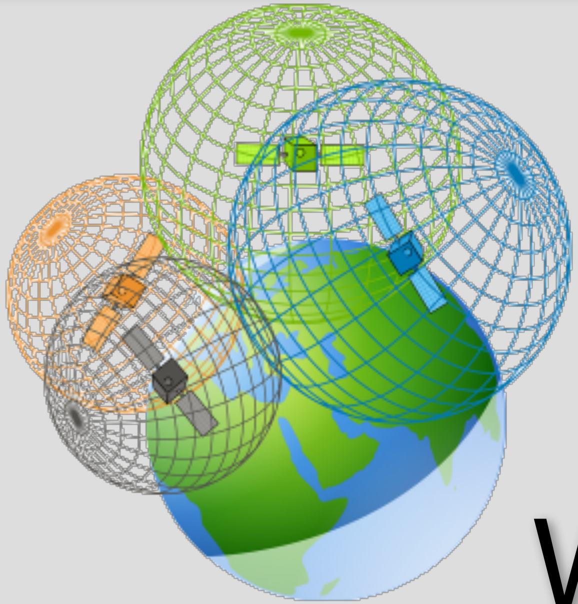
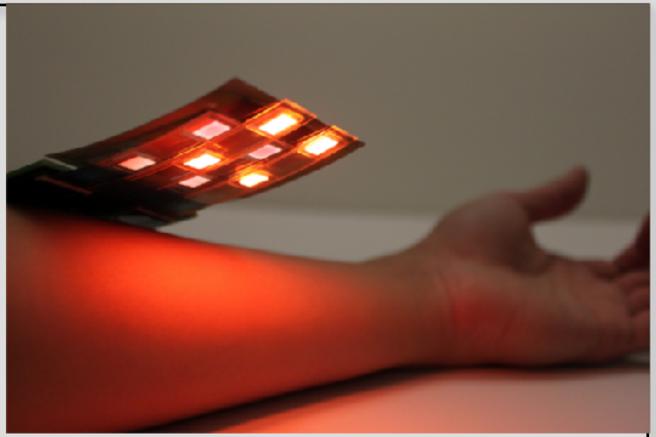






