EECS 16A Designing Information Devices and Systems I
Fall 2019 Midterm 1

Midterm 1 Solution

PRINT your student ID: ____

PRINT AND SIGN your name: ____________________, ____________________ ________________________
(last name) (first name) (signature)

PRINT your discussion section and GSI(s) (the one you attend): ____

Name and SID of the person to your left: ____

Name and SID of the person to your right: ____

Name and SID of the person in front of you: ____

Name and SID of the person behind you: ____

1. Tell us about something you did in the last year that you are proud of. (2 Points)

2. Who is your favorite superhero? Why? (2 Points)

Do not turn this page until the proctor tells you to do so. You may work on the questions above.

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3. Campfire Smores (11 points)

Patrick and SpongeBob are making smores.

There are three ingredients: **Graham Crackers, Marshmallows, and Chocolate**. To make a smore, SpongeBob needs: s_g Graham Crackers, s_m number of Marshmallows, and s_c Chocolate.

Ingredients	**Amount Needed**
Graham Crackers (s_g)	10
Marshmallows (s_m)	14
Chocolate (s_c)	20

Table 3.1: SpongeBob's smore

They find out that these ingredients are only stored in bundles as below:

Lobster Pack (p_l)
6 graham crackers
4 marshmallows
2 chocolates

Mr. Krabs Pack (p_k)
2 graham crackers
2 marshmallows
1 chocolates

Squidward Pack (p_s)
3 graham crackers
3 marshmallows
5 chocolates

Gary Pack (p_g)
1 graham crackers
4 marshmallows
5 chocolates

Pearl Pack (p_p)
2 graham crackers
3 marshmallows
2 chocolates

Table 3.2: Amount of Ingredients per Bundle

Spongebob and Patrick need to know how many of each bundle to buy: number of "Lobster" Packs, p_l, number of "Mr. Krabs" Packs, p_k, number of "Squidward" Packs, p_s, number of "Gary" Packs, p_g, and number of "Pearl" Packs, p_p.

(a) (3 points) How many equations/constraints does the information in the problem provide you with?

Solution: In this problem, we have 5 unknowns (the amounts of each pack we need to buy). We know how many ingredients Spongebob's smore needs, so we can write **three** equations: one for the quantity of graham crackers, s_g, one for the quantity of marshmallows, s_m, and one for the quantity of chocolates, s_c.

The 3 constraints/equations are:

$$6p_l + 2p_k + 3p_s + 1p_g + 2p_p = s_g = 10$$
$$4p_l + 2p_k + 3p_s + 4p_g + 3p_p = s_m = 14$$
$$2p_l + 1p_k + 5p_s + 5p_g + 2p_p = s_c = 20$$

(b) (4 points) Based on the information provided in Tables 3.1 and 3.2, **write** an equation of the form $\mathbf{A}\vec{p} = \vec{s}$ that SpongeBob can use to decide how many of each pack to buy. Here, $\vec{p} = \begin{bmatrix} p_l \\ p_k \\ p_s \\ p_g \\ p_p \end{bmatrix}$.

Solution:

$$\begin{bmatrix} 6 & 2 & 3 & 1 & 2 \\ 4 & 2 & 3 & 4 & 3 \\ 2 & 1 & 5 & 5 & 2 \end{bmatrix} \vec{p} = \begin{bmatrix} 10 \\ 14 \\ 20 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 6 & 2 & 3 & 1 & 2 \\ 4 & 2 & 3 & 4 & 3 \\ 2 & 1 & 5 & 5 & 2 \end{bmatrix}, \quad \vec{s} = \begin{bmatrix} 10 \\ 14 \\ 20 \end{bmatrix}$$

(c) (4 points) Now, the ingredients in the packets $(\mathbf{A})$ and Spongebob's receipe $(\vec{s})$ change. We have: $\mathbf{A} = \begin{bmatrix} 1 & 1 & 3 & 2 & 2 \\ 0 & 1 & 3 & 0 & 2 \\ 1 & 3 & 9 & 2 & 6 \end{bmatrix}$, and $\vec{s} = \begin{bmatrix} 3 \\ 2 \\ 10 \end{bmatrix}$.

