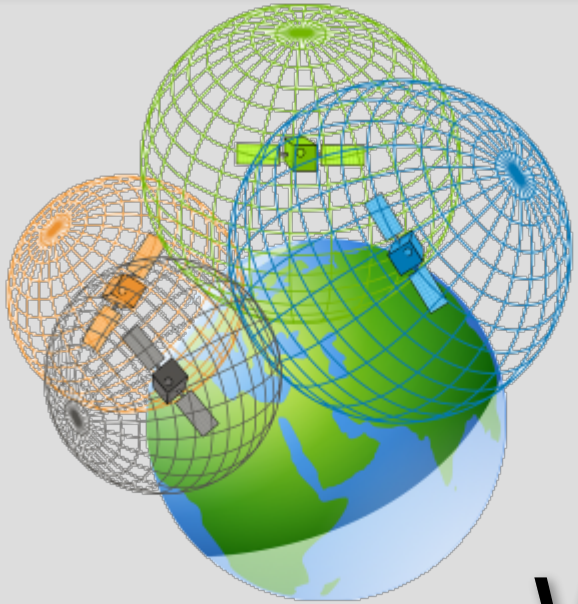
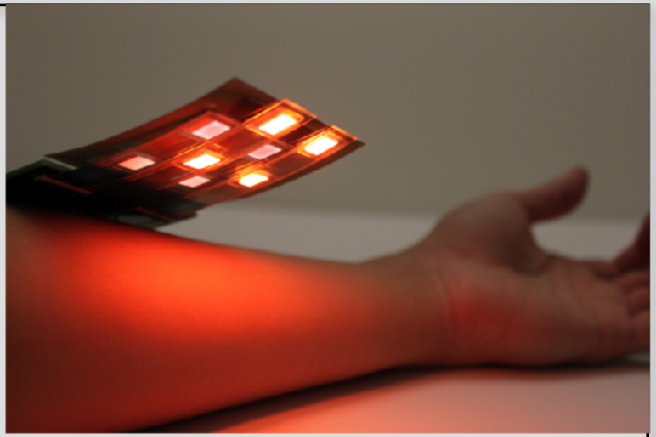




Welcome to EECS 16A!

Designing Information Devices and Systems I

Ana Arias and Miki Lustig

Sp 2022

Lecture 3A

Matrix xForms

Announcements

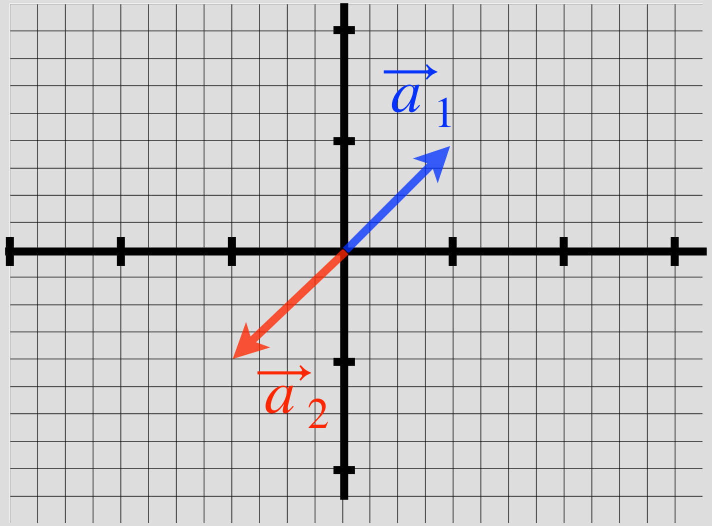
- Last time:
    - Proofs
    - Span
- Today:
    - Linear (in)dependance
    - Matrix Transformations

Linear Dependence

Recall:

$$A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{array}{cc} \swarrow & \swarrow \\ \vec{a}_1 & \vec{a}_2 \end{array}$$



$\vec{a}_1$ and $\vec{a}_2$ are linearly dependent

$$\vec{a}_1 = -\vec{a}_2$$

Linear Dependence

- Definition 1:
  A set of vectors $\{\overrightarrow{a}_1, \overrightarrow{a}_2, \cdots, \overrightarrow{a}_N\}$ are linearly dependent if $\exists \{\alpha_1, \alpha_2, \cdots, \alpha_N\} \in \mathbb{R}$, such that:

$$\overrightarrow{a}_i = \sum_{j \neq i} \alpha_j \overrightarrow{a}_j \qquad 1 \leq i, j \leq M$$

$$\downarrow$$

$\overrightarrow{a}_i$ in the span of all $\overrightarrow{a}_j$s

For example: if $\overrightarrow{a}_2 = 3\overrightarrow{a}_1 - 2\overrightarrow{a}_5 + 6\overrightarrow{a}_7$

Linear Dependence

Are these linearly dependent?

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$$

Need to solve:

Linear Dependence

Are these linearly dependent?

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$$

Are linearly dependent

Need to solve:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

but we showed that….

$$\frac{b_1+b_2}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{b_1-b_2}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

So….

$$\frac{3+1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{3-1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Linear dependence / independence

$$\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}3\\1\end{bmatrix}\right\} \quad \Rightarrow 2\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}1\\-1\end{bmatrix}-\begin{bmatrix}3\\1\end{bmatrix}=0$$

- Definition 2:
  A set of vectors $\left\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_N\right\}$ are linearly dependent if $\exists\left\{\alpha_1, \alpha_2, \cdots, \alpha_N\right\} \in \mathbb{R}$, such that:

$$\sum_{i=1}^{N} \alpha_i \vec{a}_i = 0$$

As long as not all $a_i = 0$

- Definition:
  A set of vectors $\left\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_N\right\}$ are linearly independent if they are not dependent

Linear dependence / independence

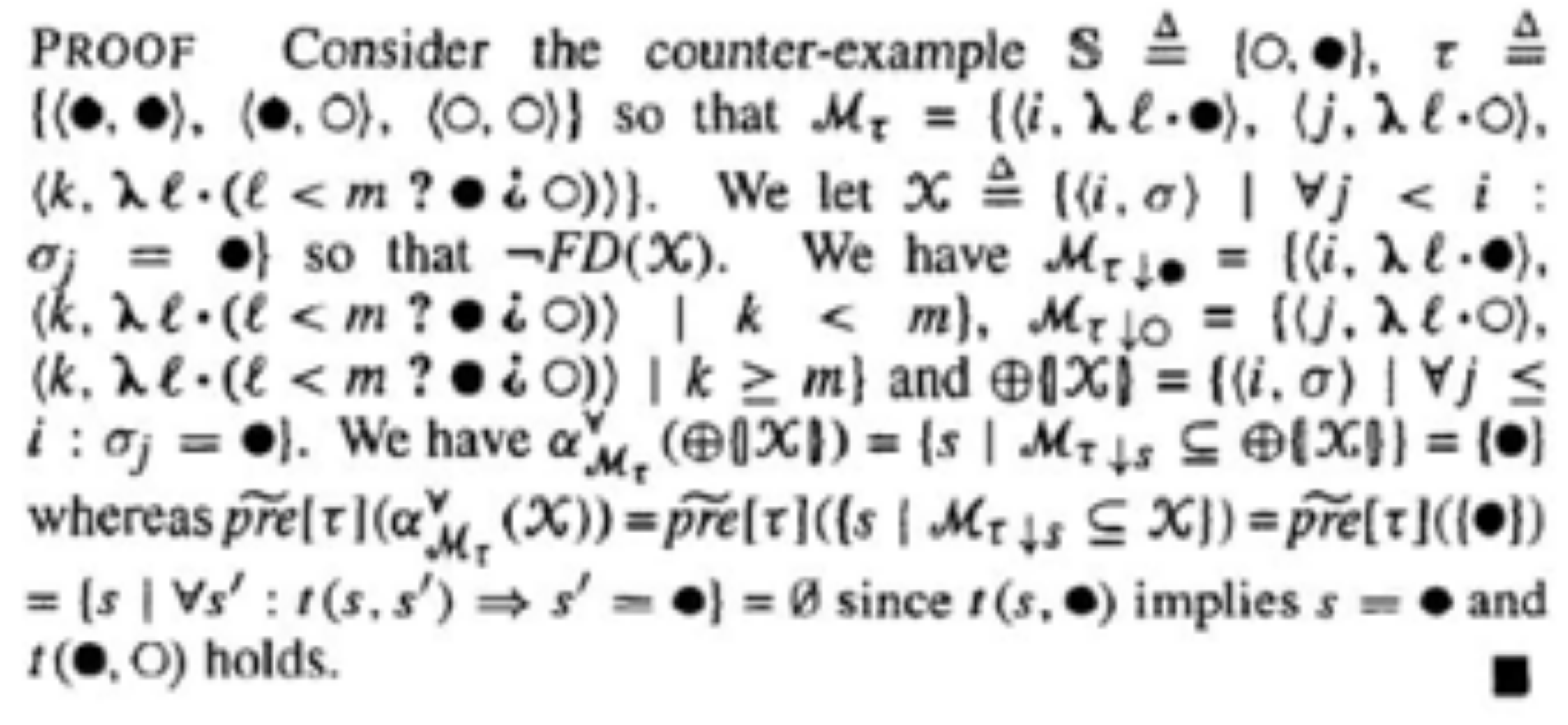
Are these linearly dependent?

$$\left\{ \underbrace{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}}_{\text{span} = \mathbb{R}^2}, \underset{\substack{\big\downarrow \\ \in \mathbb{R}^2}}{\begin{bmatrix}\pi\\\sqrt{2}\end{bmatrix}} \right\}$$

linearly dependent!