# Welcome to EECS 16A!

## Designing Information Devices and Systems I

Ana Arias and Miki Lustig

2022

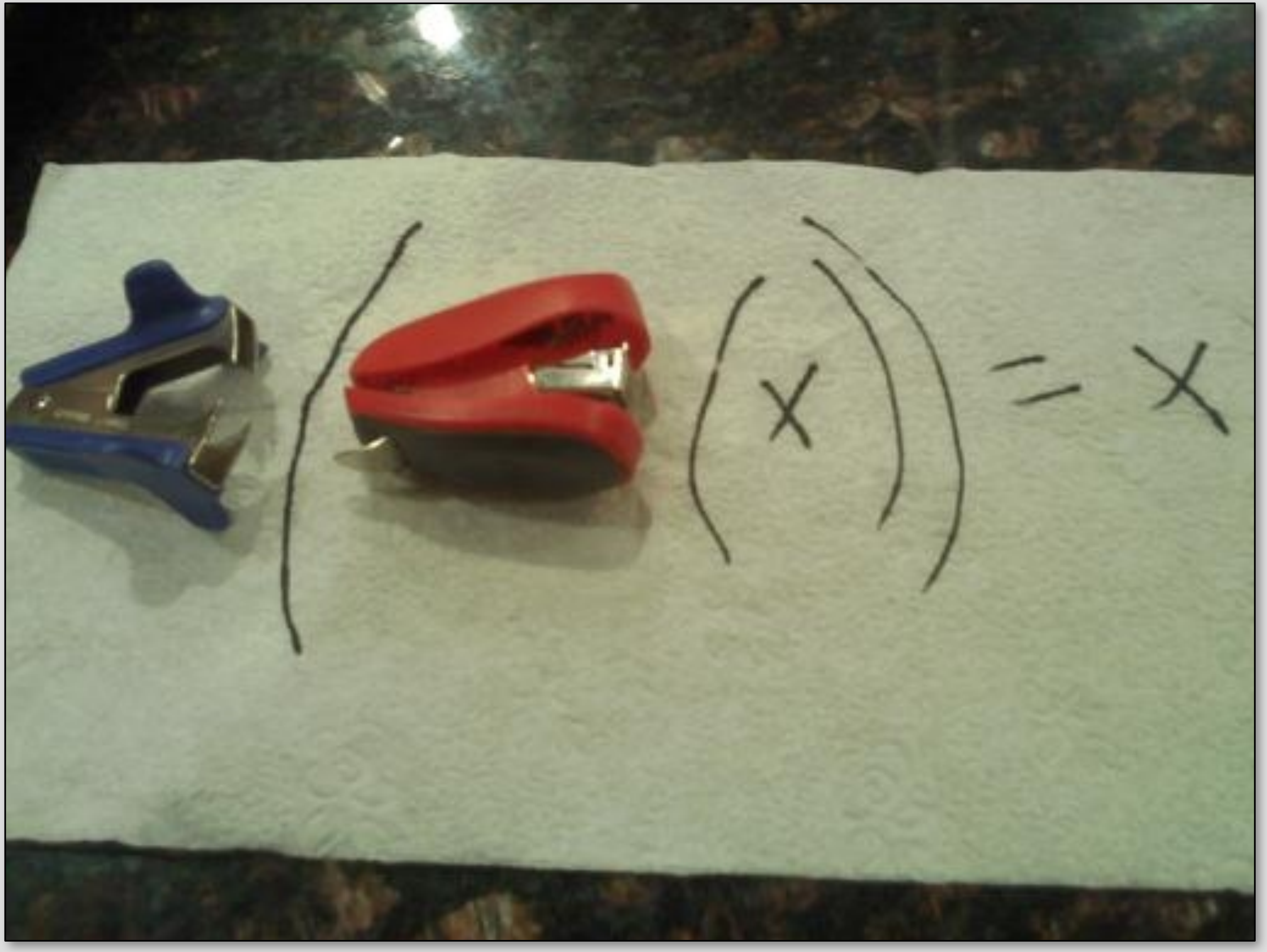
Lecture 4A

Vector Spaces

# Announcements

- Last time:
  - Continue with Matrix transformations
  - Matrix Inverse
- Today:
  - Vector spaces
  - Null spaces
  - Subspaces / Row

# Matrix Inversion



# Invertibility of Linear Transformations

- Theorem: $A$ is invertible, if and only if (iff) the columns of $A$ are linearly independent.
  1. If columns of $A$ are lin. dep. then $A^{-1}$ does not exist
  2. If $A^{-1}$ exists, then the cols. of $A$ are linearly independent

Proof concept: Assume linear dependence and invertibility and show that it is a contradiction

From linear independence: $\exists \overrightarrow{\alpha} \neq 0$ such that $A\overrightarrow{\alpha} = 0$

Assume $A^{-1}$ exists

$$A\overrightarrow{\alpha} = 0$$
$$A^{-1}A\overrightarrow{\alpha} = A^{-1}0$$
$$I\overrightarrow{\alpha} = 0$$

But $\overrightarrow{\alpha} \neq 0$ ! Hence $A^{-1}$ does not exist

# Inverse of a 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

1. Flip $a$ and $d$
2. Negate $b$ and $c$
3. Divide by $ad - bc$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Derive via Gauss Elimination!

# Equivalent Statements

- Matrix $A$ is **invertible**
- $A\overrightarrow{x} = \overrightarrow{b}$ has a unique solution
- $A$ has linearly independent columns (A is **full rank**)
- $A$ has a **trivial nullspace**
- The **determinant** of $A$ is not zero

# Today (and next time's) Jargon

- **Rank** a matrix $A$ is the number of linearly independent columns
- **Nullspace** of a matrix $A$ is the set of solutions to $A\overrightarrow{x} = 0$
- A **vector space** is a set of vectors connected by two operators (+,x)
- A vector **subspace** is a subset of vectors that have "nice properties"
- A **basis** for a vector space is a minimum set of vectors needed to represent all vectors in the space
- **Dimension** of a vector space is the number of basis vectors
- **Column space** is the span (range) of the columns of a matrix
- **Row space** is the span of the rows of a matrix

ESPIRiT—an eigenvalue approach to autocalibrating parallel MRI: where SENSE meets GRAPPA 834 2014
M Uecker, P Lai, MJ Murphy, P Virtue, M Elad, JM Pauly, SS Vasanawala, ...
Magnetic resonance in medicine 71 (3), 990-1001

https://www.ncbi.nlm.nih.gov/pmc/articles/PMC4142121/

- Basis  - 3 times
- Rank - 4 times
- Row space - 4 times
- Columns (of a matrix) - 6 times
- Subspace - 17 times
- Null Space - 29 times
- Eigen - 87 times

