse — by applying the reverse operations

Span / Column Space / Range

- Span of the columns of A is the set of all vectors $\vec{b}$ such that $A\vec{x} = \vec{b}$ has a solution

  - i.e. the set of all vectors that can be reached by all possible linear combinations of the columns of A

- Definition:

  If $\exists \vec{x}$ s.t. $A\vec{x} = \vec{b}$ then $\vec{b} \in \text{span}\{\text{cols}(A)\}$

Proof: Span

Theorem: $\operatorname{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\} = \mathbb{R}^2$

Know:

$\operatorname{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\} \Rightarrow \left\{\vec{v} \,\middle|\, \vec{v} = \alpha\begin{bmatrix}1\\1\end{bmatrix} + \beta\begin{bmatrix}1\\-1\end{bmatrix} \quad , \alpha, \beta \in \mathbb{R}\right\} = \mathbb{S}$

Need to show:

$\operatorname{span}\left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}\right\} = \mathbb{R}^2$

Concept: pick some specific $\vec{b} = \begin{bmatrix}b_1\\b_2\end{bmatrix} \in R^2$, and show that it belongs to $\mathbb{S}$

Need to solve:

Proof: Span

Theorem: $\operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\} = \mathbb{R}^2$

Know:

$\operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\} \Rightarrow \left\{\vec{v} \;\middle|\; \vec{v} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad, \alpha, \beta \in \mathbb{R}\right\} = \mathbb{S}$

Need to show:

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Concept: pick some specific $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \in R^2$, and show that it belongs to $\mathbb{S}$

Need to solve:

$\alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

unknown

Known
and $\in \mathbb{R}^2$

Proof: Span

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Gaussian Elimination:

Proof: Span

Need to solve:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\frac{b_1 + b_2}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{b_1 - b_2}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Constructive proof

Gaussian Elimination:

$$\begin{bmatrix} 1 & 1 & \bigg| & b_1 \\ 1 & -1 & \bigg| & b_2 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & \bigg| & b_1 \\ 0 & -2 & \bigg| & b_2 - b_1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & \bigg| & b_1 \\ 0 & 1 & \bigg| & \frac{b_1 - b_2}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \bigg| & \frac{b_1 + b_2}{2} \\ 0 & 1 & \bigg| & \frac{b_1 - b_2}{2} \end{bmatrix} \Rightarrow \alpha = \frac{b_1 + b_2}{2}, \beta = \frac{b_1 - b_2}{2},$$

Every $\vec{b} \in \mathbb{R}^2$ can be written as linear combinations! So also, $\vec{b} \in \mathbb{S}$

Linear Dependence
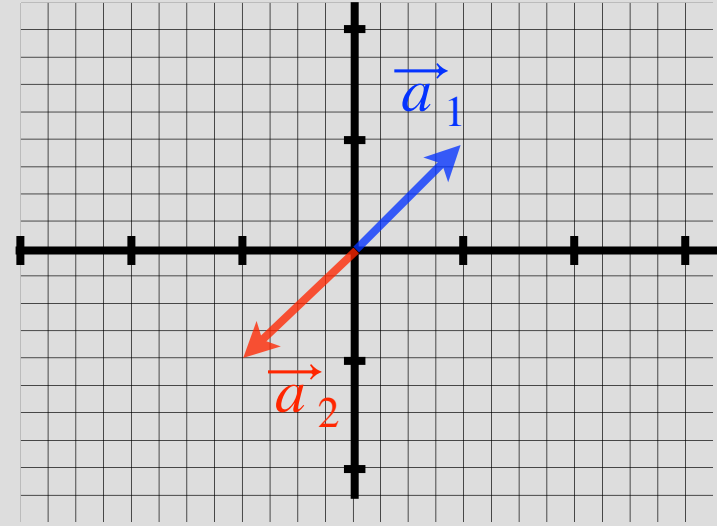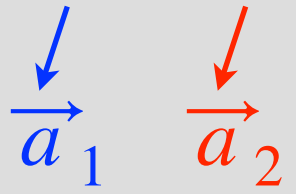
$\vec{a}_1$ $\vec{a}_2$

$\vec{a}_1$ and $\vec{a}_2$ are linearly dependent

$$\vec{a}_1 = -\vec{a}_2$$

Department of
Redundancy
Department

Linear Dependence

- Definition 1:

A set of vectors $\left\{ \overrightarrow{a}_1, \overrightarrow{a}_2, \cdots, \overrightarrow{a}_N \right\}$ are linearly dependent if

$\exists \left\{ \alpha_1, \alpha_2, \cdots, \alpha_N \right\} \in \mathbb{R}$, such that:

$$\overrightarrow{a}_i = \sum_{j \neq i} \alpha_j \overrightarrow{a}_j \qquad 1 \leq i, j \leq N$$

For example: if $\overrightarrow{a}_2 = 3\overrightarrow{a}_1 - 2\overrightarrow{a}_5 + 6\overrightarrow{a}_7$

$\overrightarrow{a}_i$ in the span of all $\overrightarrow{a}_j$s

Linear Dependence

Are these linearly dependent?

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$$

Need to solve:

Linear Dependence

Are these linearly dependent?

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$$

Are linearly dependent

Need to solve:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

but we showed that….

$$\frac{b_1 + b_2}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{b_1 - b_2}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

So….

$$\frac{3+1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{3-1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Linear dependence / independence

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\} \Rightarrow 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 0$$

- Definition 2:
  A set of vectors $\left\{ \vec{a}_1, \vec{a}_2, \cdots, \vec{a}_N \right\}$ are linearly dependent if
  $\exists \left\{ \alpha_1, \alpha_2, \cdots, \alpha_N \right\} \in \mathbb{R}$, such that:
  $$\sum_{i=1}^{N} \alpha_i \vec{a}_i = 0$$
  As long as not all $a_i = 0$

- Definition:
  A set of vectors $\left\{ \vec{a}_1, \vec{a}_2, \cdots, \vec{a}_N \right\}$ are linearly independent if they are not dependent
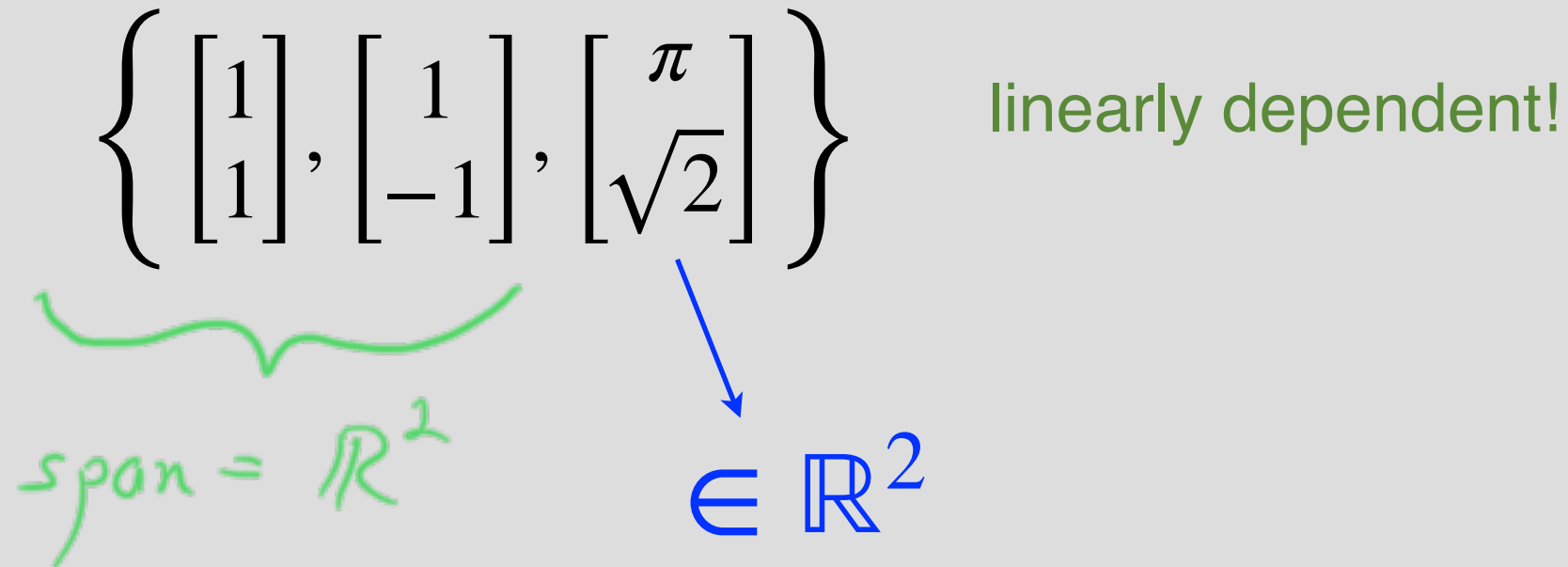
Linear dependence / independence

Are these linearly dependent?