Solutions for linear equations

- Theorem: if the columns of the matrix A are linearly dependent then, $A\overrightarrow{x} = \overrightarrow{b}$ does <u>not</u> have a unique solution



PROOF Consider the counter-example $\mathbb{S} \triangleq \{\circ, \bullet\}$, $\tau \triangleq \{\langle \bullet, \bullet \rangle, \langle \bullet, \circ \rangle, \langle \circ, \circ \rangle\}$ so that $\mathcal{M}_\tau = \{\langle i, \lambda \ell \cdot \bullet \rangle, \langle j, \lambda \ell \cdot \circ \rangle, \langle k, \lambda \ell \cdot (\ell < m \,?\, \bullet \text{ ¿ } \circ) \rangle\}$. We let $\mathcal{X} \triangleq \{\langle i, \sigma \rangle \mid \forall j < i : \sigma_j = \bullet\}$ so that $\neg FD(\mathcal{X})$. We have $\mathcal{M}_{\tau \downarrow \bullet} = \{\langle i, \lambda \ell \cdot \bullet \rangle, \langle k, \lambda \ell \cdot (\ell < m \,?\, \bullet \text{ ¿ } \circ) \rangle \mid k < m\}$, $\mathcal{M}_{\tau \downarrow \circ} = \{\langle j, \lambda \ell \cdot \circ \rangle, \langle k, \lambda \ell \cdot (\ell < m \,?\, \bullet \text{ ¿ } \circ) \rangle \mid k \geq m\}$ and $\oplus \{\!|\mathcal{X}|\!\} = \{\langle i, \sigma \rangle \mid \forall j \leq i : \sigma_j = \bullet\}$. We have $\alpha^{\forall}_{\mathcal{M}_\tau}(\oplus \{\!|\mathcal{X}|\!\}) = \{s \mid \mathcal{M}_{\tau \downarrow s} \subseteq \oplus \{\!|\mathcal{X}|\!\}\} = \{\bullet\}$ whereas $\widetilde{pre}[\tau](\alpha^{\forall}_{\mathcal{M}_\tau}(\mathcal{X})) = \widetilde{pre}[\tau](\{s \mid \mathcal{M}_{\tau \downarrow s} \subseteq \mathcal{X}\}) = \widetilde{pre}[\tau](\{\bullet\}) = \{s \mid \forall s' : t(s, s') \Rightarrow s' = \bullet\} = \emptyset$ since $t(s, \bullet)$ implies $s = \bullet$ and $t(\bullet, \circ)$ holds. ■

Solutions for linear equations

- Theorem: if the columns of the matrix A are linearly dependent then, $A\vec{x} = \vec{b}$ does <u>not</u> have a unique solution

Proof for $A \in \mathbb{R}^{3\times 3}$

know: columns are linearly ~~in~~dependent

show: more than 1 solution

Concept: pick some specific solution $\vec{x}^*$, and show that there's another one

Let: $A\vec{x}^* = \vec{b}$ and $A = [\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3]$

From linear dependence Def 2:

$$\alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \alpha_3\vec{a}_3 = 0$$

Solutions for linear equations

- Theorem: if the columns of the matrix A are linearly dependent then, $A\vec{x} = \vec{b}$ does <u>not</u> have a unique solution

Proof for $A \in \mathbb{R}^{3\times3}$

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Let: $A\vec{x}^* = \vec{b}$ and $A = [\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3]$

From linear dependence Def 2:

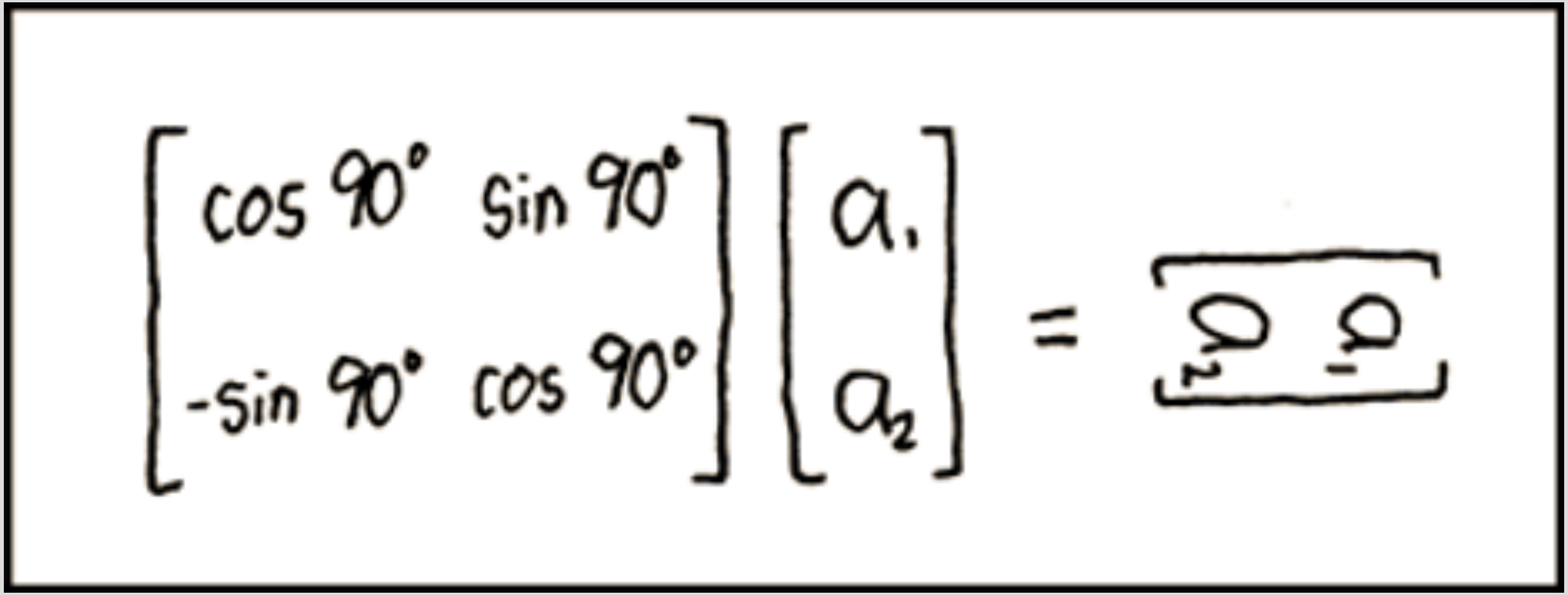
$$\alpha_1\vec{a}_1 + \alpha_2\vec{a}_2 + \alpha_3\vec{a}_3 = 0 \longrightarrow \begin{bmatrix}\vec{a}_1 & \vec{a}_2 & \vec{a}_3\end{bmatrix}\underbrace{\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{bmatrix}}_{\vec{\alpha}} = 0 \qquad \Rightarrow A\vec{\alpha} = 0$$

Set $\vec{x}^\dagger = \vec{x}^* + \vec{\alpha}$

$$\Rightarrow A\vec{x}^\dagger = A(\vec{x}^* + \vec{\alpha}) = A\vec{x}^* + A\vec{\alpha} = \vec{b} + 0$$

So $\vec{x}^\dagger$ is another solution!

Matrix Transformations

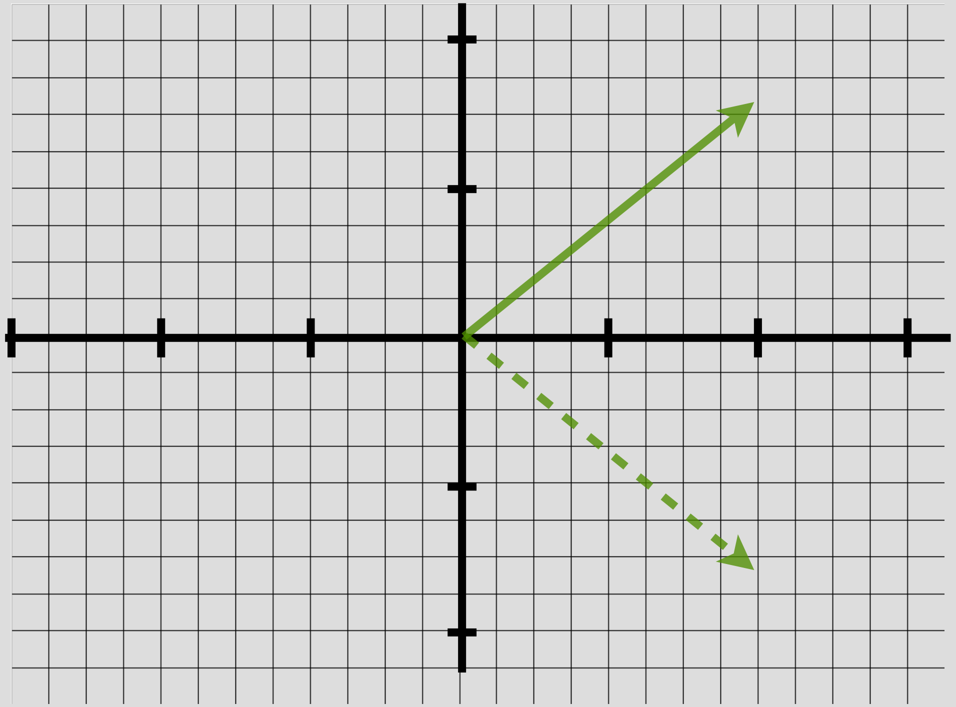


Matrices are operators that transform vectors

$$A\overrightarrow{x} = \overrightarrow{b}$$

Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$



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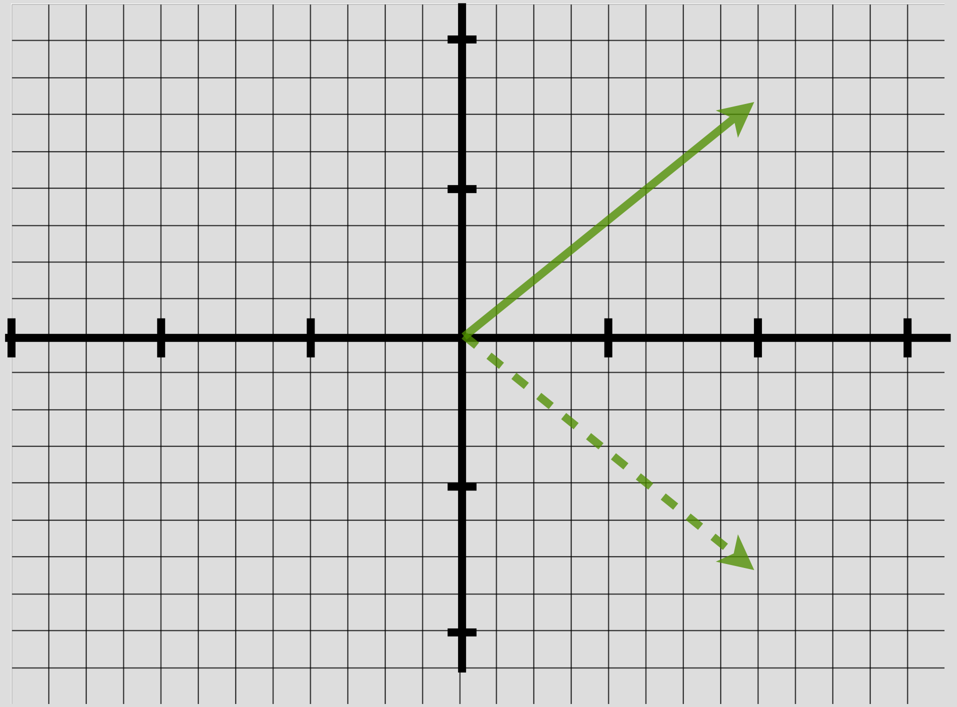
https://www.youtube.com/watch?v=LhF_56SxrGk



