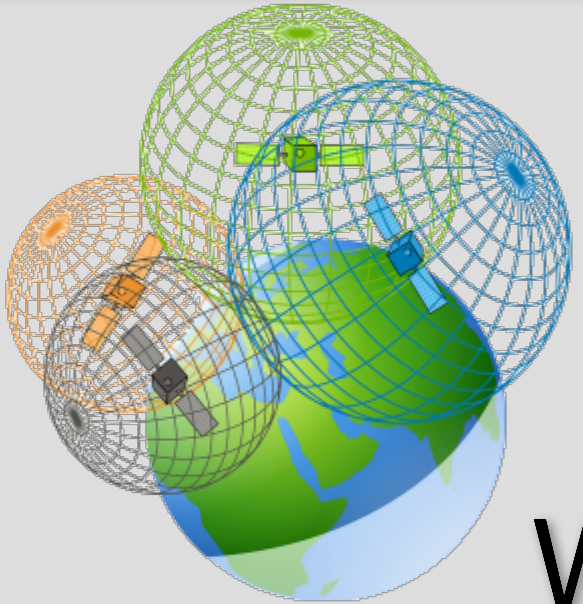
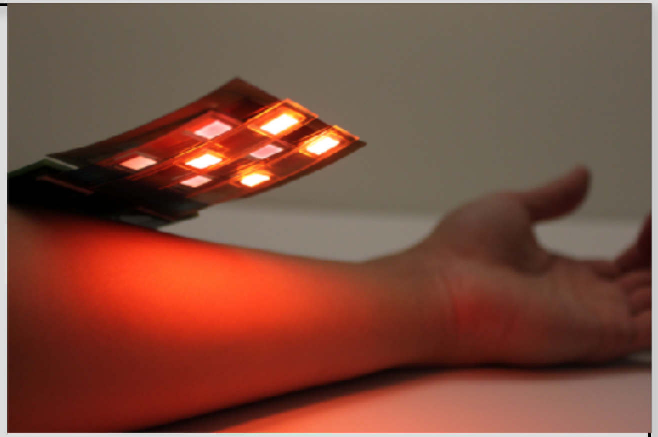




Welcome to EECS 16A!

Designing Information Devices and Systems I

Ana Arias and Miki Lustig

2022

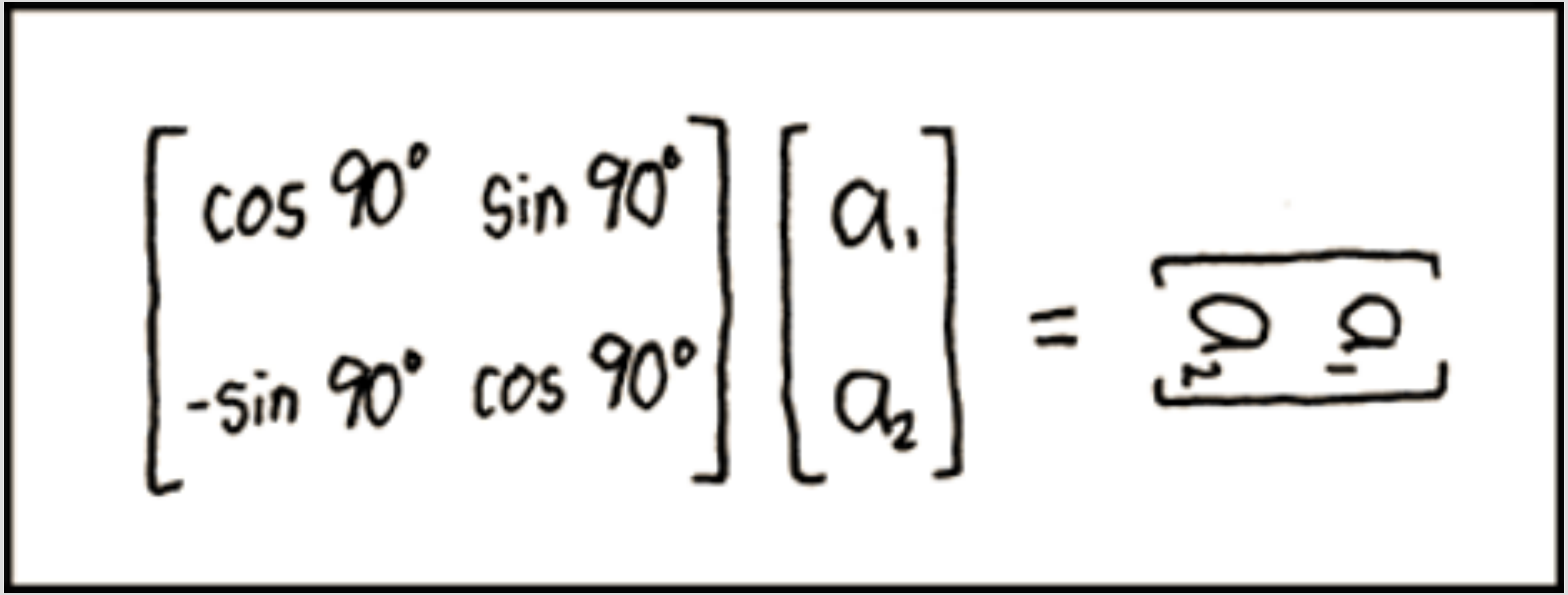
Lecture 3B

Matrix Inverse

Announcements

- Last time:
  - Linear (in)dependance
  - Matrix Transformations
- Today:
  - Continue with Matrix transformations
  - Matrix Inverse
  - Vector spaces

Matrix Transformations



Linear Transformation of vectors

f: is a linear transformation if:

$$f(\alpha \vec{x}) = \alpha f(\vec{x}) \qquad \alpha \in \mathbb{R}$$

$$f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y})$$

Claim: Matrix-vector multiplications satisfy linear transformation

$$A \cdot (\alpha \vec{x}) = \alpha A \vec{x}$$

Proof via explicitly writing the elements

$$A \cdot (\vec{x} + \vec{y}) = A \vec{x} + A \vec{y}$$

Vectors as states, Matrices as state transition

Vectors can represent states of a system

Example: The state of a car at time = t

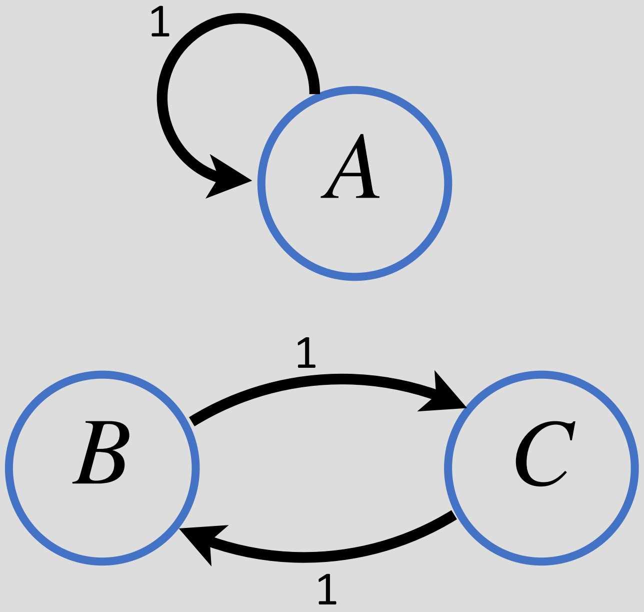
$$\vec{s}(t) = \begin{bmatrix} x(t) \\ y(t) \\ v(t) \\ \theta(t) \end{bmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix} x \\ y \end{matrix}}\right\} \text{position} \\ \left.\vphantom{v}\right\} \text{velocity} \\ \\ \end{array}$$

Q: Is that enough?