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} \pi \\ \sqrt{2} \end{bmatrix} \right\}$$

linearly dependent!

$$\underbrace{\phantom{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}}}_{}$$

$span = \mathbb{R}^2$

$\in \mathbb{R}^2$

Solutions for linear equations

- Theorem: if the columns of the matrix A are linearly dependent then, $A\vec{x} = \vec{b}$ does not have a unique solution

PROOF Consider the counter-example $S \triangleq \{O, \bullet\}$, $\tau \triangleq \{\langle \bullet, \bullet \rangle, \langle \bullet, O \rangle, \langle O, O \rangle\}$ so that $\mathcal{M}_\tau = \{\langle i, \lambda \ell \cdot \bullet \rangle, \langle j, \lambda \ell \cdot O \rangle, \langle k, \lambda \ell \cdot (\ell < m ? \bullet \mathbin{\text{¿}} O) \rangle\}$. We let $\mathcal{X} \triangleq \{\langle i, \sigma \rangle \mid \forall j < i : \sigma_j = \bullet\}$ so that $\neg FD(\mathcal{X})$. We have $\mathcal{M}_{\tau \downarrow \bullet} = \{\langle i, \lambda \ell \cdot \bullet \rangle, \langle k, \lambda \ell \cdot (\ell < m ? \bullet \mathbin{\text{¿}} O) \rangle \mid k < m\}$, $\mathcal{M}_{\tau \downarrow O} = \{\langle j, \lambda \ell \cdot O \rangle, \langle k, \lambda \ell \cdot (\ell < m ? \bullet \mathbin{\text{¿}} O) \rangle \mid k \geq m\}$ and $\oplus \llparenthesis \mathcal{X} \rrparenthesis = \{\langle i, \sigma \rangle \mid \forall j \leq i : \sigma_j = \bullet\}$. We have $\alpha^\vee_{\mathcal{M}_\tau}(\oplus \llparenthesis \mathcal{X} \rrparenthesis) = \{s \mid \mathcal{M}_{\tau \downarrow s} \subseteq \oplus \llparenthesis \mathcal{X} \rrparenthesis\} = \{\bullet\}$ whereas $\widetilde{pre}[\tau](\alpha^\vee_{\mathcal{M}_\tau}(\mathcal{X})) = \widetilde{pre}[\tau](\{s \mid \mathcal{M}_{\tau \downarrow s} \subseteq \mathcal{X}\}) = \widetilde{pre}[\tau](\{\bullet\}) = \{s \mid \forall s' : t(s, s') \Rightarrow s' = \bullet\} = \emptyset$ since $t(s, \bullet)$ implies $s = \bullet$ and $t(\bullet, O)$ holds. ∎

Solutions for linear equations

- Theorem: if the columns of the matrix A are linearly dependent then, $A\vec{x} = \vec{b}$ does <u>not</u> have a unique solution

Proof for $A \in \mathbb{R}^{3\times 3}$

know: columns are linearly ~~independent~~   show: more than 1 solution

Concept: pick some specific solution $\vec{x}*$, and show that there's another one

Let: $A\vec{x}* = \vec{b}$ and $A = \begin{bmatrix} \vec{a_1} & \vec{a}_2 & \vec{a}_3 \end{bmatrix}$

From linear dependence Def 2:

$$\alpha_1\vec{a_1} + \alpha_2\vec{a}_2 + \alpha_3\vec{a}_3 = 0$$

Solutions for linear equations

- Theorem: if the columns of the matrix A are linearly dependent then, $A\vec{x} = \vec{b}$ does not have a unique solution

Proof for $A \in \mathbb{R}^{3\times 3}$

know: columns are linearly dependent   show: more than 1 solution

Concept: pick some specific solution $\vec{x}*$, and show that there's another one

Let: $A\vec{x}* = \vec{b}$ and $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix}$

From linear dependence Def 2:

$$\alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \alpha_3\vec{a}_3 = 0 \longrightarrow \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \bigcirc \qquad \Rightarrow A\vec{\alpha} = 0$$