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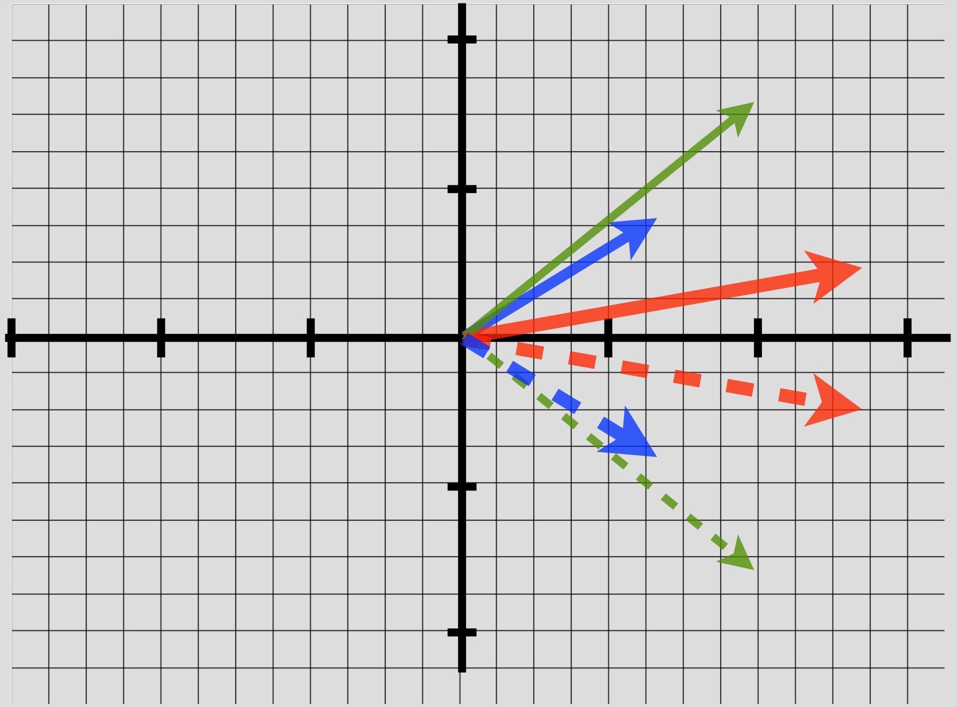
Reflection Matrix!



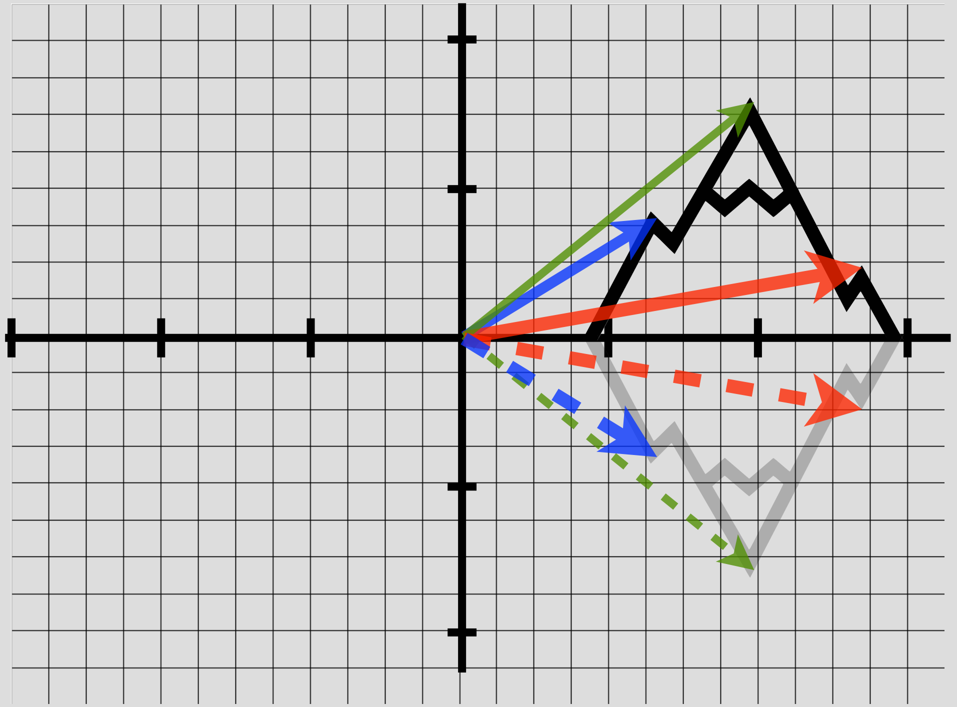
Matrices are operators that transform vectors

$$A\vec{x} = \vec{b}$$

Example:

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

Reflection Matrix!



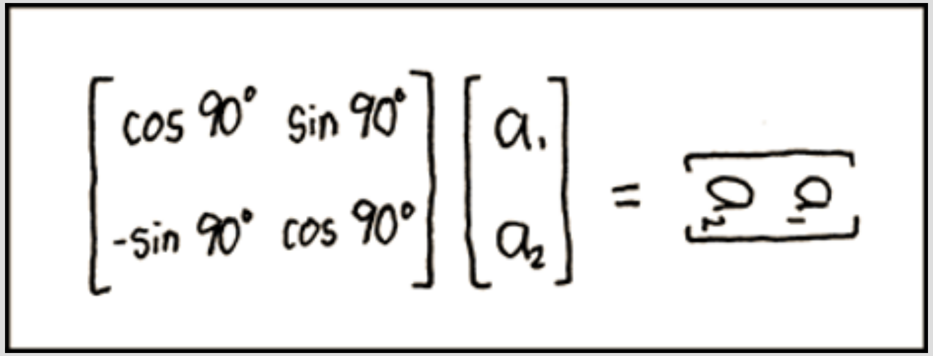
Matrices are operators that transform vectors

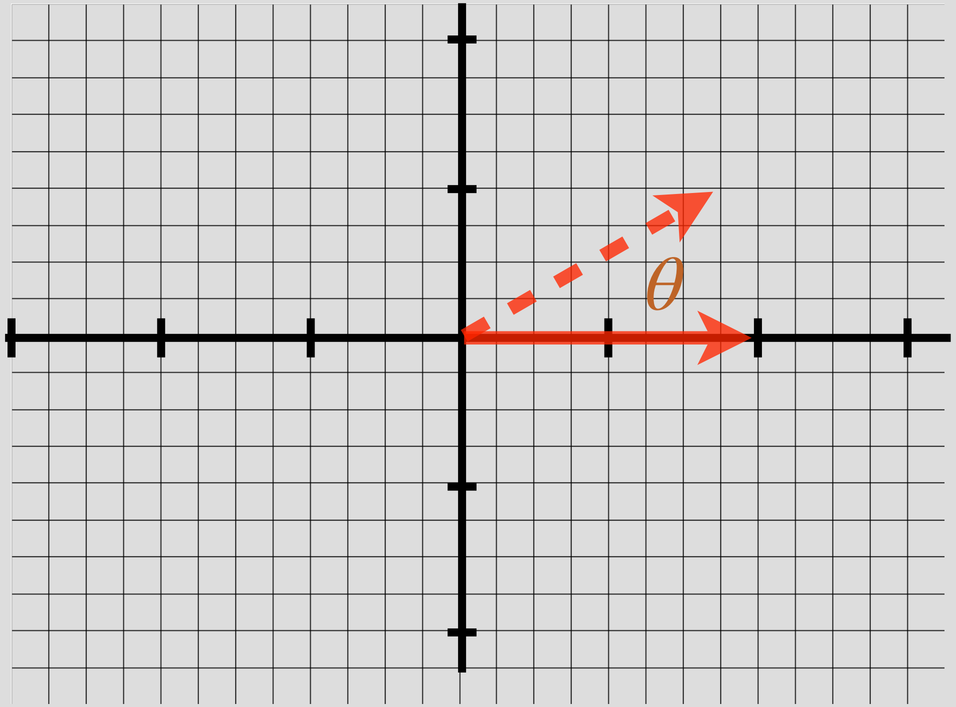
Example 2:

$$\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}\cos(\theta)x_1-\sin(\theta)x2\\ \sin(\theta)x_1+\cos(\theta)x_2\end{bmatrix}$$

$$\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}\cos(\theta)\\ \sin(\theta)\end{bmatrix}$$

Rotation Matrix!





Linear Transformation of vectors

f: is a linear transformation if:

$$f(\alpha \overrightarrow{x}) = \alpha f(\overrightarrow{x}) \qquad \alpha \in \mathbb{R}$$

$$f(\overrightarrow{x} + \overrightarrow{y}) = f(\overrightarrow{x}) + f(\overrightarrow{y})$$

Claim: Matrix-vector multiplications satisfy linear transformation

$$A \cdot (\alpha \overrightarrow{x}) = \alpha A \overrightarrow{x}$$

Proof via explicitly writing the elements

$$A \cdot (\overrightarrow{x} + \overrightarrow{y}) = A \overrightarrow{x} + A \overrightarrow{y}$$

Vectors as states, Matrices as state transition

Vectors can represent states of a system

Example: The state of a car at time = t

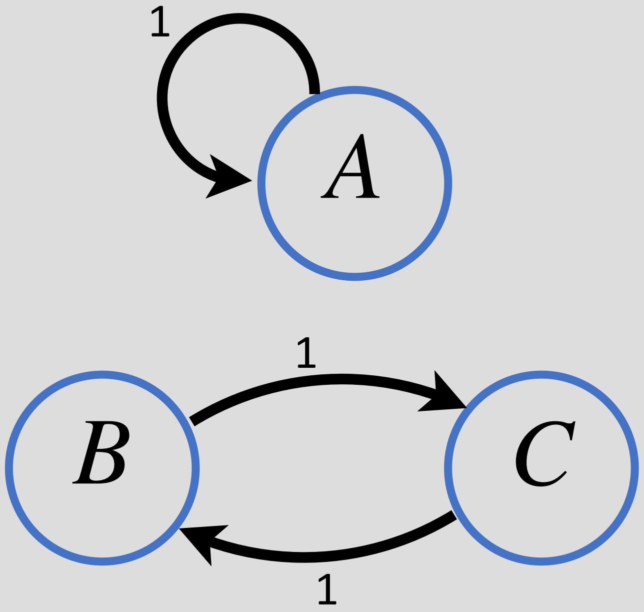
$$\vec{s}(t) = \begin{bmatrix} x(t) \\ y(t) \\ v(t) \\ \theta(t) \end{bmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} \text{position} \\ \} \text{ velocity} \\ \\ \end{array}$$

Q: Is that enough?