# Vector Space

- From Merriam Webster:

## **Definition of *vector space***

a set of vectors along with operations of addition and multiplication such that the set is a commutative group under addition, it includes a multiplicative inverse, and multiplication by scalars is both associative and distributive

# Vector Space

- A vector space, is a set of vectors and scalars ($\mathbb{V} \in \mathbb{R}^N, \mathbb{F} \in \mathbb{R}$) and two operators $\cdot\,, +$ that satisfy the following:

Axioms of closure

1. $\alpha \vec{x} \in \mathbb{V}$
2. $\vec{x} + \vec{y} \in \mathbb{V}$

Axioms of addition (+)

3. $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$ (associativity)
4. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ (commutativity)
5. $\exists\, \vec{0} \in \mathbb{V}$ s.t. $\vec{x} + \vec{0} = \vec{x}$ (additive identity)
6. $\exists (-\vec{x}) \in \mathbb{V}$ s.t. $\vec{x} + (-\vec{x}) = \vec{0}$ (additive inverse)

Axioms of scaling (·)

7. $\alpha(\vec{x} + \vec{y}) = \alpha \vec{x} + \alpha \vec{y}$ (distributivity)
8. $\alpha \cdot (\beta \vec{x}) = (\alpha\beta) \cdot \vec{x}$
9. $(\alpha + \beta)\vec{x} = \alpha \vec{x} + \beta \vec{x}$
10. $1 \cdot \vec{x} = \vec{x}$

# Vector Space

- A vector space $\mathbb{V}$ is a set of vectors and two operators $\cdot\,, +$ that satisfy the following:
    1. $\alpha \vec{x} \in \mathbb{V}$
    2. $\vec{x} + \vec{y} \in \mathbb{V}$
    3. $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$ (associativity)
    4. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ (commutativity)
    5. $\exists\, \vec{0} \in \mathbb{V}$ s.t. $\vec{x} + \vec{0} = \vec{x}$ (additive identity)
    6. $\exists (-\vec{x}) \in \mathbb{V}$ s.t. $\vec{x} + (-\vec{x}) = \vec{0}$
    7. $\alpha(\vec{x} + \vec{y}) = \alpha\vec{x} + \alpha\vec{y}$ (distributivity)
    8. $\alpha \cdot (\beta\vec{x}) = (\alpha\beta) \cdot \vec{x}$
    9. $(\alpha + \beta)\vec{x} = \alpha\vec{x} + \beta\vec{x}$
    10. $1 \cdot \vec{x} = \vec{x}$

Is $\mathbb{R}^2$ a vector space?

Is $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$ ?

Is $\alpha \in \mathbb{R}, \alpha \geq 0$ ?

Is span $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ ?

Is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ?

Is 0?

# Subspaces

- A subspace $\mathbb{U}$ consists of a subset of $\mathbb{V}$ in vector space $(\mathbb{V}, \mathbb{F}, +, \cdot)$
    - $\mathbb{U} \subset \mathbb{V}$ and have 3 properties

    1. Contains $\overrightarrow{0}$, i.e., $\overrightarrow{0} \in \mathbb{U}$
    2. Closed under vector addition: $\overrightarrow{v}_1, \overrightarrow{v}_2 \in \mathbb{U}, \Rightarrow \overrightarrow{v}_1 + \overrightarrow{v}_2 \in \mathbb{U}$
    3. Closed under scalar multiplication: $\overrightarrow{v}_1 \in \mathbb{U}, \alpha \in \mathbb{F}, \Rightarrow \alpha \overrightarrow{v} \in \mathbb{U}$