$\vec{\alpha}$

Set $\vec{x}^\dagger = \vec{x}* + \vec{\alpha}$

$$\Rightarrow A\vec{x}^\dagger = A(\vec{x}* + \vec{\alpha}) = A\vec{x}* + A\vec{\alpha} = \vec{b} + 0$$

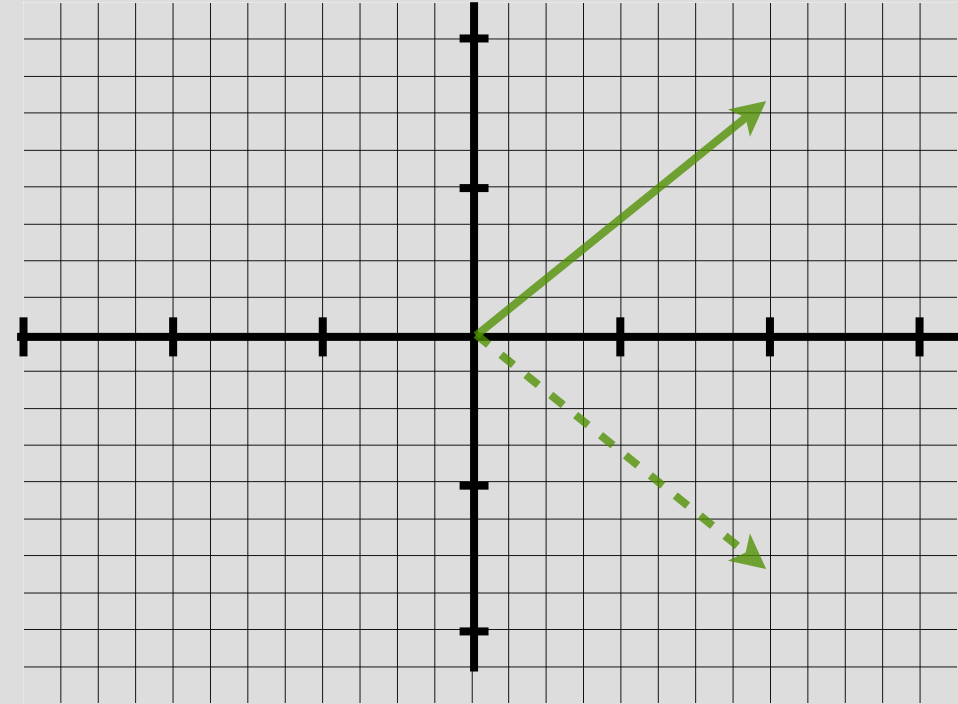So $\vec{x}^\dagger$ is another solution!

Matrix Transformations

$$\begin{bmatrix} \cos 90° & \sin 90° \\ -\sin 90° & \cos 90° \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_2 \\ a_1 \end{bmatrix}$$

Matrices are operators that transform vectors

$$A\overrightarrow{x} = \overrightarrow{b}$$

Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$
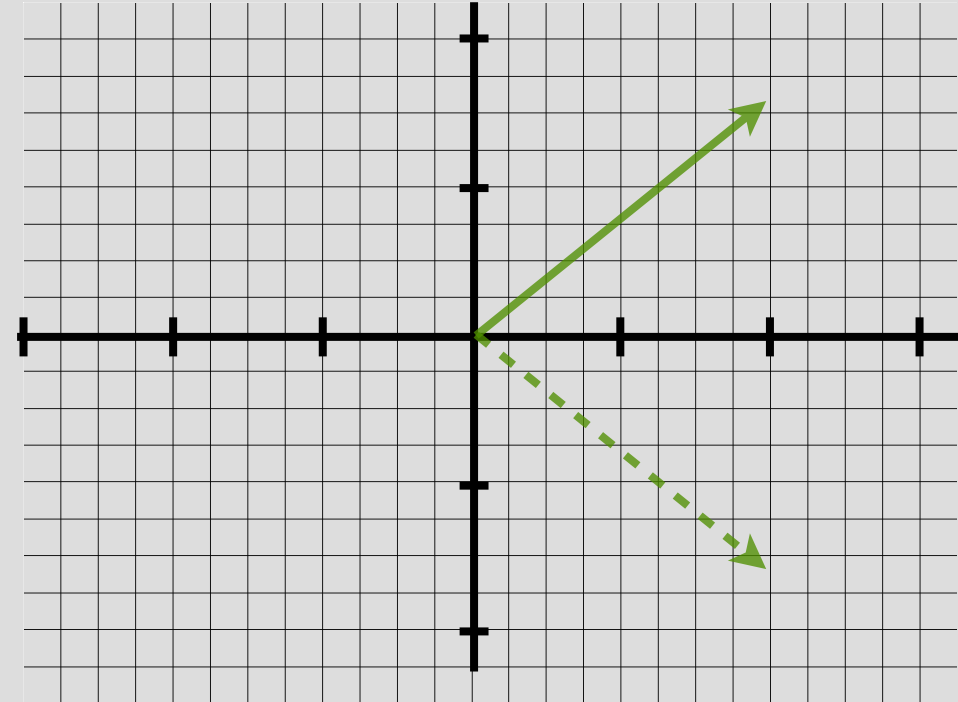
Matrices are operators that transform vectors