A: need orientation or $v_x(t), v_y(t)$

Graph Transition Matrices

Example: Reservoirs and Pumps



Q: What is the state?

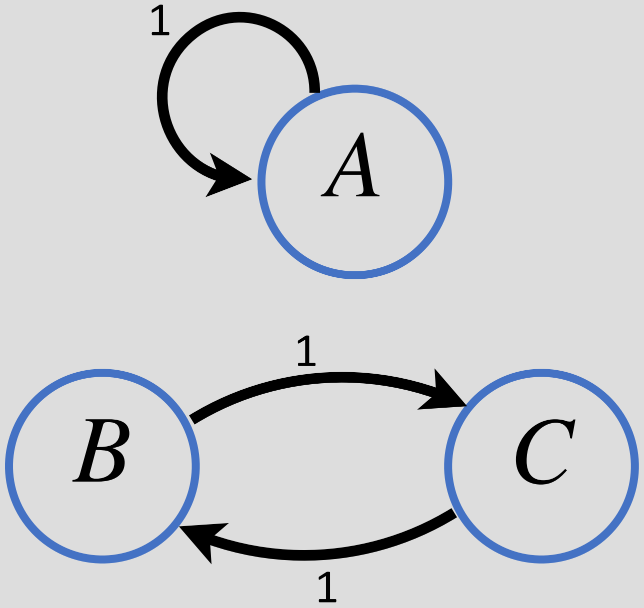
A: Water in each reservoir

$$\vec{x}(t) = \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

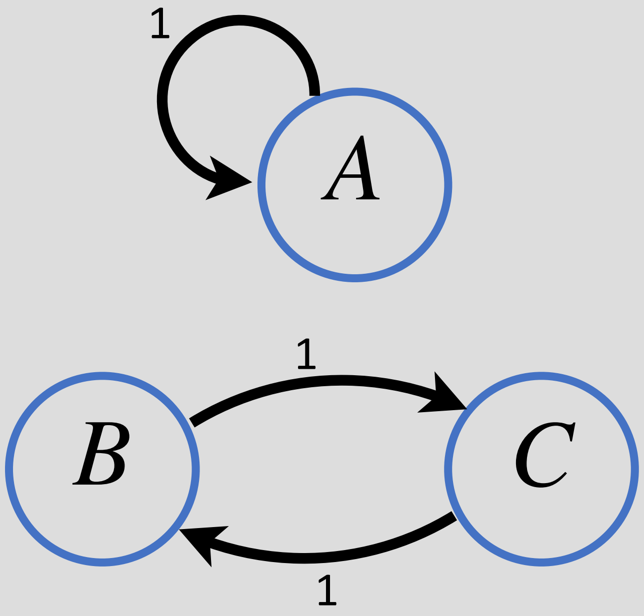
Pumps move water…

What would the state be tomorrow?

State Transition Matrices



State Transition Matrices

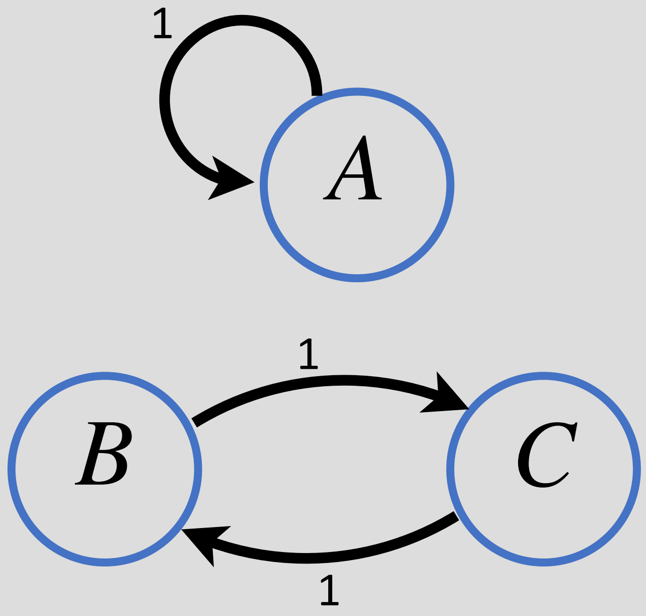


$$x_A(t+1) = x_A(t)$$
$$x_B(t+1) = x_C(t)$$
$$x_C(t+1) = x_B(t)$$

Write as a matrix-vector multiplication:

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

State Transition Matrices



$$x_A(t+1) = x_A(t)$$
$$x_B(t+1) = x_C(t)$$
$$x_C(t+1) = x_B(t)$$

Write as a matrix-vector multiplication:

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

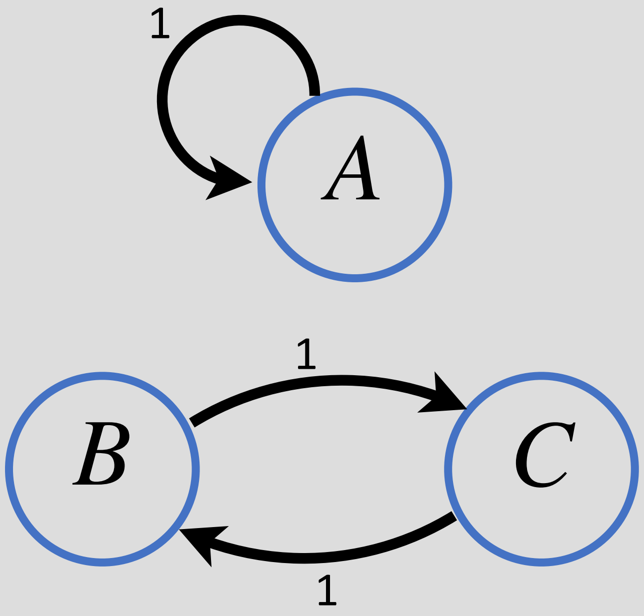
or $\overrightarrow{x}(t+1) = Q\overrightarrow{x}(t)$

What is the state after 2 times?

$$\overrightarrow{x}(t+2) = Q\overrightarrow{x}(t+1) = QQ\overrightarrow{x}(t) = Q^2\overrightarrow{x}(t)$$

State Transition Matrices

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

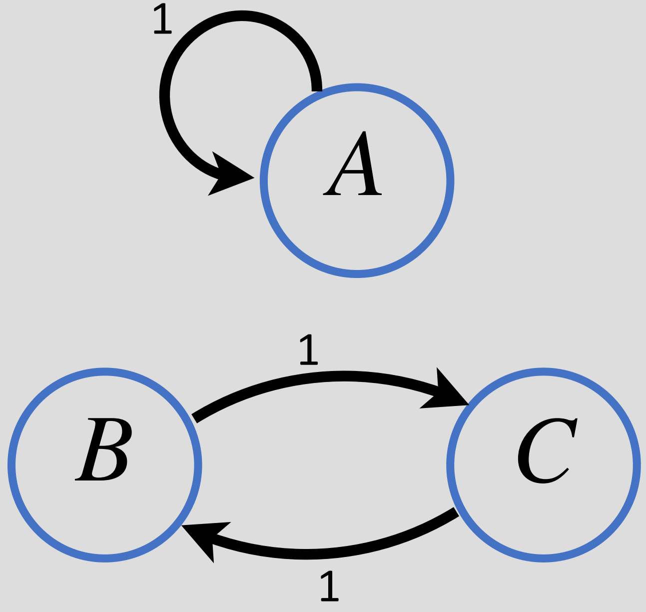


$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

What is the state after at t=1, 2?