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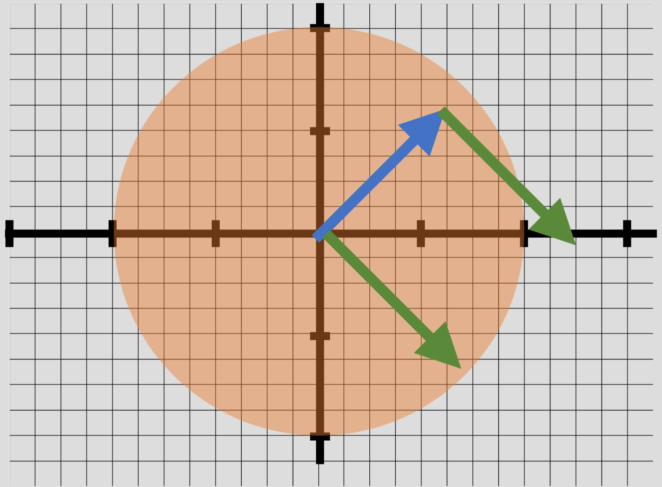
    3. Closed under scalar multiplication: $\overrightarrow{v}_1 \in \mathbb{U}, \alpha \in \mathbb{F}, \Rightarrow \alpha \overrightarrow{v} \in \mathbb{U}$

Q: Consider all vectors $\overrightarrow{v}$ who's length $< 1$. Is this a subspace?



A: not closed under addition,
nor scalar mult.

# Subspaces

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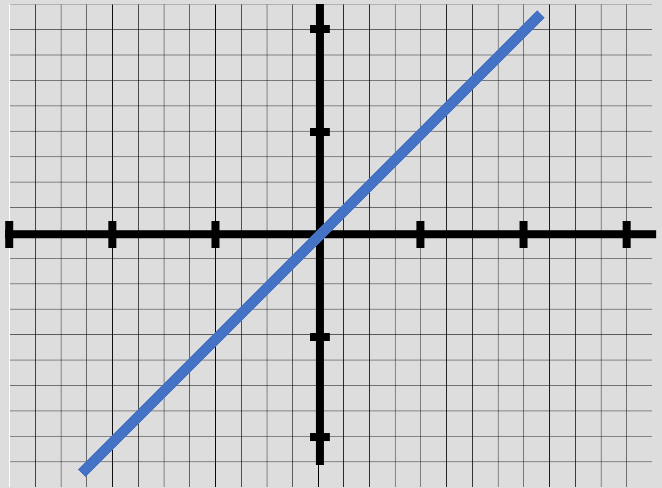
  3. Closed under scalar multiplication: $\vec{v}_1 \in \mathbb{U}, \alpha \in \mathbb{F}, \Rightarrow \alpha \vec{v} \in \mathbb{U}$

Q: Is $\text{Span}\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ a subspace?



A: Yes!

# Subspaces

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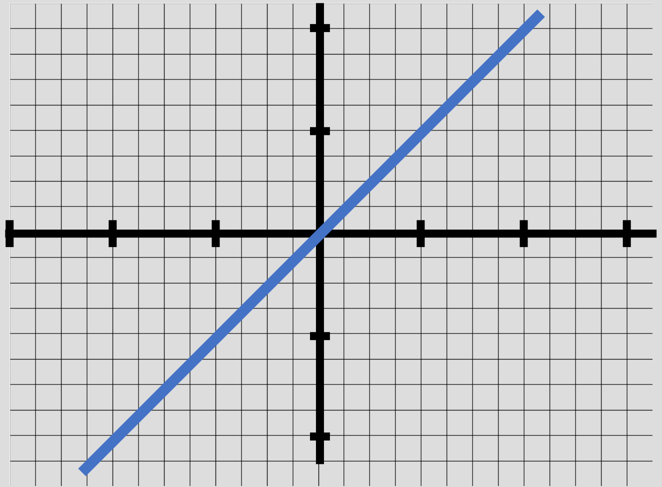
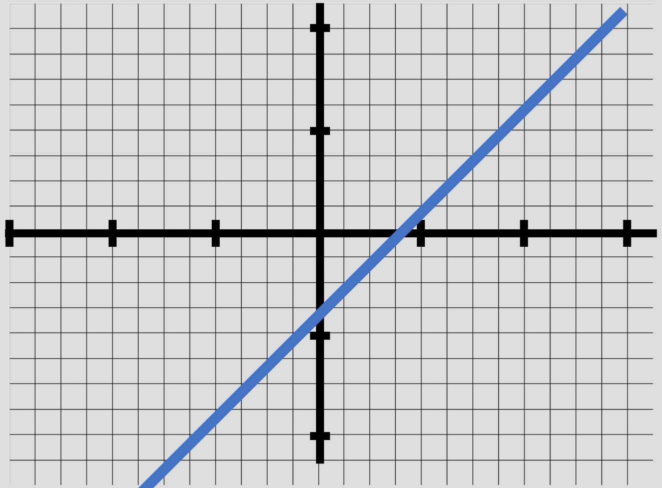


A: Yes!