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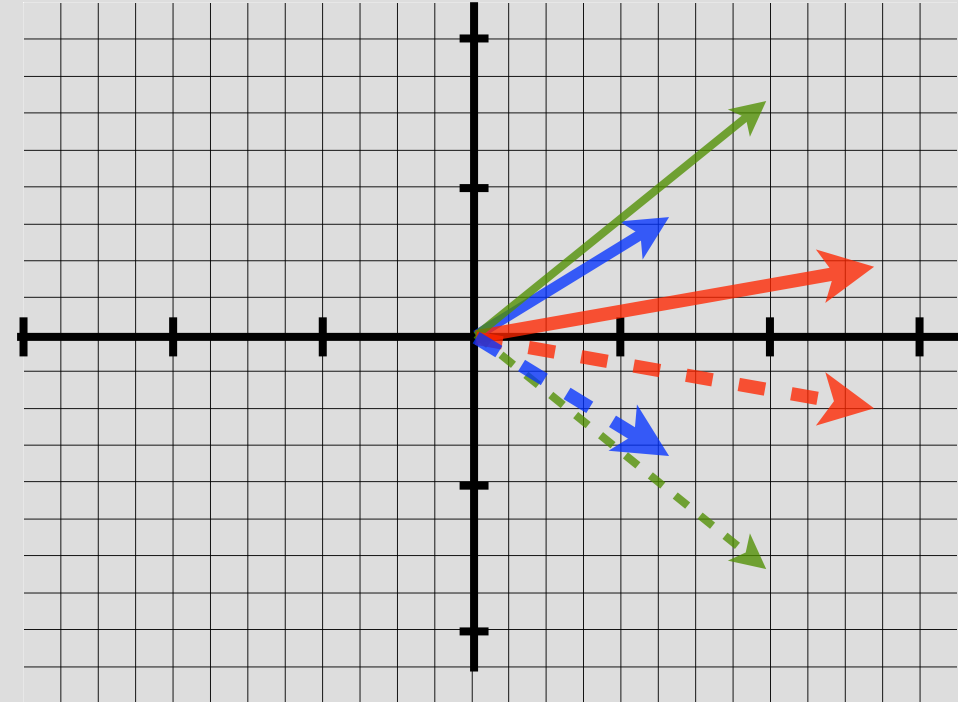
https://www.youtube.com/watch?v=LhF_56SxrGk

Matrices are operators that transform vectors

$$A \vec{x} = \vec{b}$$

Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$
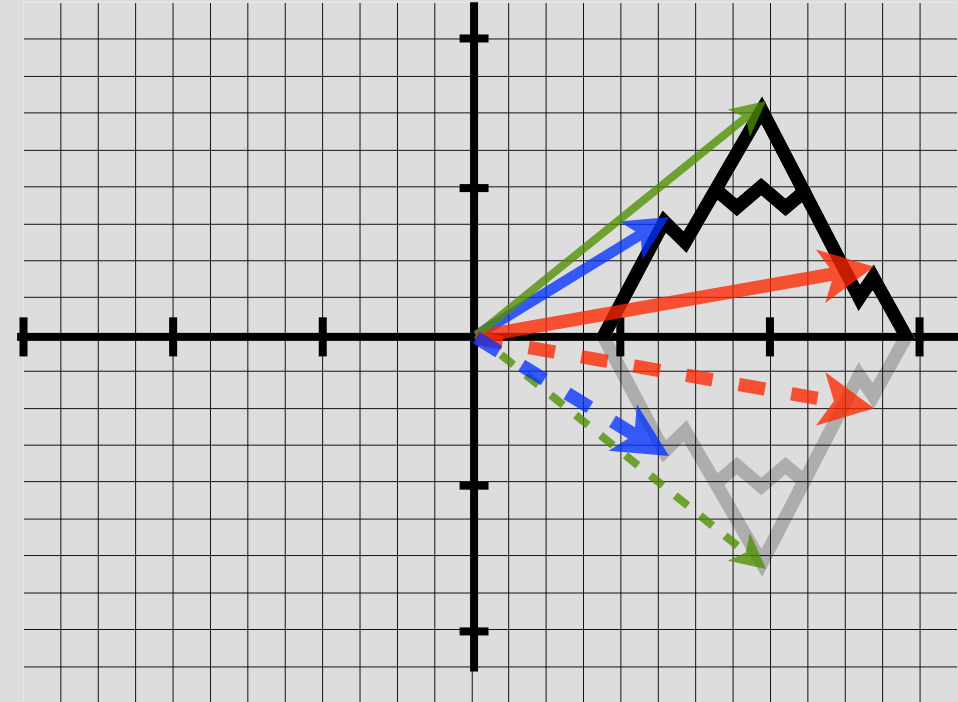
Reflection Matrix!