State Transition Matrices

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



$$\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

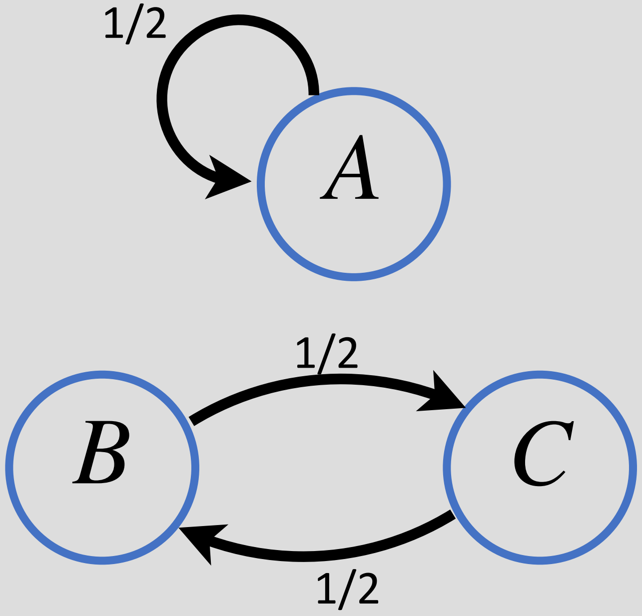
What is the state after at t=1, 2?

① $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

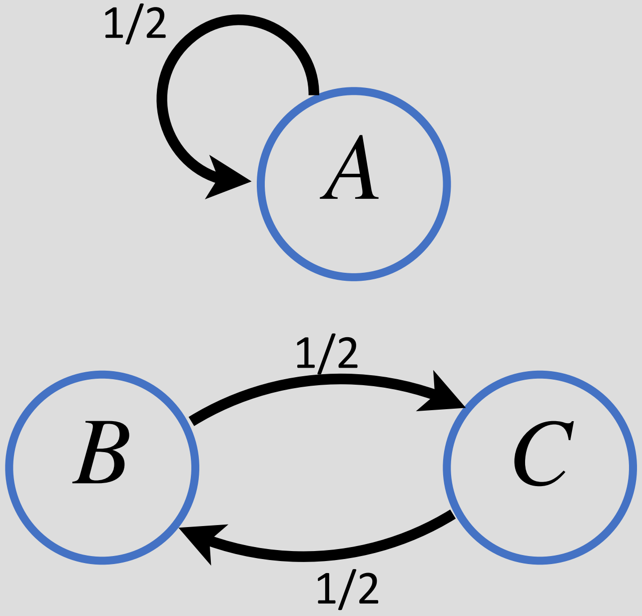
② $$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\Phi \cdot \Phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

State Transition Matrices



State Transition Matrices



$$x[t+1] = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} x(t)$$

Non-conservative!

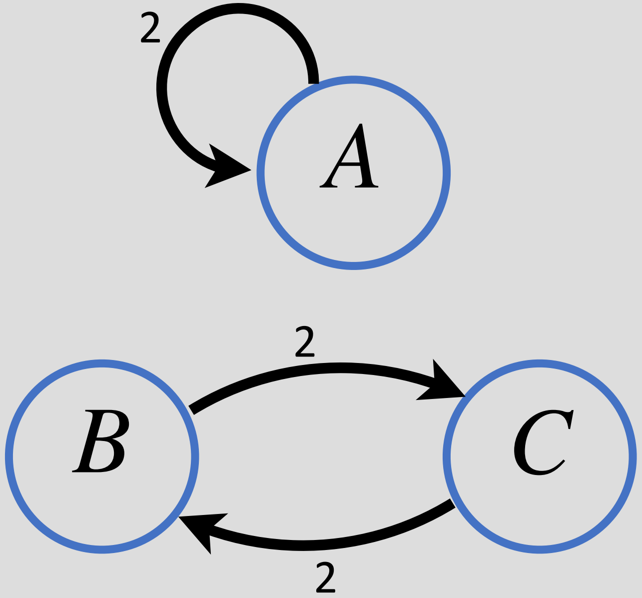
$$Q^2 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$$

Q) What will happen if we keep going?

A) Numbers will diminish to zero


Google
DEAD SEA SOUTH
1984

State Transition Matrices



$$x(t+1) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} x(t)$$

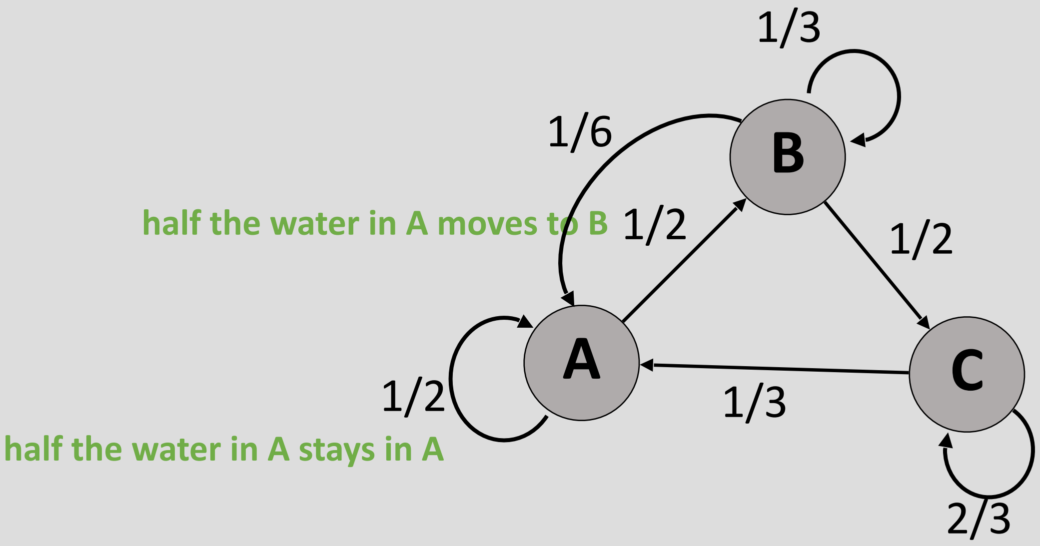
$$\Phi^2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

Q) What will happen if we keep going?

A) Numbers will explode to infinity

Graph Representation

Ex: Reservoirs and Pumps



Nodes

I have 3 reservoirs: A,B,C
and I want to keep track of how much water is in each

When I turn on some pumps, water moves between the reservoirs.

Where the water moves and what fraction is represented by arrows.

Edge weights **Edges**

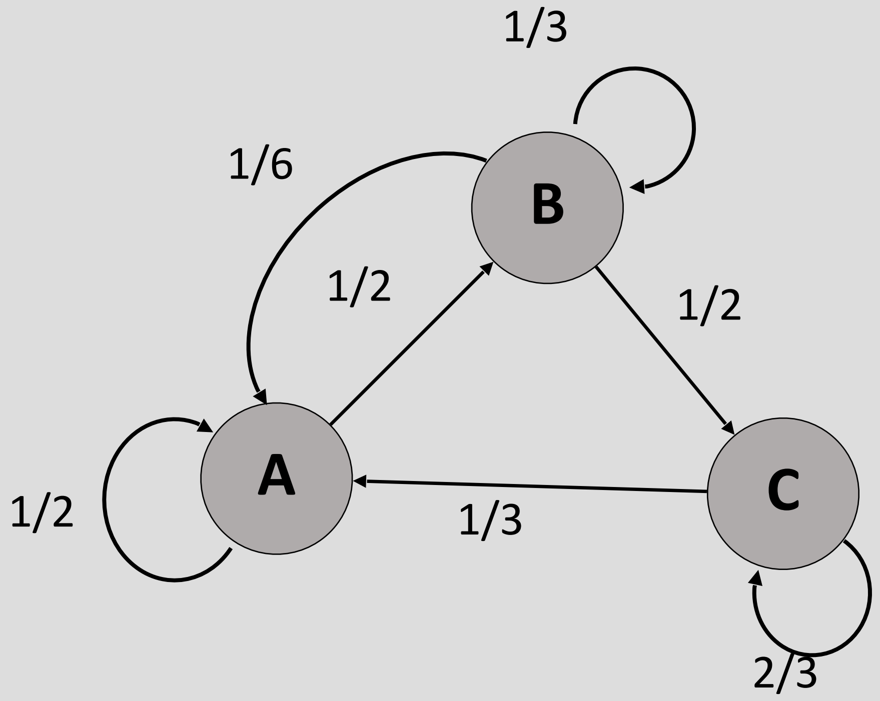
"directed" graph because arrows have a direction