A: need orientation or $v_x(t), v_y(t)$

Graph Transition Matrices

Example: Reservoirs and Pumps



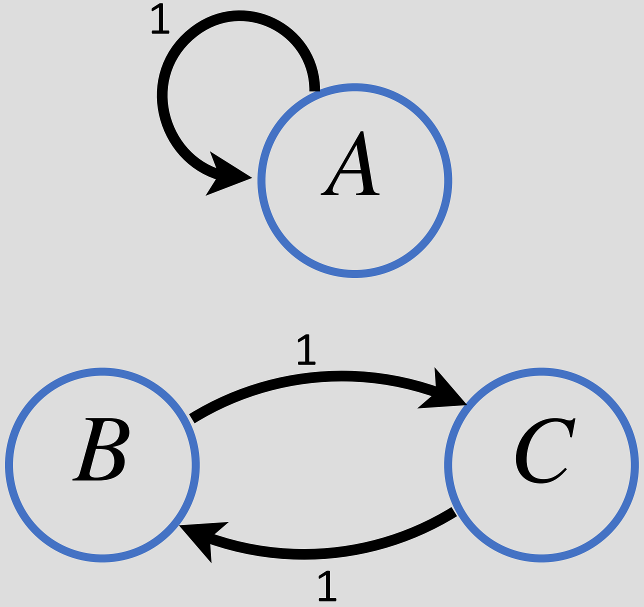
Q: What is the state?

A: Water in each reservoir

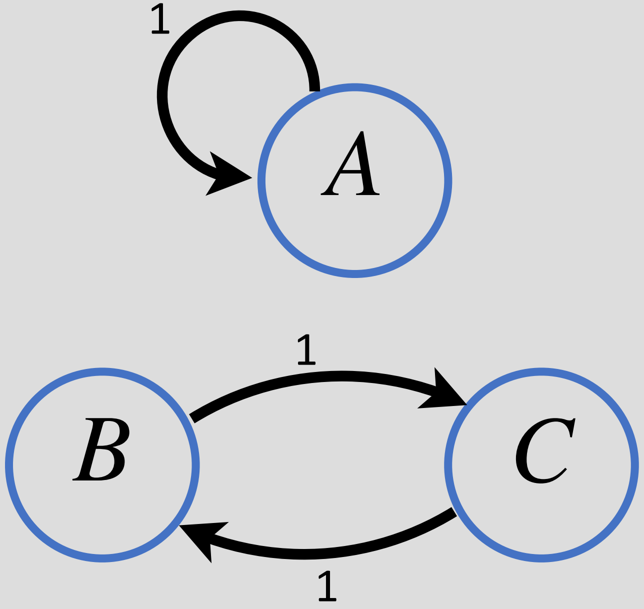
$$\vec{x}(t) = \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

Pumps move water…
What would the state be tomorrow?

State Transition Matrices



State Transition Matrices

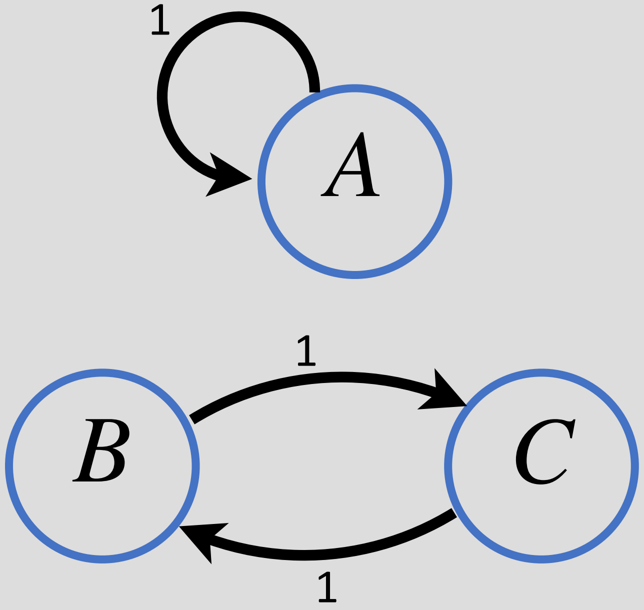


$$x_A(t+1) = x_A(t)$$
$$x_B(t+1) = x_C(t)$$
$$x_C(t+1) = x_B(t)$$

Write as a matrix-vector multiplication:

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

State Transition Matrices



$$x_A(t+1) = x_A(t)$$
$$x_B(t+1) = x_C(t)$$
$$x_C(t+1) = x_B(t)$$

Write as a matrix-vector multiplication:

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

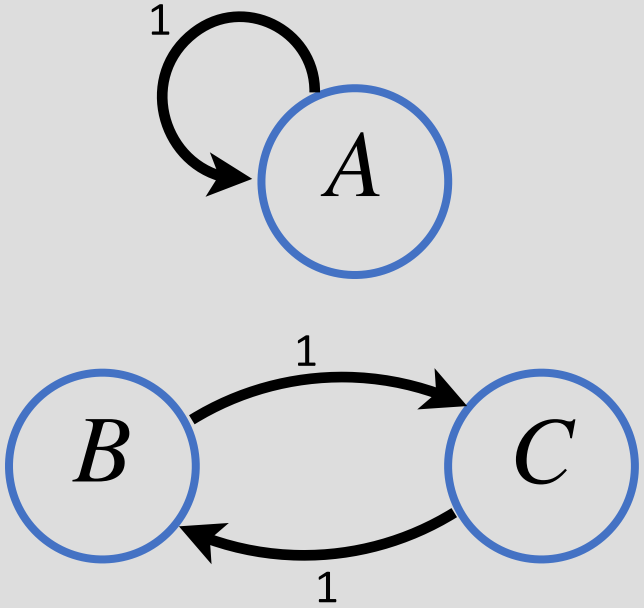
or $\overrightarrow{x}(t+1) = Q\overrightarrow{x}(t)$

What is the state after 2 times?

$$\overrightarrow{x}(t+2) = Q\overrightarrow{x}(t+1) = QQ\overrightarrow{x}(t) = Q^2\overrightarrow{x}(t)$$

State Transition Matrices

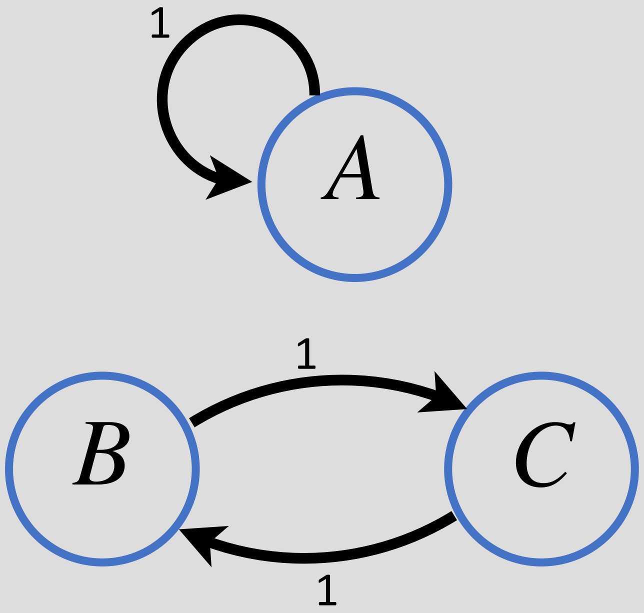
$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

What is the state after at t=1, 2?

State Transition Matrices



$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

What is the state after at t=1, 2?

①

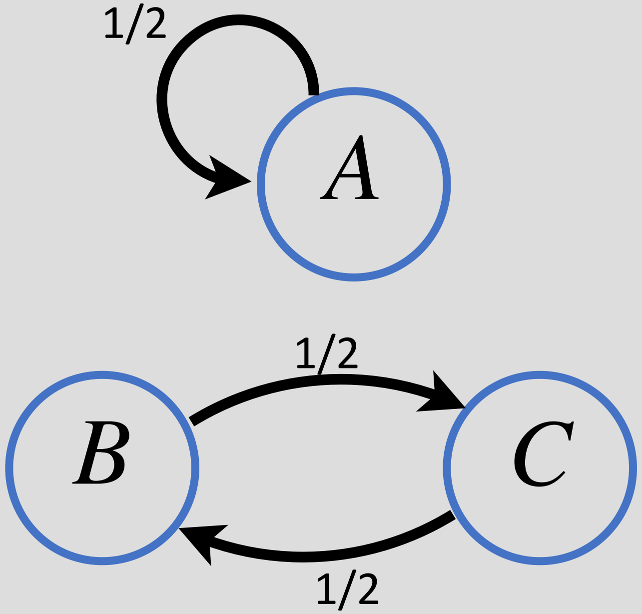
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

②

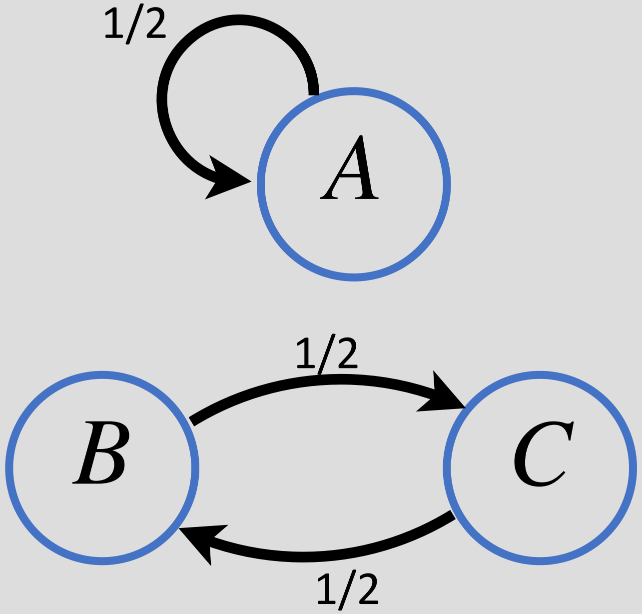
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\phi \cdot \phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

State Transition Matrices



State Transition Matrices



$$x[t+1] = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} x(t)$$

Non-conservative!

$$Q^2 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

Q) What will happen if we keep going?