Q: What about this?



A: $\vec{0} \notin \mathbb{U}$

No!

# Subspaces

- A subspace $\mathbb{U}$ consists of a subset of $\mathbb{V}$ in vector space $(\mathbb{V}, \mathbb{F}, +, \cdot)$
  - $\mathbb{U} \subset \mathbb{V}$ and have 3 properties
  1. Contains $\overrightarrow{0}$, i.e., $\overrightarrow{0} \in \mathbb{U}$
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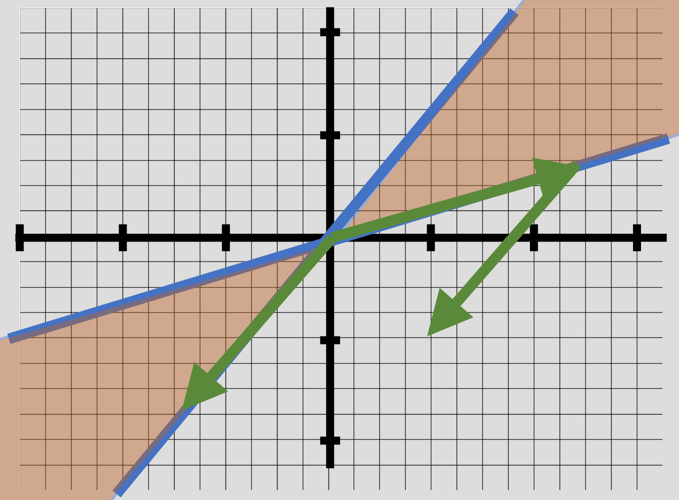
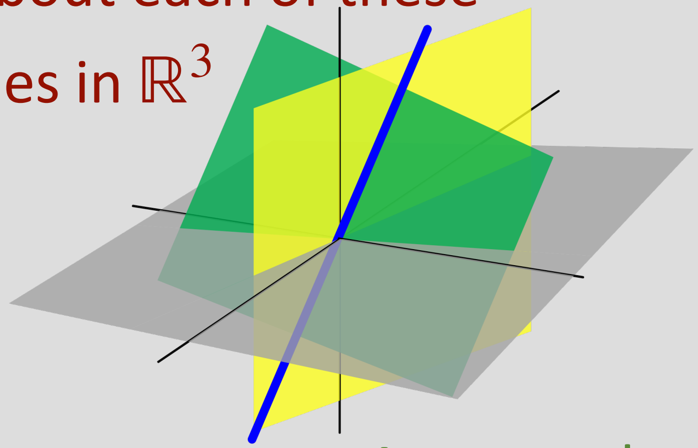
Q: What about this?



A: Not closed under addition!

Q: What about each of these 2D planes in $\mathbb{R}^3$



A: yes, as long as passing through 0

By Alksentrs at en.wikipedia -

# Subspaces

## Example:

$$\mathbb{W} = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| \; a, b, d \in \mathbb{R} \right\}, \quad \mathbb{V} = \mathbb{R}^{2\times 2}$$

Is $\mathbb{W} \subset \mathbb{V}$?

 1. Zero vector?

 2. Closed under addition?

 3. Closed under scalar multiplication?

# Bases

- In words: Minimum set of vectors that spans a vector space

- Definition: given $\mathbb{V}$, a set of vectors $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_N\}$ is a basis of the vector space, if it satisfies:
  - $\{\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_N\}$ are linearly independent
  - $\forall \vec{v} \in \mathbb{V}, \quad \exists \quad \alpha_1, \alpha_2, \cdots, \alpha_N \in \mathbb{R}^N$ such that $\vec{v} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_N \vec{v}_N$

# Bases examples

Q: Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ a basis for $\mathbb{V} = \mathbb{R}^3$?

Q: Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ a basis for $\mathbb{V} = \mathbb{R}^3$?

Q: Is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ a basis for $\mathbb{V} = \mathbb{R}^3$?

# Bases examples

Q: Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$ a basis for $\mathbb{V} = \mathbb{R}^3$?

Q: Is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix} \right\}$ a basis for $\mathbb{V} = \mathbb{R}^3$?