Find a $\vec{p}$ that satisfies $\mathbf{A}\vec{p} = \vec{s}$. If no solution exists, explain why not.

Solution: We can solve this using Gaussian elimination:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 3 & 2 & 2 & 3 \\ 0 & 1 & 3 & 0 & 2 & 2 \\ 1 & 3 & 9 & 2 & 6 & 10 \end{array}\right] \overset{R_3 - R_1 \mapsto R_3}{\overbrace{\Rightarrow}} \left[\begin{array}{ccccc|c} 1 & 1 & 3 & 2 & 2 & 3 \\ 0 & 1 & 3 & 0 & 2 & 2 \\ 0 & 2 & 6 & 0 & 4 & 7 \end{array}\right] \overset{R_3 - 2R_2 \mapsto R_3}{\overbrace{\Rightarrow}} \left[\begin{array}{ccccc|c} 1 & 1 & 3 & 2 & 2 & 3 \\ 0 & 1 & 3 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{array}\right]$$

Spongebob's smore yields an inconsistent system, therefore we cannot find a solution.

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4. Operations on polynomials (8 points)

Matrix multiplication is quite powerful, and can be used to represent operations such as differentiation and integration. Here we focus on cubic polynomials:

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3.$$

where the c_i are real scalar coefficients that do not depend on x.

We represent these cubic polynomials as 4-dimensional vectors by stacking the c_i — for instance, we will represent $f(x)$ as the vector $\vec{f} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$.

Recall that the derivative of $f(x)$ is $f'(x) = c_1 + 2c_2 x + 3c_3 x^2$. The matrix, $\mathbf{D_3}$,

$$\mathbf{D_3} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

represents differentiation, i.e.:

$$\mathbf{D_3} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ 2c_2 \\ 3c_3 \\ 0 \end{bmatrix}.$$

(a) (4 points) Now we consider the integration of quadratic polynomials. For a quadratic polynomial $g(t) = c_0 + c_1 t + c_2 t^2$, the definite integral from 0 to x is given by the cubic polynomial:

$$h(x) = \int_0^x g(t)dt = c_0 x + \frac{c_1}{2}x^2 + \frac{c_2}{3}x^3.$$

The quadratic polynomial $g(\cdot)$ can be represented as a cubic polynomial by the vector of coefficients $\vec{g} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ 0 \end{bmatrix}$. Note that the last entry of this vector will always be 0.

Find a matrix 4×4 matrix $\mathbf{E}_3$ such that $\mathbf{E}_3\vec{g}$ is a vector representing the integral of the quadratic polynomial $g(\cdot)$. Because we are representing a quadratic polynomial as a cubic, and the last entry of $\vec{g}$ is always 0, we set the last column of $\mathbf{E}_3$ to all zeros, i.e. it is of the form:

$$\mathbf{E}_3 = \begin{bmatrix} e_{11} & e_{12} & e_{13} & 0 \\ e_{21} & e_{22} & e_{23} & 0 \\ e_{31} & e_{32} & e_{33} & 0 \\ e_{41} & e_{42} & e_{43} & 0 \end{bmatrix}.$$

Solution: Let c_0, c_1, and c_2 be such that $f(x) = c_0 + c_1 x + c_2 x^2$. Now, we know that

$$\begin{aligned} F(x) &= \int_0^x f(a)\,\mathrm{d}a \\ &= \int_0^x (c_0 + c_1 a + c_2 a^2)\,\mathrm{d}a \\ &= \left[c_0 a + \frac{1}{2}c_1 a^2 + \frac{1}{3}c_2 a^3\right]_{a=0}^{a=x} \\ &= c_0 x + \frac{c_1}{2}x^2 + \frac{c_2}{3}x^3. \end{aligned}$$

Applying $\texttt{vec3}$ to both $f(x)$ and $F(x)$, we find that

$$\begin{aligned} \texttt{vec3}[f(x)] &= \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ 0 \end{bmatrix} \\ \texttt{vec3}[F(x)] &= \begin{bmatrix} 0 \\ c_0 \\ c_1/2 \\ c_2/3 \end{bmatrix}. \end{aligned}$$