Where does the rest of the water in A go? Need to label that too…

Can you tell me how much water in each after pumps start? Need to know initial amounts

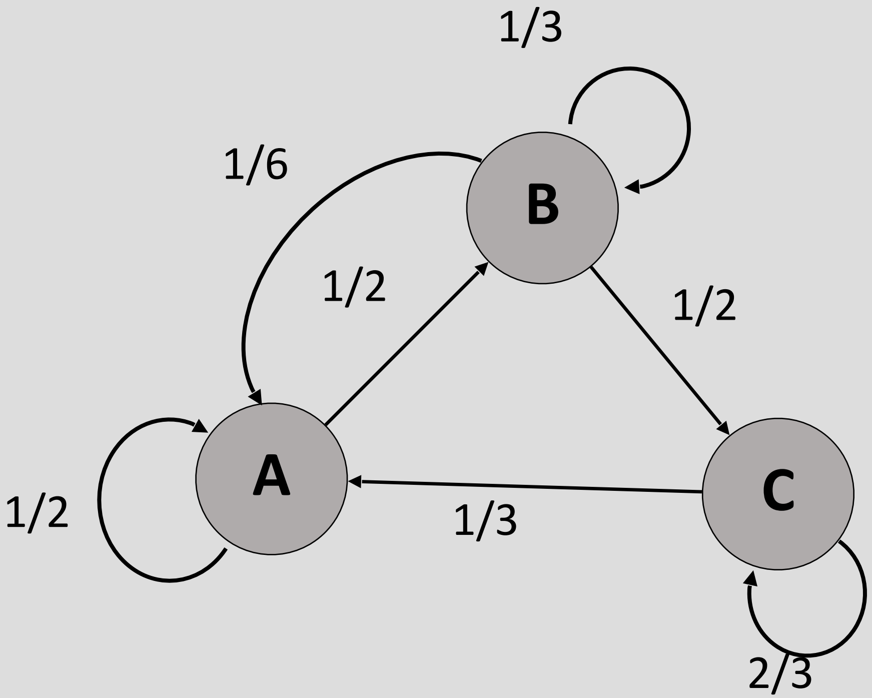
Exercise:

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} A \to A & B \to A & C \to A \\ A \to B & B \to B & C \to B \\ A \to C & B \to C & C \to C \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



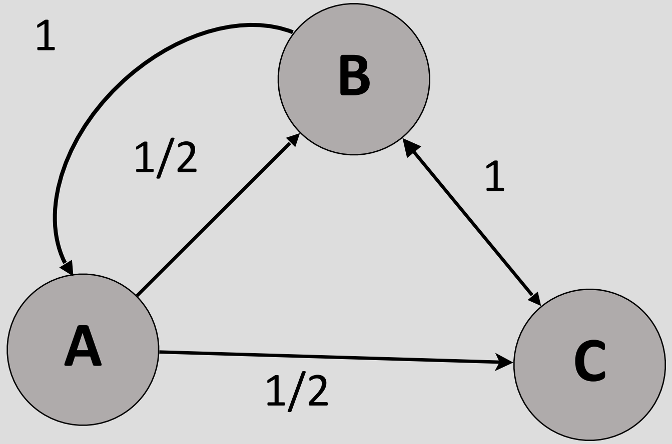
Exercise:

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & 0 \\ 0 & \frac{1}{2} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



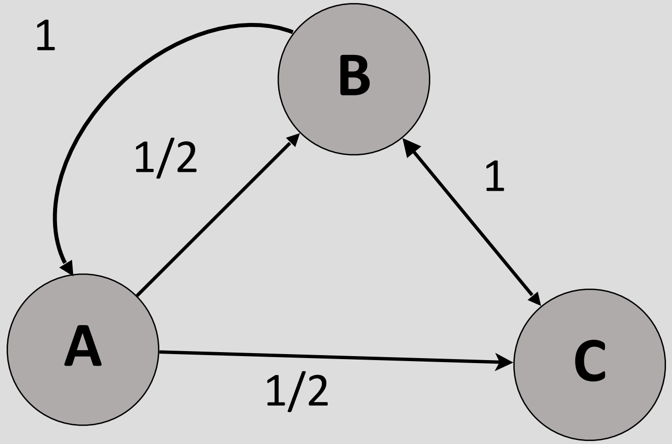
Example 2:

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} A \to A & B \to A & C \to A \\ A \to B & B \to B & C \to B \\ A \to C & B \to C & C \to C \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



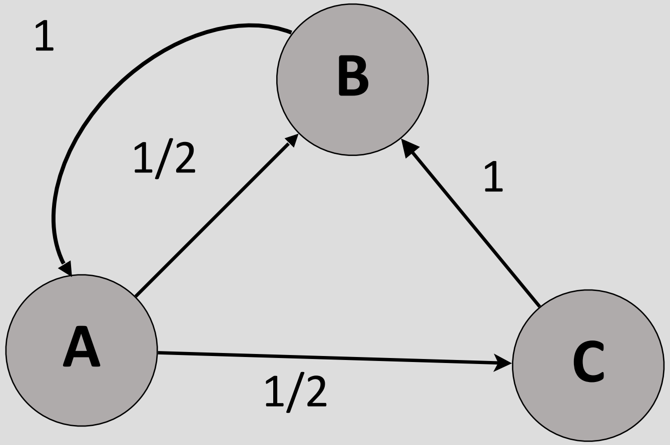
Example 2:

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



What about the reverse?

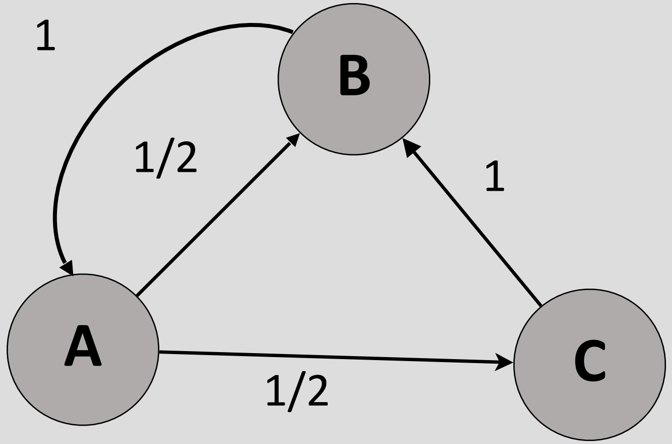
$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



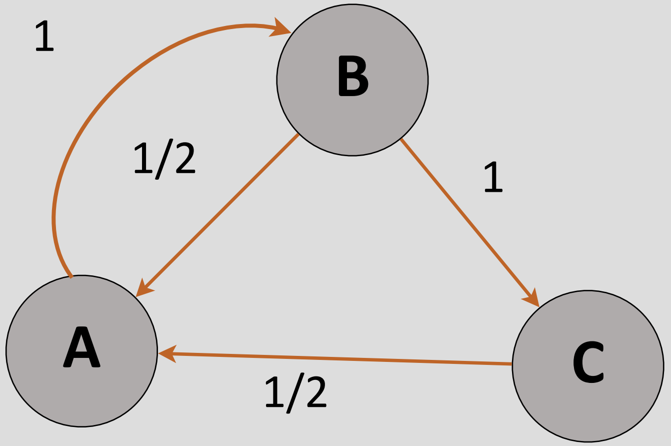
Q) Will flipping the arrows make us go back in time?