A) Numbers will diminish to zero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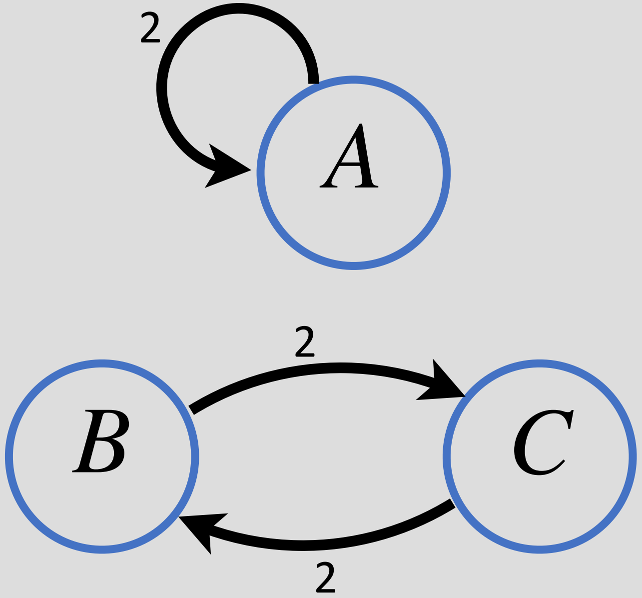
DEAD SEA SOUTH
1984

State Transition Matrices



$$x(t+1) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} x(t)$$

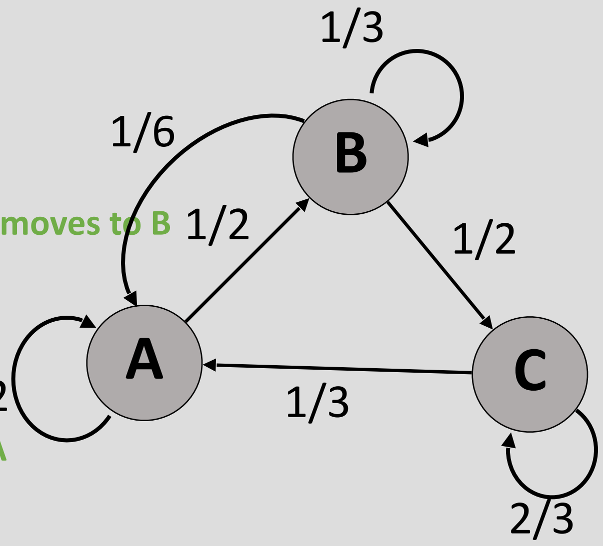
$$\Phi^2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Q) What will happen if we keep going?

A) Numbers will explode to infinity

Graph Representation

Ex: Reservoirs and Pumps



half the water in A moves to B

half the water in A stays in A

Nodes

I have 3 reservoirs: A,B,C
and I want to keep track of how much water is in each

When I turn on some pumps, water moves between the reservoirs.

Where the water moves and what fraction is represented by arrows.

Edge weights **Edges**

“directed” graph because arrows have a direction

Where does the rest of the water in A go?

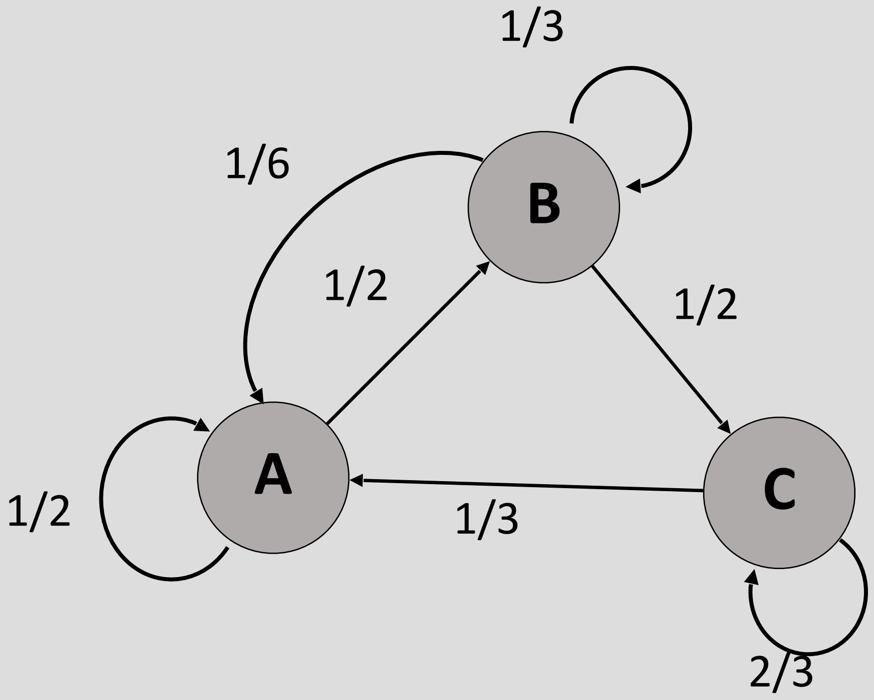
Need to label that too…

Can you tell me how much water in each after pumps start?

Need to know initial amounts

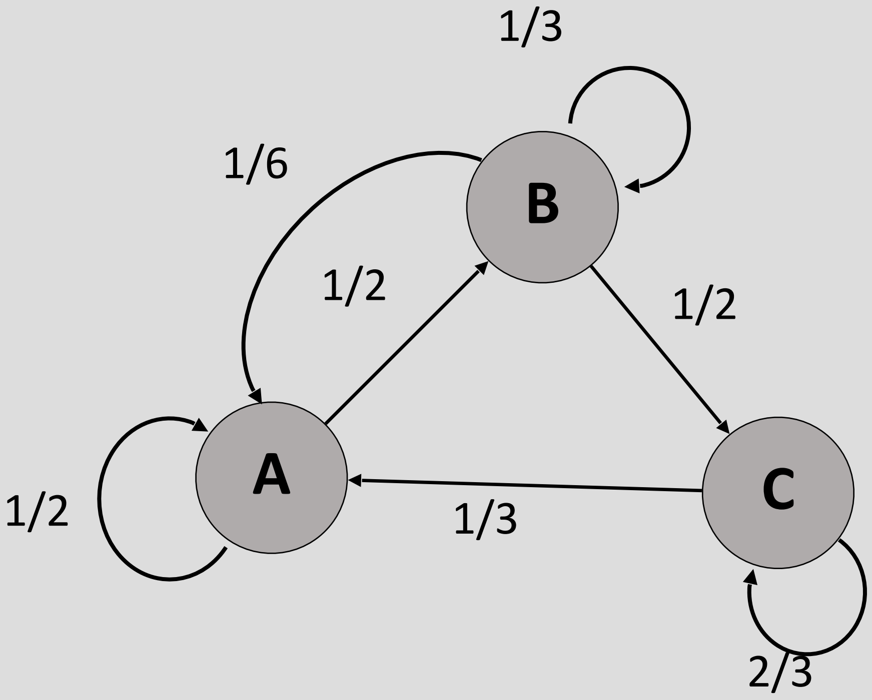
Exercise:

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} A \rightarrow A & B \rightarrow A & C \rightarrow A \\ A \rightarrow B & B \rightarrow B & C \rightarrow B \\ A \rightarrow C & B \rightarrow C & C \rightarrow C \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



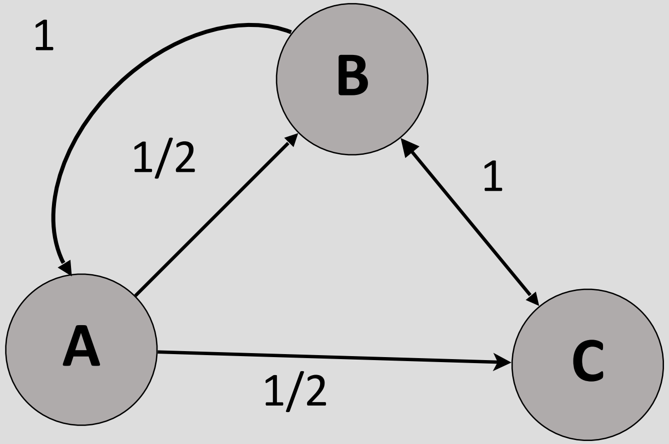
Exercise:

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\; A \to A & \frac{1}{6}\; B \to A & \frac{1}{3}\; C \to A \\ \frac{1}{2}\; A \to B & \frac{1}{3}\; B \to B & 0\; C \to B \\ 0\; A \to C & \frac{1}{2}\; B \to C & \frac{2}{3}\; C \to C \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



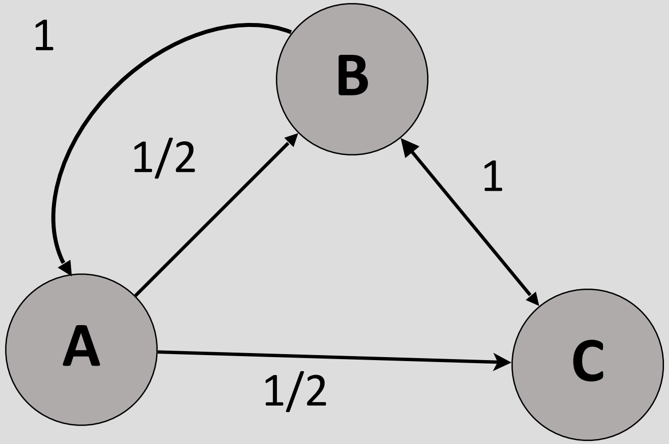
Example 2:

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} A \to A & B \to A & C \to A \\ A \to B & B \to B & C \to B \\ A \to C & B \to C & C \to C \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



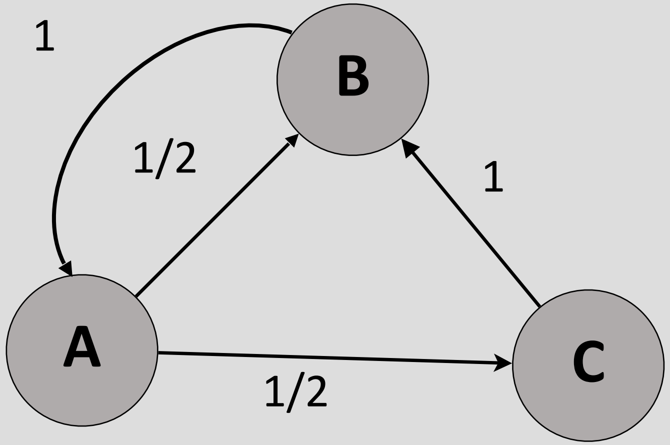
Example 2:

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} \overset{A \to A}{0} & \overset{B \to A}{1} & \overset{C \to A}{0} \\ \overset{A \to B}{\frac{1}{2}} & \overset{B \to B}{0} & \overset{C \to B}{1} \\ \overset{A \to C}{\frac{1}{2}} & \overset{B \to C}{0} & \overset{C \to C}{0} \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



What about the reverse?