Thus, using the linear combination interpretation of matrix multiplication,

$$\begin{aligned} \texttt{vec3}[F(x)] &= \begin{bmatrix} 0 \\ c_0 \\ c_1/2 \\ c_2/3 \end{bmatrix} \\ &= c_0 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/3 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & ? \\ 1 & 0 & 0 & ? \\ 0 & 1/2 & 0 & ? \\ 0 & 0 & 1/3 & ? \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ 0 \end{bmatrix}, \end{aligned}$$

where the ?s in the last column indicate that the values are indeterminate, since they could be anything and the equation would still hold. Thus, we know that E_3 must be of the form

$$E_3 = \begin{bmatrix} 0 & 0 & 0 & ? \\ 1 & 0 & 0 & ? \\ 0 & 1/2 & 0 & ? \\ 0 & 0 & 1/3 & ? \end{bmatrix}$$

with unknown ?s.

However, recall that we were also told that

$$E_3(\texttt{vec3}[x^3]) = E_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \vec{0}.$$

This tells us that the last column of E_3 must be $\vec{0}$, so we can solve for the ?s to determine that

$$E_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \end{bmatrix}.$$

(b) (4 points) $\mathbf{E}_3$ is a matrix representing the integration of a quadratic polynomial, and $\mathbf{D}_3$ is a matrix representing the differentiation of a cubic polynomial. **Explicitly write a matrix such that $\mathbf{M}\vec{f}$ calculates the result of first differentiating a cubic polynomial $f(x)$ and then integrating it.** What do you notice about this matrix? *For convenience:*

$$\mathbf{D}_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

If you prefer, you may write out your answer in terms of the entries of E_3 given by:

$$\mathbf{E}_3 = \begin{bmatrix} e_{11} & e_{12} & e_{13} & 0 \\ e_{21} & e_{22} & e_{23} & 0 \\ e_{31} & e_{32} & e_{33} & 0 \\ e_{41} & e_{42} & e_{43} & 0 \end{bmatrix},$$

before explicitly computing M as this may help with partial credit. However, you are not required to.

Solution: Composing the transformations represented by first differentiating and then integrating, we obtain the matrix E_3D_3. Evaluating it, we obtain

$$\begin{aligned} E_3D_3 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \end{aligned}$$

This matrix looks very similar to the identity matrix, but is not the same! Specifically, it knocks out the first term of a vector it is applied to, so

$$E_3D_3(\texttt{vec3}[c_0 + c_1x + c_2x^2 + c_3x^3]) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \texttt{vec3}[c_1x + c_2x^2 + c_3x^3].$$

Expressed algebraically, this tells us that differentiating, and then integrating a cubic polynomial will cause us to lose information about the constant term c_0, but we will still be able to retain the other coefficients.

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5. Drone Dynamics (11 points)

Professor Boser is characterizing the motion of drones with only three propellers, called tri-rotor drones. He provides commands through motor inputs, $\vec{m} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}$. The drone position is given by $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. The motor inputs, $\vec{m}$, affect the position, $\vec{v}$, through the matrix, $\mathbf{D}$. That is, $\vec{v} = \mathbf{D}\vec{m}$.

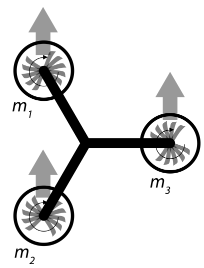

Figure 5.1: The lifting forces acting on a tri-rotor drone.

(a) (5 points) Professor Boser considers the matrix $\mathbf{D} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -2 & -3 \\ 1 & 0 & 1 \end{bmatrix}$. Can the drone reach any position in $\mathbb{R}^3$ using this matrix? Justify your answer.

Solution: No, the drone cannot reach any position in $\mathbb{R}^3$. We can check this by Gaussian elimination on the following augmented matrix, to see if we can solve for motor inputs $\vec{m}$ that allow for every position.

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & x \\ 1 & -2 & -3 & y \\ 1 & 0 & 1 & z \end{array}\right]$$

If it so happens that we get expressions for m_1, m_2, and m_3, the left hand side of the augmented matrix will eliminate to the identity. So by performing the elimination only on the coefficient matrix, we have:

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & -2 & -3 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & -2 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

We see that we have a row of zeroes. This indicates that the columns are linearly dependent, and that for certain x, y, and z, it is not possible to choose m_1, m_2, and m_3 that yield them.