What about the reverse?

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} \underset{A\to A}{0} & \underset{B\to A}{1} & \underset{C\to A}{0} \\ \underset{A\to B}{\frac{1}{2}} & \underset{B\to B}{0} & \underset{C\to B}{1} \\ \underset{A\to C}{\frac{1}{2}} & \underset{B\to C}{0} & \underset{C\to C}{0} \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$

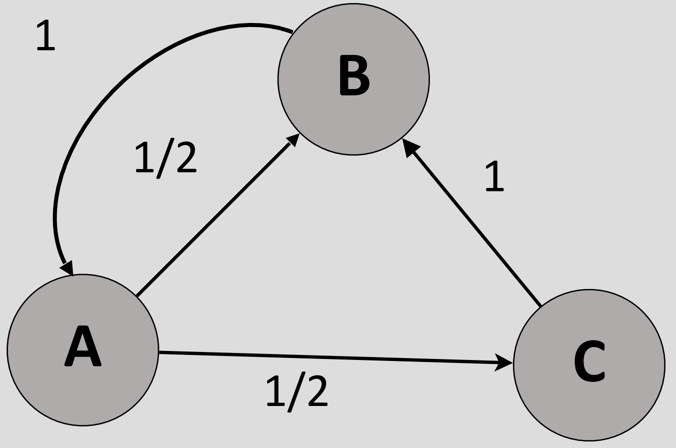


Q) Will flipping the arrows make us go back in time?



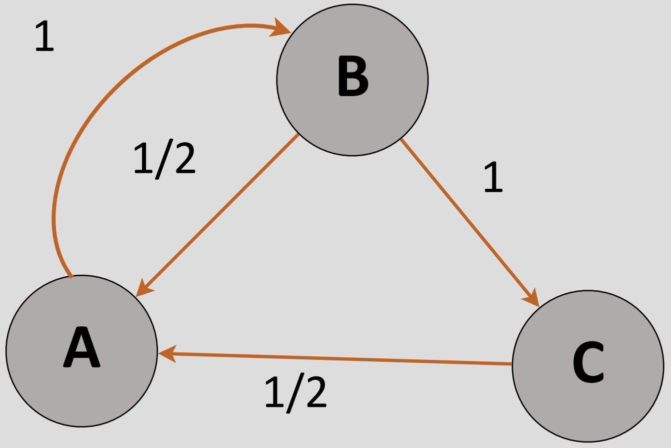
What about the reverse?

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} \underset{A\to A}{0} & \underset{B\to A}{1} & \underset{C\to A}{0} \\ \underset{A\to B}{\frac{1}{2}} & \underset{B\to B}{0} & \underset{C\to B}{1} \\ \underset{A\to C}{\frac{1}{2}} & \underset{B\to C}{0} & \underset{C\to C}{0} \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



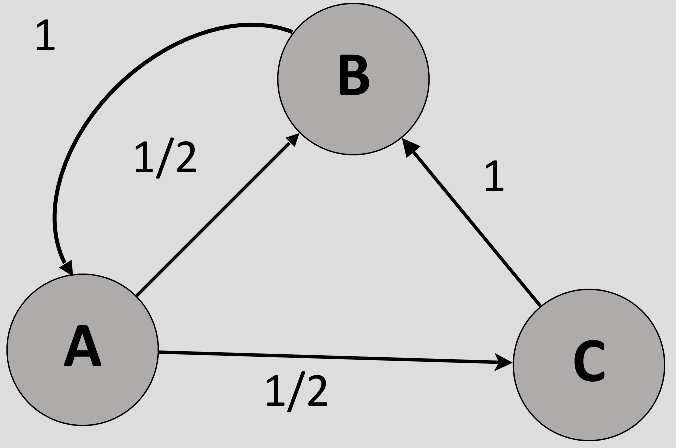
Q) Will flipping the arrows make us go back in time?

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$



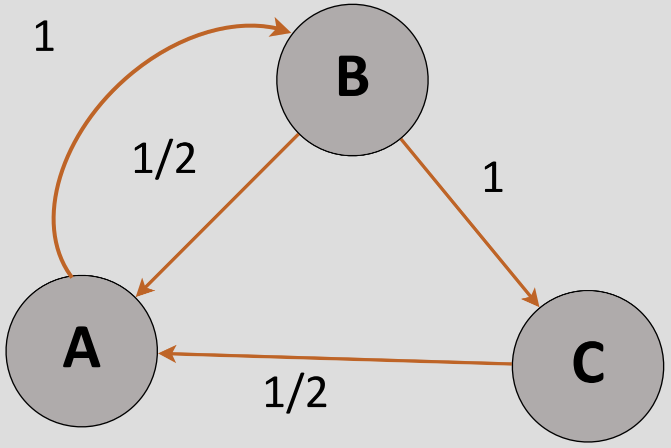
What about the reverse?

$$\begin{array}{c} 6 \\ 10 \\ 2 \end{array} \begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix} \begin{array}{c} 4 \\ 6 \\ 8 \end{array}$$



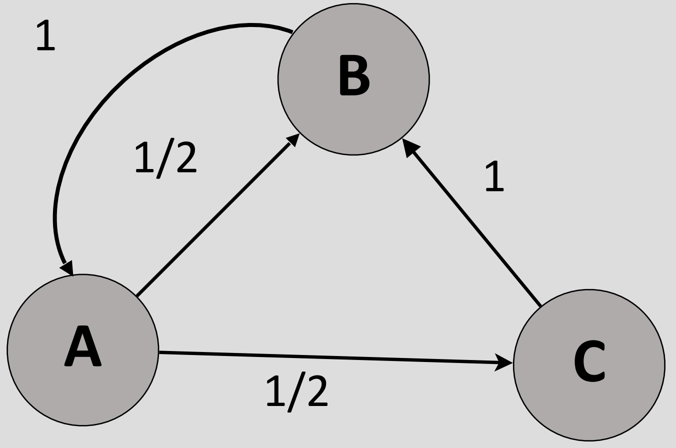
Q) Will flipping the arrows make us go back in time?

$$\begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \\ \\ \\ \end{bmatrix}$$



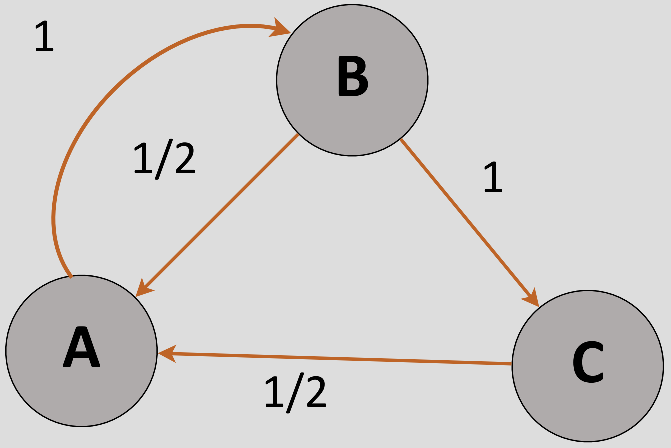
What about the reverse?

$$\begin{array}{c} 6\\ 10\\ 2 \end{array}\begin{bmatrix} x_A(t+1)\\ x_B(t+1)\\ x_C(t+1) \end{bmatrix} = \begin{bmatrix} \underset{A\to A}{0} & \underset{B\to A}{1} & \underset{C\to A}{0}\\ \underset{A\to B}{\frac{1}{2}} & \underset{B\to B}{0} & \underset{C\to B}{1}\\ \underset{A\to C}{\frac{1}{2}} & \underset{B\to C}{0} & \underset{C\to C}{0} \end{bmatrix}\begin{bmatrix} x_A(t)\\ x_B(t)\\ x_C(t) \end{bmatrix}\begin{array}{c} 4\\ 6\\ 8 \end{array}$$



Q) Will flipping the arrows make us go back in time?

$$\begin{bmatrix} 7\\ 6\\ 10 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2}\\ 1 & 0 & 0\\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 6\\ 10\\ 2 \end{bmatrix}$$



A) In general, no!