# Column Space

- The range/span/columnspace of a set of vectors is a set of all possible linear combinations:

$$\text{span}\left\{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_M\right\} = \triangleq \left\{ \sum_{m=1}^{M} \alpha_m \vec{a}_m \,\middle|\, \alpha_1, \alpha_2, \cdots, \alpha_M \in \mathbb{R} \right\}$$

Consider:

$$A = \begin{bmatrix} a & 0 \\ 0 & b \\ 0 & 0 \end{bmatrix}$$

Q: Are the columns of $A$, a basis? 

Q: Is the column space of $A$, a subspace?

# Column Space

1. Zero vector?
2. Closed under addition?
3. Closed under scalar multiplication?

Consider:

$$A = \begin{bmatrix} a & 0 \\ 0 & b \\ 0 & 0 \end{bmatrix} \qquad \vec{v}_1 = A\vec{u}_1, \quad \vec{v}_2 = A\vec{u}_2$$

Q: Is the column space of $A$, a subspace?

$$A\vec{0} = \vec{0}$$

$$\vec{v}_1 + \vec{v}_2 = A\vec{u}_1 + A\vec{u}_2 = A(\vec{u}_1 + \vec{u}_2)$$

$$\alpha\vec{v}_1 = \alpha A\vec{u}_1 = A(\alpha\vec{u}_1)$$

# Rank

- USA Today University Ranking for Cal:
  - #1 (joint) in Computer Science
  - #2 in Electrical Engineering (joint)
  - #3 in Computer Engineering

# Rank

- $A \in \mathbb{R}^{N \times M}, \ \text{Rank}\{A\} = \dim\left\{\text{Span}\{A\}\right\}$

$$A = \begin{bmatrix} a & 0 \\ 0 & b \\ 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{bmatrix}$$

2   2   1

- $\text{Rank}\{A\} = \dim\left\{\text{Span}\{A\}\right\} \leq \min(M, N)$

# Null Space

- Definition: The null-space of $A \in \mathbb{R}^{N \times M}$ is the set of all vectors $\overrightarrow{x} \in \mathbb{R}^M$ such that: $A\overrightarrow{x} = 0$

$$A\overrightarrow{x} = 0$$

How many solutions for $\overrightarrow{x}$ satisfy the above?

# Examples

$$\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Linearly independent!

$$\overrightarrow{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\overrightarrow{0}$ is always in the null space — trivial Null space

# Examples

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Linearly dependent!

Gaussian elimination:

$$\left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 = 2x_2$$

$$\Rightarrow \vec{x} = \begin{bmatrix} 2\alpha \\ \alpha \end{bmatrix}$$

$$\vec{x} = \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

A has a non-trivial null-space, $\text{span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