(b) (6 points) Professor Boser is simultaneously testing multiple drones. The first drone can occupy positions in the columnspace of $\mathbf{D}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and the second drone can occupy positions in the columnspace of $\mathbf{D}_2 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

The drones will crash if the columnspaces of $\mathbf{D}_1$ and $\mathbf{D}_2$ intersect. **Find a basis for the subspace that both drones can reach**, i.e. the intersection of the columnspaces of $\mathbf{D}_1$ and $\mathbf{D}_2$. Where can the drones crash into each other?

Hint #1: Observe the columns of $\mathbf{D}_1$ *and* $\mathbf{D}_2$*. Hint #2: For partial credit, find bases for the columnspaces of both* $\mathbf{D}_1$ *and* $\mathbf{D}_2$ *individually.*

Solution: $\mathbf{D}_1$ has the standard basis as its columns. It will span all of $\mathbb{R}^3$. Since the first drone can go anywhere in $\mathbb{R}^3$, the intersection of the first and second drone's columnspaces will be the second drone's columnspace. An observation that can be made about the second drone's columns is that the third column is a sum of the first two columns. Thus to get a basis, which has to be linearly independent, we can select only the first two columns of $\mathbf{D}_2$. One valid basis is: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. The drones can crash into each other for all positions in $\text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

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6. Aragorn's Odyssey (22 points)

In a desperate attempt to save Minas Tirith, Aragorn is trying to maneuver your ship in a 2D plane around the fleet of the Corsairs of the South. The position of your ship in two dimensions (x, y) is represented as a vector, $\begin{bmatrix} x \\ y \end{bmatrix}$.

(a) (5 points) In order to evade the Witch-King of Angmar, Gandalf provides Aragorn with linear transformation spell. The spell first reflects your ship along the X-axis (i.e. multiplies the Y-coordinate by -1) and then rotates it by 30 degrees counterclockwise. Express the transformation Gandalf's spell performed on the ship's location as a 2×2 matrix.

Hint: Recall that the matrix $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ *rotates a vector counterclockwise by* θ.

Solution:

$$\mathbf{G_{spell}} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} * \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\mathbf{G_{spell}} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}$$

(b) (3 points) If the ship was initially 1 unit distance away from the origin $(0, 0)$, how far is it from the origin after the transformation above? Justify your answer.

Solution: 1 unit still - the transformation should not affect the distance from the origin

(c) (6 points) Having evaded the Witch-King and the Corsairs, Aragorn needs to quickly reach Minas Tirith. To do so, he uses the wind spell, $\mathbf{B}_{\text{spell}}$, ten times, where his position $\vec{x}[t]$ changes according to the equation

$$\vec{x}[t+1] = \mathbf{B}_{\text{spell}}\vec{x}[t],$$

where $\mathbf{B}_{\text{spell}} = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$.

The initial location of your ship is $\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. **What is the location of your ship at time $t = 10$, i.e. what is $\vec{x}[10]$?** Explicitly compute your final solution and justify your answer.

Solution: $\lambda_1 = 2, \lambda = 3$

Eigenvector $\vec{v}_1 = [10]^T$ corresponds to eigenvalue $\lambda_1 = 2$

So at time 10 we have: $\vec{x}[10] = 2^{10}\vec{x}[0]$

(d) (8 points) The ship is now moving in an n dimensional space. The position of the ship at time t is represented by $\vec{x}[t] \in \mathbb{R}^n$. The ship starts at the origin $\vec{0}$.

Aragorn tries a new spell, $\mathbf{C}_{\text{spell}} \in \mathbb{R}^{n \times n}$, $\mathbf{C}_{\text{spell}} \neq 0$. In addition to the spell, the ship is given some ability to steer using the scalar input $u[t] \in \mathbb{R}$. The location of the ship at the next time step is described by the equation:

$$\vec{x}[t+1] = \mathbf{C}_{\text{spell}}\vec{x}[t] + \vec{b}u[t],$$

where $\vec{b} \in \mathbb{R}^n$ is fixed.