Matrix Transpose

If the elements of the matrix $A \in \mathbb{R}^{N \times M}$ are a_{ij}

The elements of $A^T \in \mathbb{R}^{M \times N}$ are a_{ji}

Matrix transpose is not (generally) an inverse!

$$A \in \mathbb{R}^{N \times M} \quad \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_M \end{bmatrix}$$

$$\begin{bmatrix} \vec{a}_1^{\,T} \\ \vec{a}_2^{\,T} \\ \vdots \\ a_M^T \end{bmatrix} \quad A^T \in \mathbb{R}^{M \times N}$$

Matrix Transpose

If the elements of the matrix $A \in \mathbb{R}^{N \times M}$ are a_{ij}

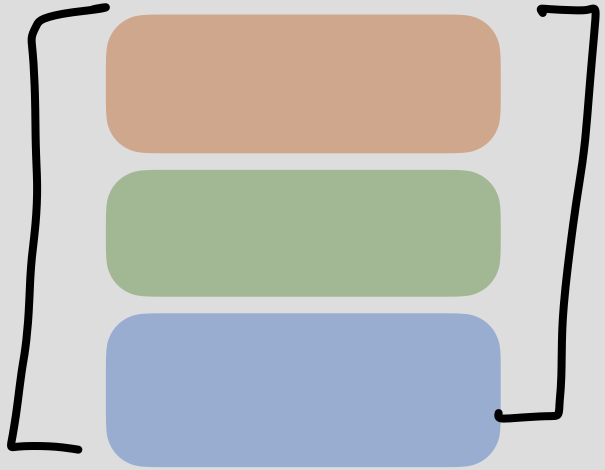
The elements of $A^T \in \mathbb{R}^{M \times N}$ are a_{ji}

Matrix transpose is not (generally) an inverse!

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_m \end{bmatrix} \qquad \begin{bmatrix} \vec{a}_1^{\,T} \\ \vec{a}_2^{\,T} \\ \vdots \\ a_m^T \end{bmatrix}$$

$A \in \mathbb{R}^{N \times M}$ $\qquad$ $A^T \in \mathbb{R}^{M \times N}$

$$\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}$$



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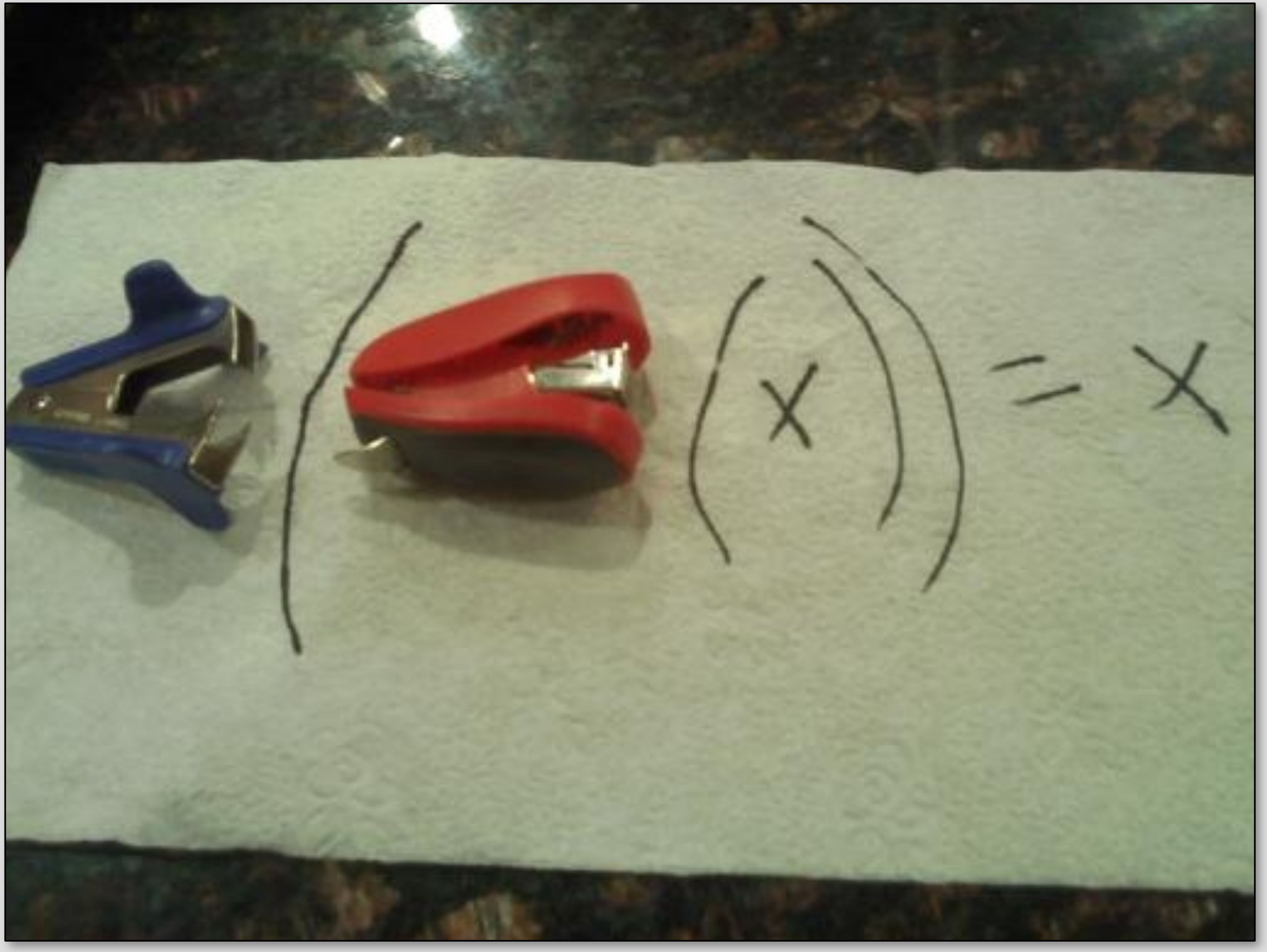
$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_m \end{bmatrix} \qquad \begin{bmatrix} \vec{a}_1^{\,T} \\ \vec{a}_2^{\,T} \\ \vdots \\ a_m^T \end{bmatrix}$$

$A \in \mathbb{R}^{N \times M}$ $\qquad$ $A^T \in \mathbb{R}^{M \times N}$

$$\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Matrix Inversion



Matrix Inverse

$$\overrightarrow{x}(t+1) = Q\overrightarrow{x}(t)$$

Is there a square matrix P such that we can go back in time?

$$\overrightarrow{x}(t) = P\overrightarrow{x}(t+1)$$

Yes, if : $PQ = I$

$$P\overrightarrow{x}(t+1) = PQ\overrightarrow{x}(t)$$

$$P\overrightarrow{x}(t+1) = I\overrightarrow{x}(t)$$

As consequence : $QP = I$

$$\overrightarrow{x}(t+1) = Q\overrightarrow{x}(t)$$

$$\overrightarrow{x}(t+1) = QP\overrightarrow{x}(t+1)$$

$$\overrightarrow{x}(t+1) = I\overrightarrow{x}(t+1)$$

Matrix Inverse - Formal definition

- Definition: Let $P, Q \in \mathbb{R}^{N \times N}$ be square matrices.
  - P is the inverse of Q if $PQ = QP = I$

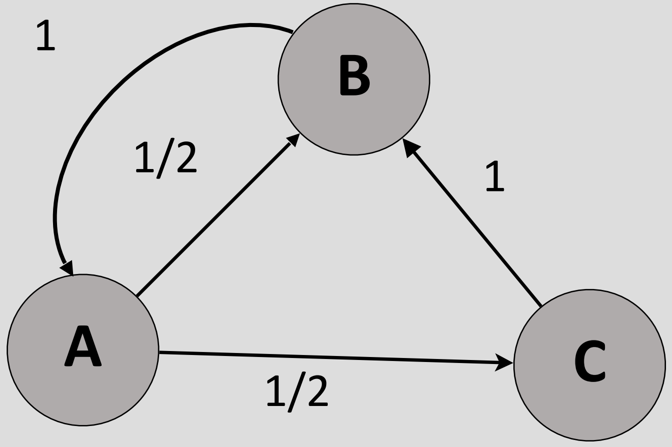
We say that $P = Q^{-1}$ and $Q = P^{-1}$

Q: What about non-square matrices?