Matrices are operators that transform vectors

$$A \vec{x} = \vec{b}$$

Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$
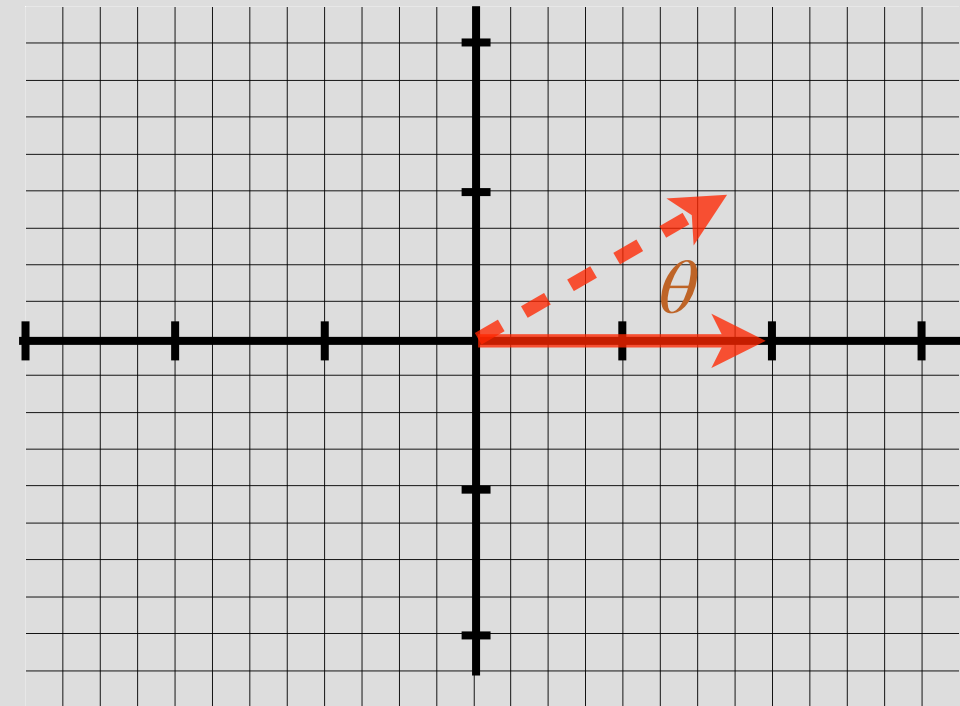
Reflection Matrix!

Matrices are operators that transform vectors

Example 2:

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos(\theta)x_1 - \sin(\theta)x2 \\ \sin(\theta)x_1 + \cos(\theta)x_2 \end{bmatrix}$$

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

Rotation Matrix!

$$\begin{bmatrix} \cos 90° & \sin 90° \\ -\sin 90° & \cos 90° \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_2 & a_1 \end{bmatrix}$$

Linear Transformation of vectors

f: is a linear transformation if:

$$f(\alpha \vec{x}) = \alpha f(\vec{x}) \qquad \alpha \in \mathbb{R}$$

$$f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y})$$

Claim: Matrix-vector multiplications satisfy linear transformation

$$A \cdot (\alpha \vec{x}) = \alpha A \vec{x}$$

Proof via explicitly writing the elements

$$A \cdot (\vec{x} + \vec{y}) = A \vec{x} + A \vec{y}$$