# Example

$$A\vec{x} = \vec{b}$$

We know that $\vec{v}_0 \in \mathrm{Null}(A)$

$$\rightarrow A\vec{v}_0 = \vec{0}$$

We know 1 solution: $\vec{x}_0$

$$\rightarrow A\vec{x}_0 = b$$

# Example

$$A\vec{x} = \vec{b}$$

We know that $\vec{v}_0 \in \text{Null}(A)$

$$\rightarrow A\vec{v}_0 = \vec{0}$$

We know 1 solution: $\vec{x}_0$

$$\rightarrow A\vec{x}_0 = b$$

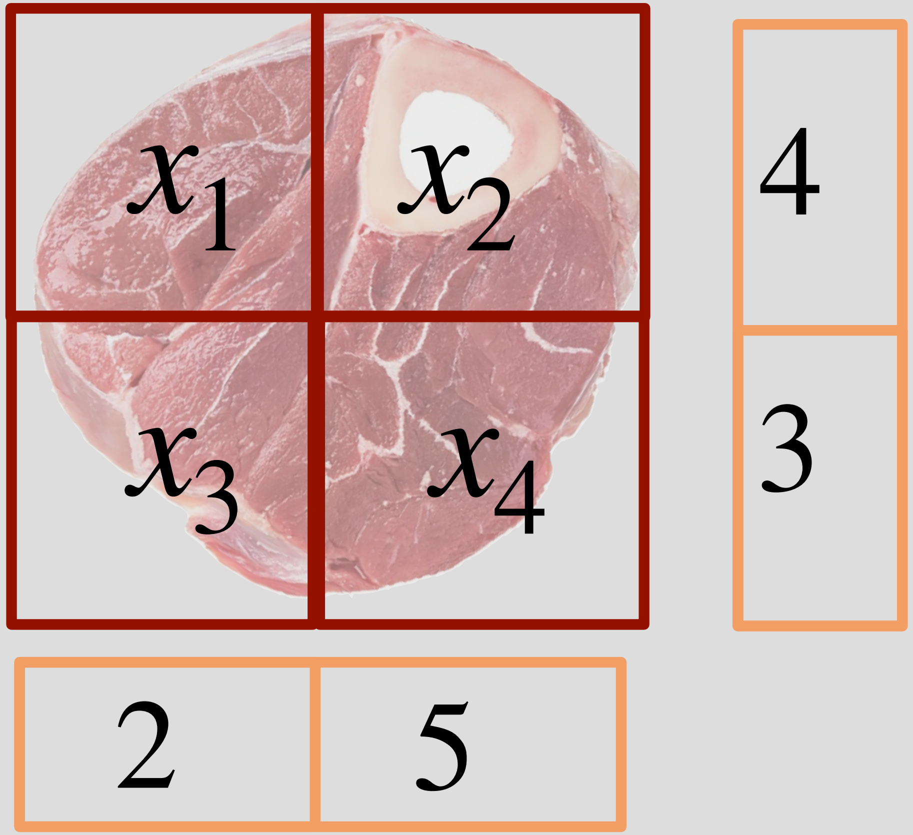
Then: $\vec{x}_0 + \alpha\vec{v}_0$ is also a solution

$$\rightarrow A(\vec{x}_0 + \alpha\vec{v}_0) = A\vec{x}_0 + A(\alpha\vec{v}_0)$$

$$= \vec{b} + \alpha A\vec{v}_0$$

$$= \vec{b}$$

# Back to Tomography




$$1 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 0 \cdot x_4 = 4$$
$$0 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 1 \cdot x_4 = 3$$
$$1 \cdot x_1 + 0 \cdot x_2 + 1 \cdot x_3 + 0 \cdot x_4 = 2$$
$$0 \cdot x_1 + 1 \cdot x_2 + 0 \cdot x_3 + 1 \cdot x_4 = 5$$

$A$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 5 \end{array}\right]$$

# Null Space of the Tomography System (4 measur.)

Step I

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

Step II

$$\begin{array}{l} \\ \\ (3)-(1) \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

Step III

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Step IV

$$\begin{array}{l} \\ \\ (3)+(2) \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

Step V

$$\begin{array}{l} \\ \\ \\ (4)-(3) \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Step VI

$$\begin{array}{l} (1)-(2) \\ \\ \\ \\ \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

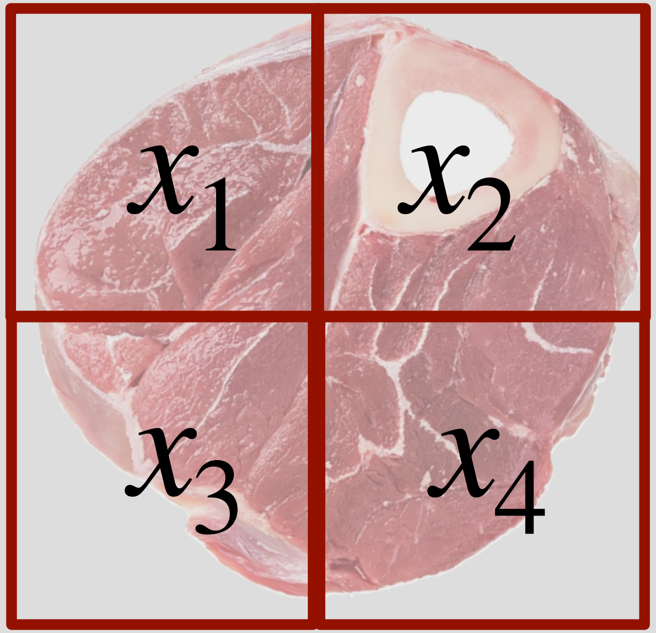
# Null Space of the Tomography System (4 measur.)

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_4$ is the free variable:

$$\Rightarrow \vec{x} = \alpha \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Possible reconstruction




$$= \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} + \alpha \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$