A: EECS16B!

Computing the Matrix Inverse

$$\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}}^{Q} \begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}$$



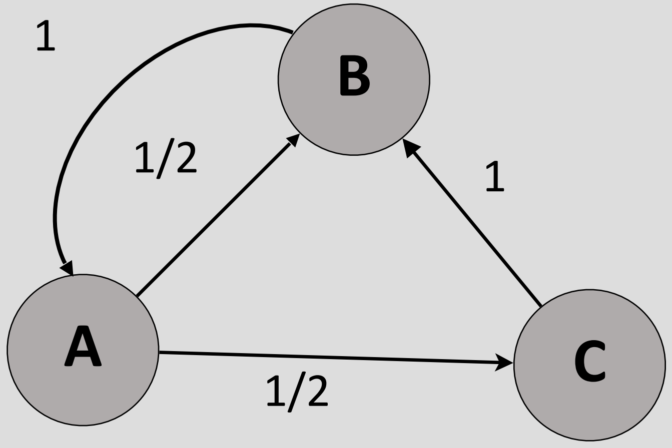
- Want $P = Q^{-1}$ such that $\overrightarrow{x}(t) = P\overrightarrow{x}(t+1)$
  - Need: $QP = I$

Computing the Matrix Inverse

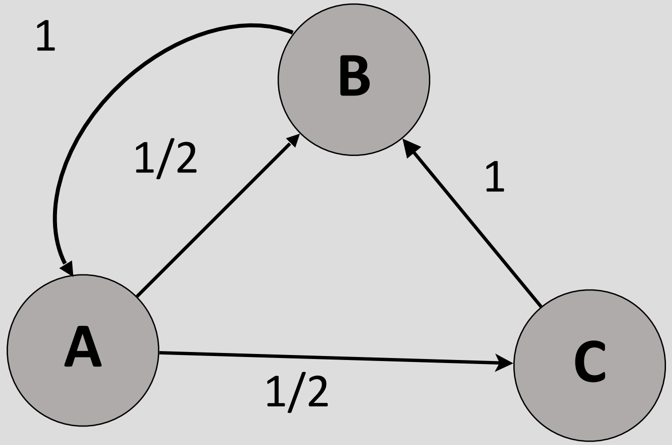
Need: $QP = I$

Pose as a linear set of equations.

Solve with Gaussian Elimination



Computing the Matrix Inverse



Need: $QP = I$

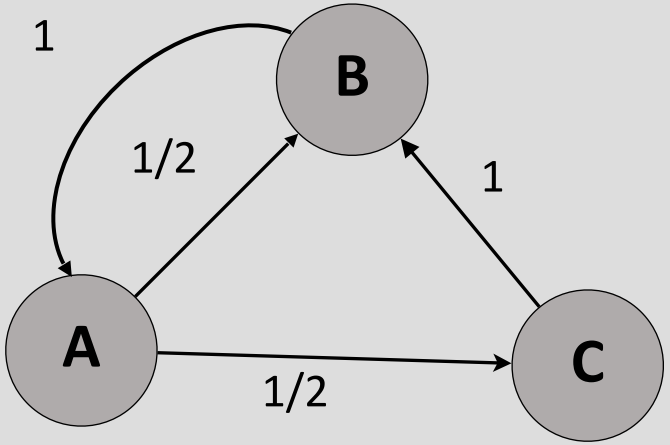
Pose as a linear set of equations.

Solve with Gaussian Elimination

$$\overset{Q}{\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}} \overset{P}{\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}} = \overset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$\vec{p}_1$ $\qquad$ $\vec{b}_1$

Computing the Matrix Inverse



Need: $QP = I$

Pose as a linear set of equations.

Solve with Gaussian Elimination

$$\underset{Q}{\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & 0 & 0 \end{bmatrix}} \underset{P}{\begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}} = \underset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$\vec{p}_1 \quad \vec{p}_2 \quad \vec{p}_3 \qquad \vec{b}_1 \quad \vec{b}_2 \quad \vec{b}_3$

Matrix Inverse via Gaussian Elimination

$$\begin{array}{c} \quad Q \qquad\qquad I \\ \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$\left[\begin{array}{ccc|ccc} \frac{1}{2} & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

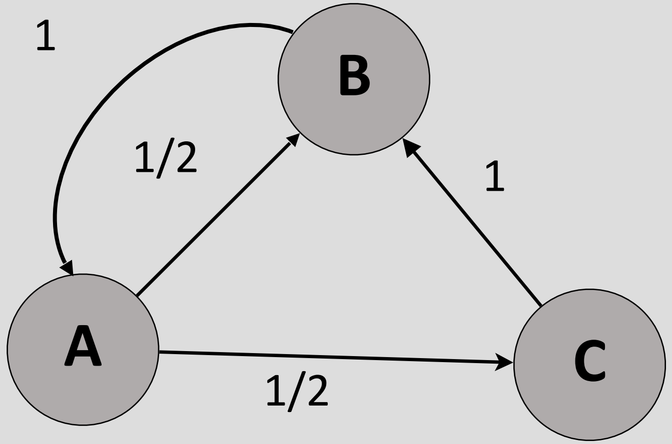
$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & -2 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right]$$

$$\begin{array}{c} \quad I \qquad\qquad P \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 \end{array}\right] \end{array}$$

Let's check

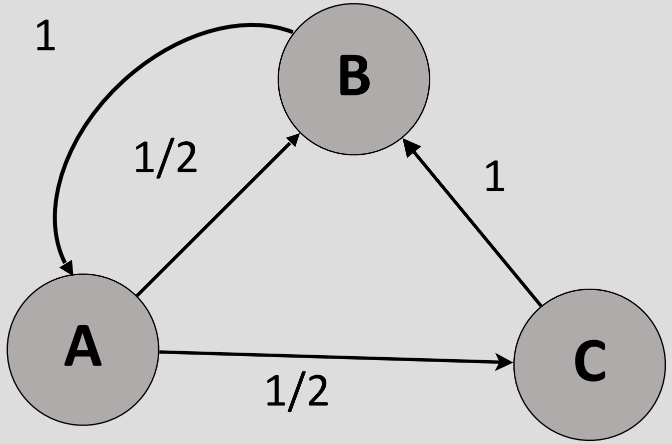
$$
\begin{array}{c}6\\10\\2\end{array}
\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix}
=
\begin{bmatrix} \underset{A\to A}{0} & \underset{B\to A}{1} & \underset{C\to A}{0} \\ \underset{A\to B}{\frac{1}{2}} & \underset{B\to B}{0} & \underset{C\to B}{1} \\ \underset{A\to C}{\frac{1}{2}} & \underset{B\to C}{0} & \underset{C\to C}{0} \end{bmatrix}
\begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}
\begin{array}{c}4\\6\\8\end{array}
$$



$$
\begin{bmatrix} \\ \\ \\ \end{bmatrix}
=
\begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}
\begin{bmatrix} 6 \\ 10 \\ 2 \end{bmatrix}
$$