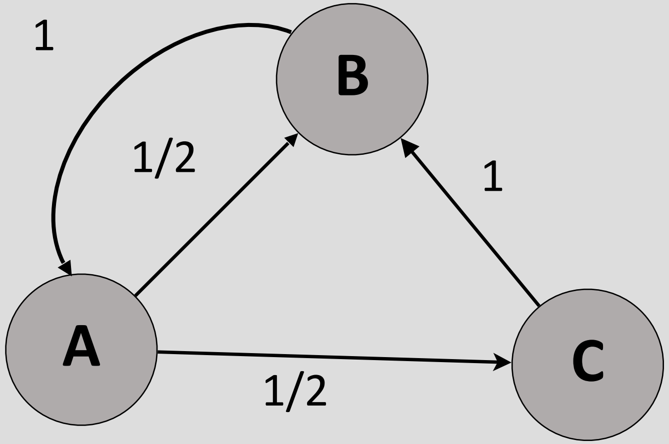
$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} \underset{A \to A}{0} & \underset{B \to A}{1} & \underset{C \to A}{0} \\ \underset{A \to B}{\frac{1}{2}} & \underset{B \to B}{0} & \underset{C \to B}{1} \\ \underset{A \to C}{\frac{1}{2}} & \underset{B \to C}{0} & \underset{C \to C}{0} \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



Q) Will flipping the arrows make us go back in time?

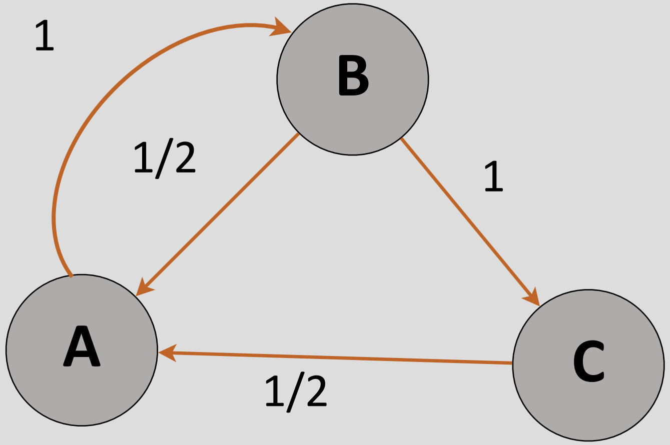
What about the reverse?

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} \underset{A\to A}{0} & \underset{B\to A}{1} & \underset{C\to A}{0} \\ \underset{A\to B}{\frac{1}{2}} & \underset{B\to B}{0} & \underset{C\to B}{1} \\ \underset{A\to C}{\frac{1}{2}} & \underset{B\to C}{0} & \underset{C\to C}{0} \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



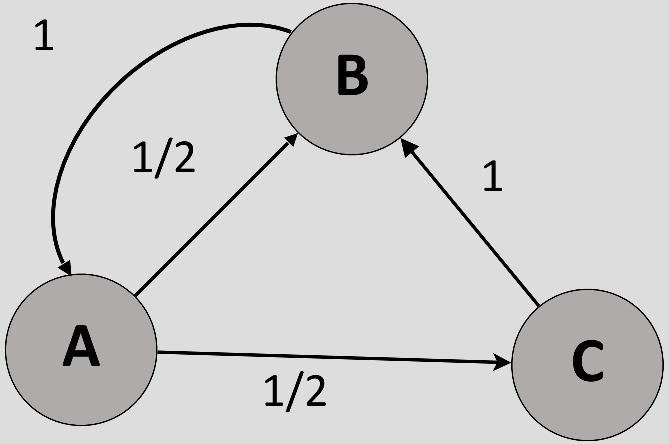
Q) Will flipping the arrows make us go back in time?

$$\begin{bmatrix} \\ \\ \\ \end{bmatrix}$$



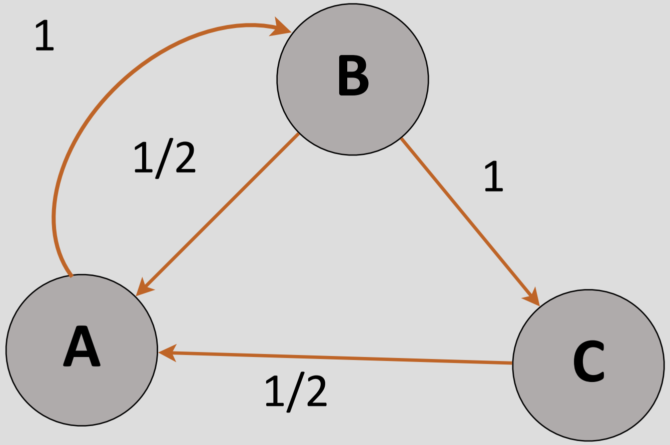
What about the reverse?

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} \overset{A \to A}{0} & \overset{B \to A}{1} & \overset{C \to A}{0} \\ \overset{A \to B}{\frac{1}{2}} & \overset{B \to B}{0} & \overset{C \to B}{1} \\ \overset{A \to C}{\frac{1}{2}} & \overset{B \to C}{0} & \overset{C \to C}{0} \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



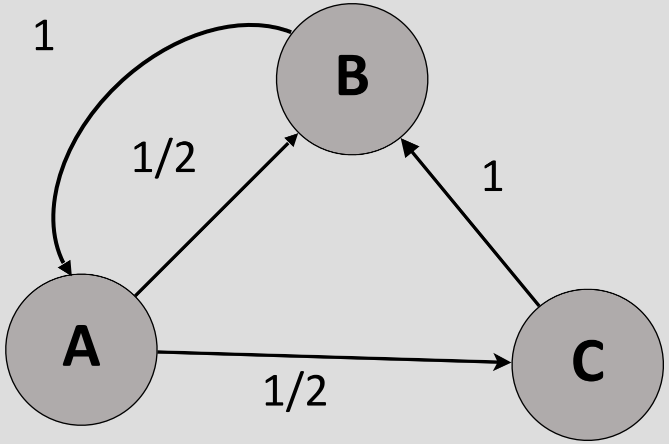
Q) Will flipping the arrows make us go back in time?

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$



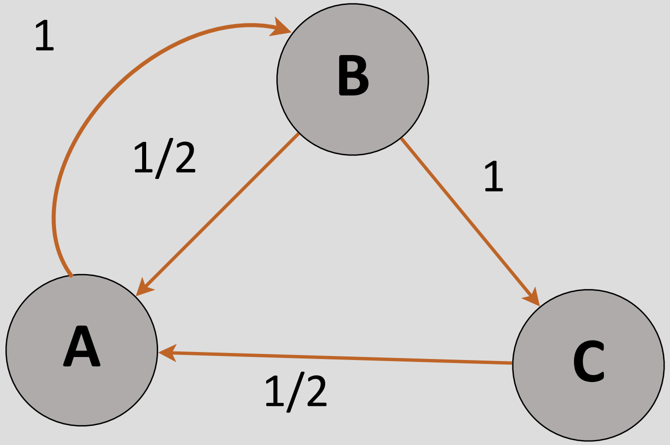
What about the reverse?

$$\begin{array}{c} 6 \\ 10 \\ 2 \end{array} \begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} \underset{A\to A}{0} & \underset{B\to A}{1} & \underset{C\to A}{0} \\ \underset{A\to B}{\frac{1}{2}} & \underset{B\to B}{0} & \underset{C\to B}{1} \\ \underset{A\to C}{\frac{1}{2}} & \underset{B\to C}{0} & \underset{C\to C}{0} \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix} \begin{array}{c} 4 \\ 6 \\ 8 \end{array}$$



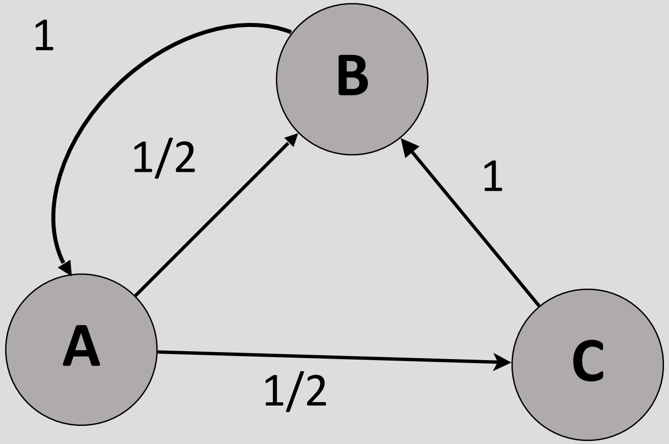
Q) Will flipping the arrows make us go back in time?

$$\begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$



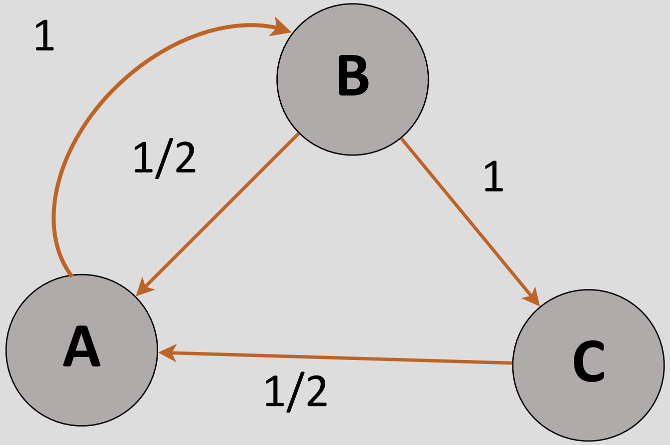
What about the reverse?

$$\begin{array}{c} 6 \\ 10 \\ 2 \end{array} \begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} \underset{A\to A}{0} & \underset{B\to A}{1} & \underset{C\to A}{0} \\ \underset{A\to B}{\frac{1}{2}} & \underset{B\to B}{0} & \underset{C\to B}{1} \\ \underset{A\to C}{\frac{1}{2}} & \underset{B\to C}{0} & \underset{C\to C}{0} \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix} \begin{array}{c} 4 \\ 6 \\ 8 \end{array}$$



Q) Will flipping the arrows make us go back in time?

$$\begin{bmatrix} 7 \\ 6 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 10 \\ 2 \end{bmatrix}$$



A) In general, no!

Matrix Transpose

If the elements of the matrix $A \in \mathbb{R}^{N\times M}$ are a_{ij}

The elements of $A^T \in \mathbb{R}^{M\times N}$ are a_{ji}

Matrix transpose is not (generally) an inverse!

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_m \\ & & & \end{bmatrix}$$

$A \in \mathbb{R}^{N\times M}$

$$\begin{bmatrix} \vec{a}_1^{\,T} \\ \vec{a}_2^{\,T} \\ \vdots \\ a_m^T \end{bmatrix}$$

$A^T \in \mathbb{R}^{M\times N}$

Matrix Transpose

If the elements of the matrix $A \in \mathbb{R}^{N\times M}$ are a_{ij}

The elements of $A^T \in \mathbb{R}^{M\times N}$ are a_{ji}

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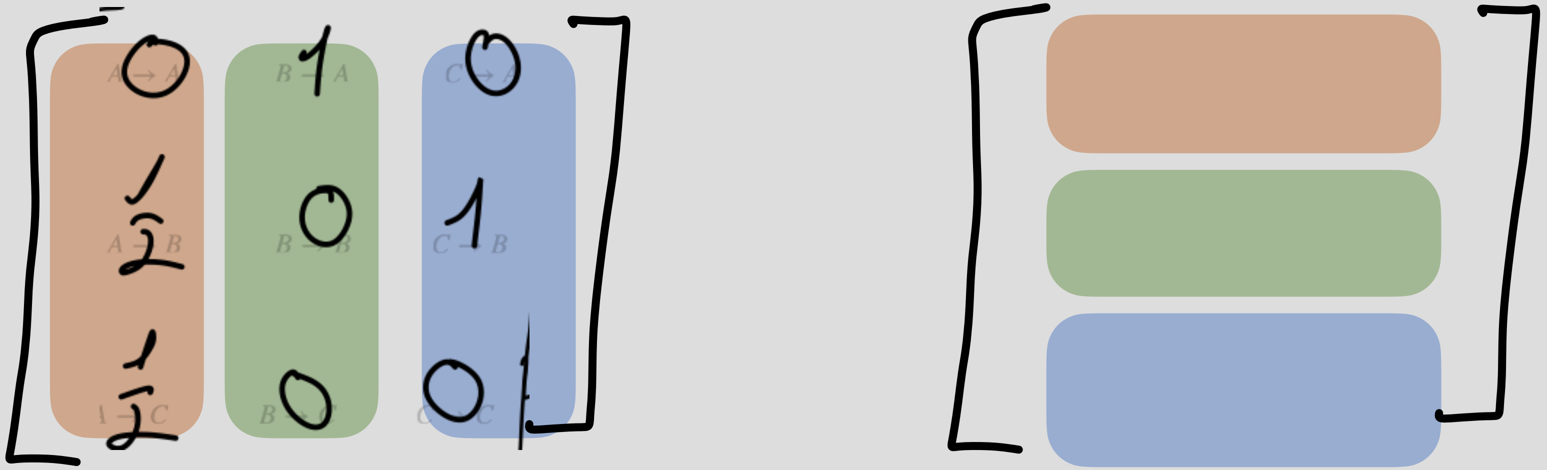
$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_m \end{bmatrix} \qquad \begin{bmatrix} \vec{a}_1^{\,T} \\ \vec{a}_2^{\,T} \\ \vdots \\ a_m^T \end{bmatrix}$$

$A \in \mathbb{R}^{N\times M}$

$A^T \in \mathbb{R}^{M\times N}$



$$\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$$

Matrix Transpose

If the elements of the matrix $A \in \mathbb{R}^{N\times M}$ are a_{ij}

The elements of $A^T \in \mathbb{R}^{M\times N}$ are a_{ji}

Matrix transpose is not (generally) an inverse!

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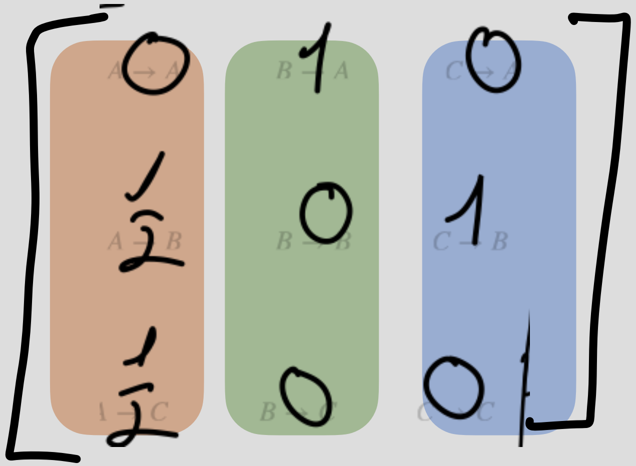
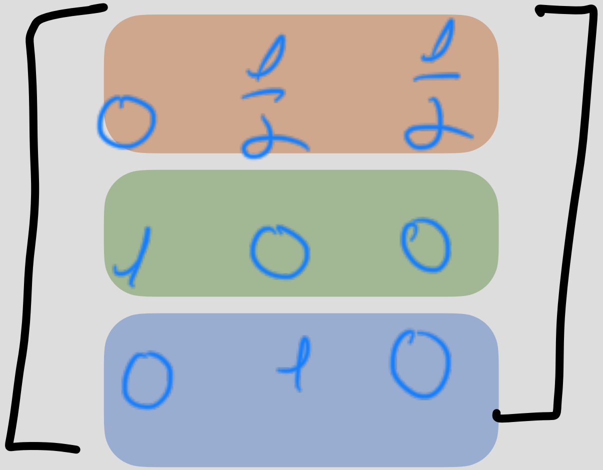
$A \in \mathbb{R}^{N\times M}$

$A^T \in \mathbb{R}^{M\times N}$

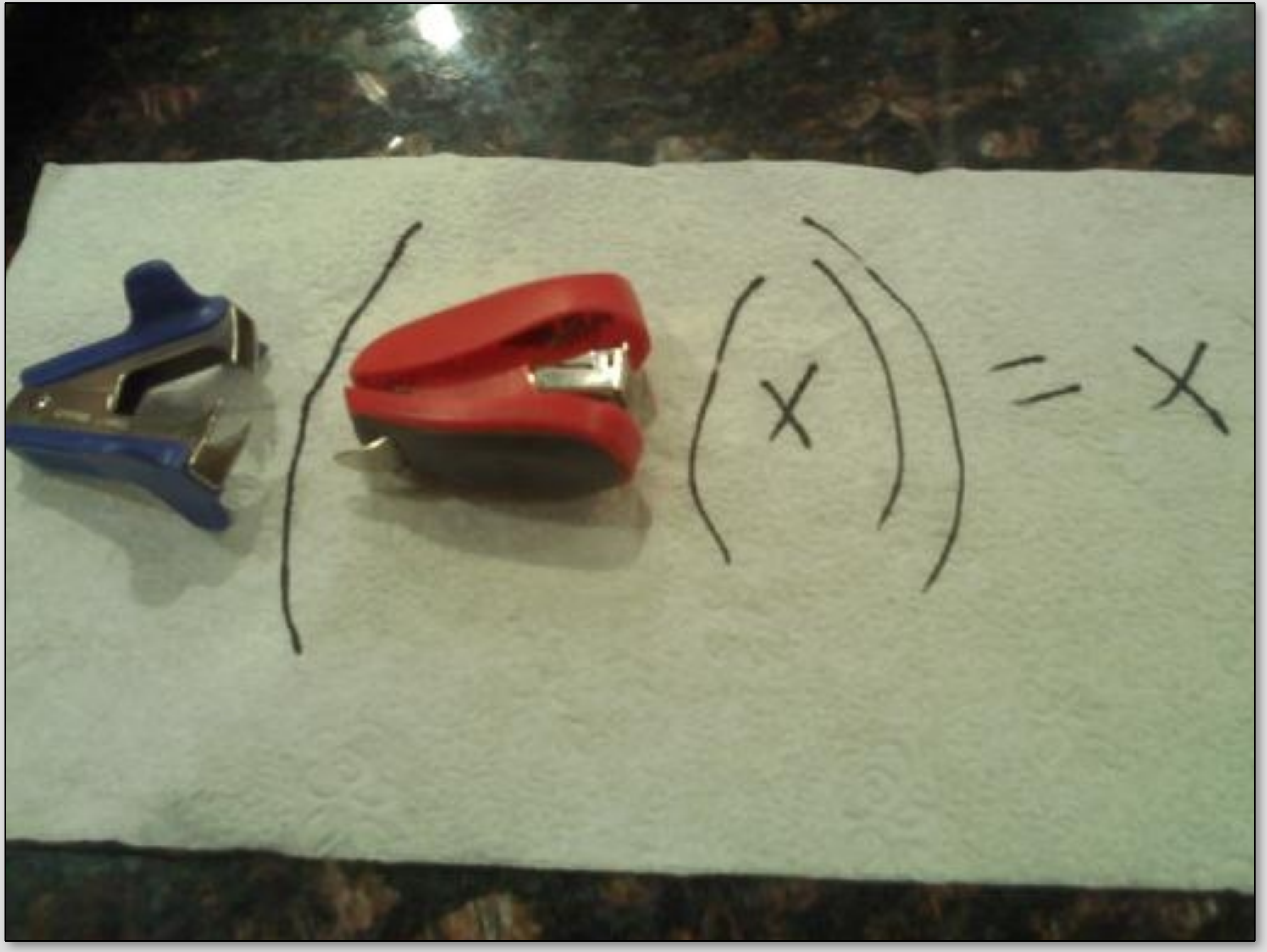


$$\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$$



$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Matrix Inversion



Matrix Inverse

$$\overrightarrow{x}(t+1) = Q\overrightarrow{x}(t)$$

Is there a square matrix P such that we can go back in time?