Let's check

$$
\begin{array}{c}6\\10\\2\end{array}
\begin{bmatrix} x_A(t+1) \\ x_B(t+1) \\ x_C(t+1) \end{bmatrix}
=
\begin{bmatrix} \underset{A\to A}{0} & \underset{B\to A}{1} & \underset{C\to A}{0} \\ \underset{A\to B}{\frac{1}{2}} & \underset{B\to B}{0} & \underset{C\to B}{1} \\ \underset{A\to C}{\frac{1}{2}} & \underset{B\to C}{0} & \underset{C\to C}{0} \end{bmatrix}
\begin{bmatrix} x_A(t) \\ x_B(t) \\ x_C(t) \end{bmatrix}
\begin{array}{c}4\\6\\8\end{array}
$$



$$
\begin{bmatrix} 4 \\ 6 \\ 8 \end{bmatrix}
=
\begin{bmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}
\begin{bmatrix} 6 \\ 10 \\ 2 \end{bmatrix}
$$

And now we can take any number of steps backwards!

Can we always invert a function?

- Can we always invert a function .....$f^{-1}\left(f(\overrightarrow{x})\right) = \overrightarrow{x}$?
  - $f(x) = x^2$?
  - $f(x) = ax$?
  - $f(x) = Ax$?

Invertibility of Linear Transformations

- Theorem: A is invertible, if and only if (iff) the columns of A are linearly independent.
    1. If columns of A are lin. dep. then A^{-1} does not exist
    2. If A^{-1} exists, then the cols. of A are linearly independent

Proof concept: Assume linear dependence and invertibility and show that it is a contradiction

From linear independence: $\exists \overrightarrow{\alpha} \neq 0$ such that $A\overrightarrow{\alpha} = 0$

Assume A^{-1} exists

$$A\overrightarrow{\alpha} = 0$$
$$A^{-1}A\overrightarrow{\alpha} = A^{-1}0$$
$$I\overrightarrow{\alpha} = 0$$

But $\overrightarrow{\alpha} \neq 0$! Hence A^{-1} does not exist

Inverse of a 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

1. Flip a and d
2. Negate b and c
3. Divide by $ad - bc$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Derive via Gauss Elimination!

Equivalent Statements

- Matrix A is **invertible**
- $A\overrightarrow{x} = \overrightarrow{b}$ has a unique solution
- A has linearly independent columns (A is **full rank**)
- A has a **trivial nullspace**
- The **determinant** of A is not zero

Today (and next time's) Jargon

- **Rank** a matrix A is the number of linearly independent columns
- **Nullspace** of a matrix A is the set of solutions to $A\overrightarrow{x} = 0$
- A **vector space** is a set of vectors connected by two operators (+,x)
- A vector **subspace** is a subset of vectors that have "nice properties"
- A **basis** for a vector space is a minimum set of vectors needed to represent all vectors in the space
- **Dimension** of a vector space is the number of basis vectors
- **Column space** is the span (range) of the columns of a matrix
- **Row space** is the span of the rows of a matrix

ESPIRiT—an eigenvalue approach to autocalibrating parallel MRI: where SENSE meets GRAPPA | 834 | 2014
M Uecker, P Lai, MJ Murphy, P Virtue, M Elad, JM Pauly, SS Vasanawala, ...
Magnetic resonance in medicine 71 (3), 990-1001

https://www.ncbi.nlm.nih.gov/pmc/articles/PMC4142121/

- Basis - 3 times
- Rank - 4 times
- Row space - 4 times
- Columns (of a matrix) - 6 times
- Subspace - 17 times
- Null Space - 29 times
- Eigen - 87 times

Vector Space

- From Merriam Webster:

Definition of *vector space*

a set of vectors along with operations of addition and multiplication such that the set is a commutative group under addition, it includes a multiplicative inverse, and multiplication by scalars is both associative and distributive

Vector Space

- A vector space, is a set of vectors and scalars ($\mathbb{V} \in \mathbb{R}^N, \mathbb{F} \in \mathbb{R}$) and two operators $\cdot\,, +$ that satisfy the following:

Axioms of closure

1. $\alpha \vec{x} \in \mathbb{V}$
2. $\vec{x} + \vec{y} \in \mathbb{V}$

Axioms of addition (+)

3. $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$ (associativity)
4. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ (commutativity)
5. $\exists\, \vec{0} \in \mathbb{V}$ s.t. $\vec{x} + \vec{0} = \vec{x}$ (additive identity)
6. $\exists (-\vec{x}) \in \mathbb{V}$ s.t. $\vec{x} + (-\vec{x}) = \vec{0}$ (additive inverse)

Axioms of scaling (·)

7. $\alpha(\vec{x} + \vec{y}) = \alpha \vec{x} + \alpha \vec{y}$ (distributivity)
8. $\alpha \cdot (\beta \vec{x}) = (\alpha\beta) \cdot \vec{x}$
9. $(\alpha + \beta)\vec{x} = \alpha \vec{x} + \beta \vec{x}$
10. $1 \cdot \vec{x} = \vec{x}$

Vector Space

- A vector space $\mathbb{V}$ is a set of vectors and two operators $\cdot\,, +$ that satisfy the following:
    1. $\alpha \overrightarrow{x} \in \mathbb{V}$
    2. $\overrightarrow{x} + \overrightarrow{y} \in \mathbb{V}$
    3. $\overrightarrow{x} + (\overrightarrow{y} + \overrightarrow{z}) = (\overrightarrow{x} + \overrightarrow{y}) + \overrightarrow{z}$ (associativity)
    4. $\overrightarrow{x} + \overrightarrow{y} = \overrightarrow{y} + \overrightarrow{x}$ (commutativity)
    5. $\exists\, \overrightarrow{0} \in \mathbb{V}$ s.t. $\overrightarrow{x} + \overrightarrow{0} = \overrightarrow{x}$ (additive identity)
    6. $\exists (-\overrightarrow{x}) \in \mathbb{V}$ s.t. $\overrightarrow{x} + (-\overrightarrow{x}) = \overrightarrow{0}$
    7. $\alpha(\overrightarrow{x} + \overrightarrow{y}) = \alpha \overrightarrow{x} + \alpha \overrightarrow{y}$ (distributivity)
    8. $\alpha \cdot (\beta \overrightarrow{x}) = (\alpha\beta) \cdot \overrightarrow{x}$
    9. $(\alpha + \beta)\overrightarrow{x} = \alpha \overrightarrow{x} + \beta \overrightarrow{x}$
    10. $1 \cdot \overrightarrow{x} = \overrightarrow{x}$


Is $\mathbb{R}^2$ a vector space?


Is $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$?


Is $\alpha \in \mathbb{R}, \alpha \geq 0$?


Is span $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$?


Is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$?


Is 0?

Subspaces

- A subspace $\mathbb{U}$ consists of a subset of $\mathbb{V}$ in vector space $(\mathbb{V}, \mathbb{F}, +, \cdot)$
    - $\mathbb{U} \subset \mathbb{V}$ and have 3 properties

    1. Contains $\overrightarrow{0}$, i.e., $\overrightarrow{0} \in \mathbb{U}$
    2. Closed under vector addition: $\overrightarrow{v}_1, \overrightarrow{v}_2 \in \mathbb{U}, \ \Rightarrow \overrightarrow{v}_1 + \overrightarrow{v}_2 \in \mathbb{U}$
    3. Closed under scalar multiplication: $\overrightarrow{v}_1 \in \mathbb{U}, \alpha \in \mathbb{F}, \ \Rightarrow \alpha \overrightarrow{v} \in \mathbb{U}$