$$\overrightarrow{x}(t) = P\overrightarrow{x}(t+1)$$

Yes, if : $PQ = I$

As consequence : $QP = I$

$$P\overrightarrow{x}(t+1) = PQ\overrightarrow{x}(t)$$
$$P\overrightarrow{x}(t+1) = I\overrightarrow{x}(t)$$

$$\overrightarrow{x}(t+1) = Q\overrightarrow{x}(t)$$
$$\overrightarrow{x}(t+1) = QP\overrightarrow{x}(t+1)$$
$$\overrightarrow{x}(t+1) = I\overrightarrow{x}(t+1)$$

Matrix Inverse - Formal definition

- Definition: Let $P, Q \in \mathbb{R}^{N \times N}$ be square matrices.
  - P is the inverse of Q if $PQ = QP = I$

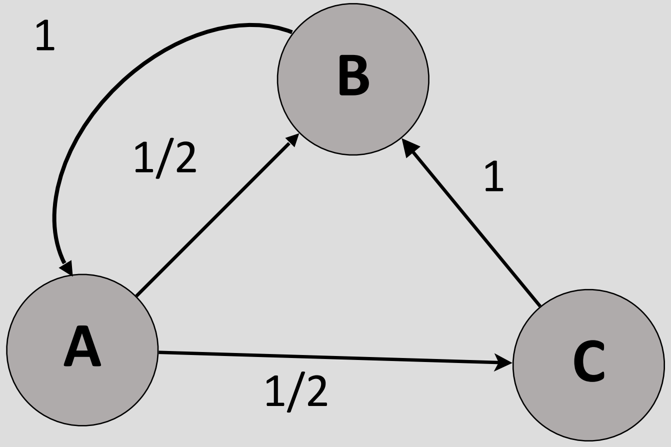
We say that $P = Q^{-1}$ and $Q = P^{-1}$

Q: What about non-square matrices?

A: EECS16B!

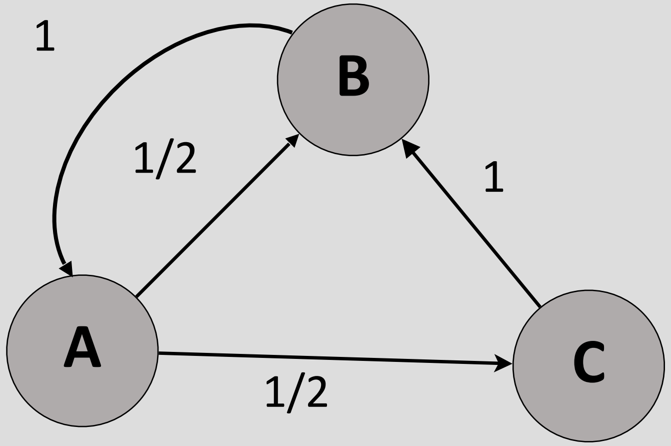
Computing the Matrix Inverse

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \overset{Q}{\begin{bmatrix} \underset{A\to A}{0} & \underset{B\to A}{1} & \underset{C\to A}{0} \\ \underset{A\to B}{\frac{1}{2}} & \underset{B\to B}{0} & \underset{C\to B}{1} \\ \underset{A\to C}{\frac{1}{2}} & \underset{B\to C}{0} & \underset{C\to C}{0} \end{bmatrix}} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



- Want $P = Q^{-1}$ such that $\overrightarrow{x}(t) = P\overrightarrow{x}(t+1)$
  - Need: $QP = I$

Computing the Matrix Inverse

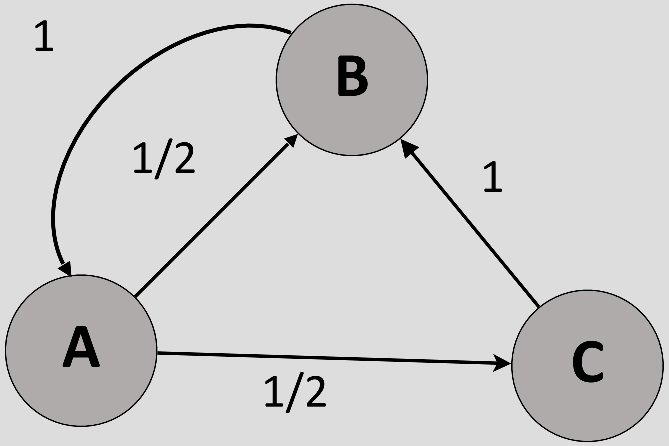


Need: $QP = I$

Pose as a linear set of equations.

Solve with Gaussian Elimination

Computing the Matrix Inverse



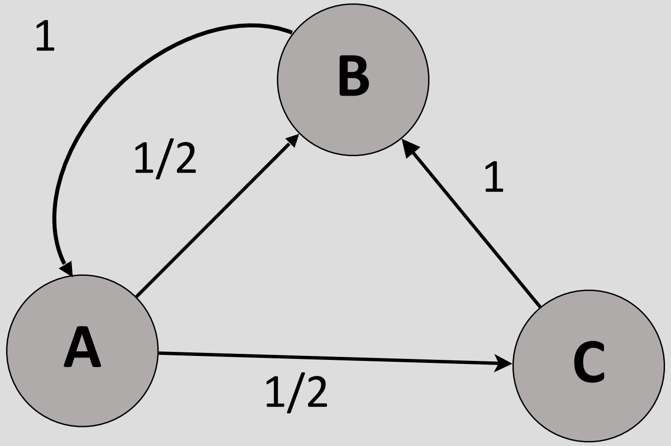
Need: $QP = I$

Pose as a linear set of equations.

Solve with Gaussian Elimination

$$\overset{Q}{\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}} \underset{\vec{p}_1}{\overset{P}{\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}}} = \underset{\vec{b}_1}{\overset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}}$$

Computing the Matrix Inverse



Need: $QP = I$

Pose as a linear set of equations.

Solve with Gaussian Elimination

$$\begin{array}{c} Q \\ \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \\ \end{array} \begin{array}{c} P \\ \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \\ \vec{p}_1 \quad \vec{p}_2 \quad \vec{p}_3 \end{array} = \begin{array}{c} I \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \vec{b}_1 \quad \vec{b}_2 \quad \vec{b}_3 \end{array}$$

Matrix Inverse via Gaussian Elimination

$$\begin{array}{cc} Q & I \\ \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$\left[\begin{array}{ccc|ccc} \frac{1}{2} & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

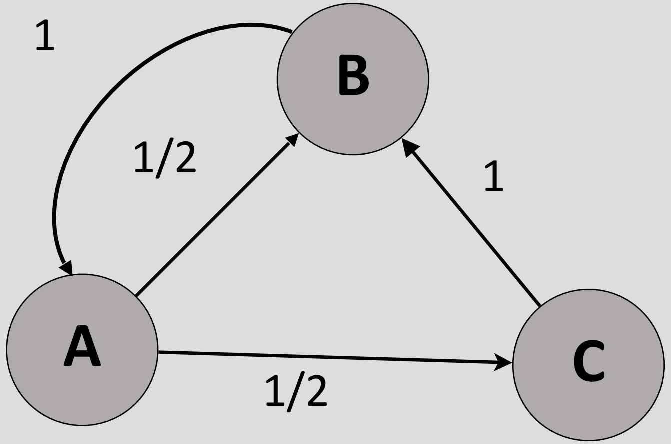
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & -2 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

$$\begin{array}{cc} I & P \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right] \end{array}$$

Let's check

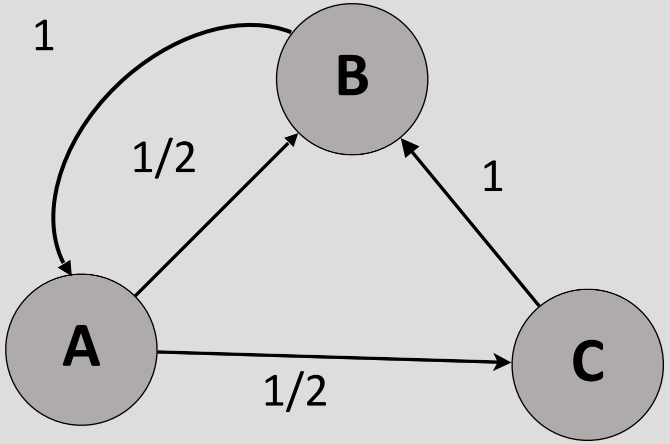
$$\begin{array}{c} 6 \\ 10 \\ 2 \end{array} \begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} \underset{A\to A}{0} & \underset{B\to A}{1} & \underset{C\to A}{0} \\ \underset{A\to B}{\frac{1}{2}} & \underset{B\to B}{0} & \underset{C\to B}{1} \\ \underset{A\to C}{\frac{1}{2}} & \underset{B\to C}{0} & \underset{C\to C}{0} \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix} \begin{array}{c} 4 \\ 6 \\ 8 \end{array}$$



$$\begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 10 \\ 2 \end{bmatrix}$$

Let’s check

$$\begin{array}{c} 6 \\ 10 \\ 2 \end{array}\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} \underset{A \to A}{0} & \underset{B \to A}{1} & \underset{C \to A}{0} \\ \underset{A \to B}{\frac{1}{2}} & \underset{B \to B}{0} & \underset{C \to B}{1} \\ \underset{A \to C}{\frac{1}{2}} & \underset{B \to C}{0} & \underset{C \to C}{0} \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}\begin{array}{c} 4 \\ 6 \\ 8 \end{array}$$



$$\begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 10 \\ 2 \end{bmatrix}$$

And now we can take any number of steps backwards!

Can we always invert a function?

- Can we always invert a function .....$f^{-1}\left(f(\overrightarrow{x})\right) = \overrightarrow{x}$?
    - $f(x) = x^2$?
    - $f(x) = ax$?
    - $f(x) = Ax$?

Invertibility of Linear Transformations

- Theorem: A is invertible, if and only if (iff) the columns of A are linearly independent.

  1. If columns of A are lin. dep. then A^{-1} does not exist
  2. If A^{-1} exists, then the cols. of A are linearly independent

Proof concept: Assume linear dependence and invertibility and show that it is a contradiction

From linear independence: $\exists \overrightarrow{\alpha} \neq 0$ such that $A\overrightarrow{\alpha} = 0$

Assume A^{-1} exists

$$A\overrightarrow{\alpha} = 0$$

$$A^{-1}A\overrightarrow{\alpha} = A^{-1}0$$

$$I\overrightarrow{\alpha} = 0$$

But $\overrightarrow{\alpha} \neq 0$! Hence A^{-1} does not exist

Inverse of a 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

1. Flip a and d
2. Negate b and c
3. Divide by $ad - bc$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Derive via Gauss Elimination!