You know from the Segway problem on the homework that the ship can reach all locations in the $\text{span}\{\vec{b}, \mathbf{C}_{\text{spell}}\vec{b}, \mathbf{C}_{\text{spell}}^2\vec{b}, \cdots, \mathbf{C}_{\text{spell}}^9\vec{b}\}$ in ten time steps. Given that $\vec{b} \neq 0$ is an eigenvector of $\mathbf{C}_{\text{spell}}$, **what is the maximum dimension of the subspace of locations the ship can reach?** Justify your answer.

Solution: Since $\vec{b}$ is an eigenvector of $\mathbf{C}_{\mathbf{spell}}$ we will have $\mathbf{C}^{\mathbf{k}}\vec{b} = \lambda_b^k\vec{b}$, this means that:
$\dim(\text{span}\{\vec{b}, \mathbf{C}_{\text{spell}}\vec{b}, \mathbf{C}_{\text{spell}}^2\vec{b}, \cdots, \mathbf{C}_{\text{spell}}^9\vec{b}\}) = 1$

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7. Color vision (22 points)

This problem will explore how our eyes see color.

Let $\vec{x} = \begin{bmatrix} x_{\text{violet}} \\ x_{\text{blue}} \\ x_{\text{green}} \\ x_{\text{yellow}} \\ x_{\text{red}} \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_{\text{short}} \\ y_{\text{medium}} \\ y_{\text{long}} \end{bmatrix}$. Let $\vec{y} = \mathbf{A}\vec{x}$.

For this problem, light from a point in the world is an **input light vector**, $\vec{x}$, as above, where the entries represent the intensities of different colors of light. Our eye has three types of cone cells, short, medium and long, and the light recorded by the eye can be represented by the **eye vector**, $\vec{y}$, as above.

Our vision system **can be represented as a linear transformation** $(\mathbf{A})$ of the input light vector $(\vec{x})$ onto the cone cells in our eyes to form the eye vector, $(\vec{y})$. That is, $\vec{y} = \mathbf{A}\vec{x}$.

Distinct light vectors, $\vec{x}_1, \vec{x}_2 \in \mathbb{R}^5$, $\vec{x}_1 \neq \vec{x}_2$, can result in the same eye vector, $\vec{y}$. That is $\mathbf{A}\vec{x}_1 = \mathbf{A}\vec{x}_2 = \vec{y}$. This concept is referred to as a metamerism.

(a) (4 points) For this subpart $\mathbf{A} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. You are given four light vectors, $\vec{x}$, below. Some of them result in the same eye vector, $\vec{y}$. **Fill in the circles (completely) to the left of each light vector, $\vec{x}$, that result in the same eye vector, $\vec{y}$.** (*There is no partial credit for this subpart. Space in the below box is for scratch and will not be graded.*)

$$\bigcirc \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad \bigcirc \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} \quad \bigcirc \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad \bigcirc \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Solution: 2nd and 4th both give $\vec{y} = [.25, 1, 1]^T$

(b) (6 points) Analyzing the null space of $\mathbf{A}$ will help us explain the concept of metamerism. **Find the null space of A and provide a set of basis vectors that span it.**

$$\mathbf{A} = \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Solution:

$$\text{Null}(\mathbf{A}) = \text{span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

(c) (8 points) People with color blindness cannot see as many unique colors. Color blindness can be modeled by left-multiplying the color blindness matrix, $\mathbf{B}$, with the vision's matrix, $\mathbf{A}$, so that $\vec{y} = \mathbf{B}\mathbf{A}\vec{x}$. In this part, consider generic matrices $\mathbf{A}$ and $\mathbf{B}$, unrelated to the earlier parts.

Prove that the dimension of the null space of BA is greater than or equal to the dimension of the null space of A. That is:

$$\dim(\mathrm{Null}(\mathbf{BA})) \geq \dim(\mathrm{Null}(\mathbf{A})).$$

Hint: Can you show that every vector from one nullspace must belong to the other?

Solution: Let $\vec{w} \in \mathrm{Null}(\mathbf{A})$,

$$\mathbf{BA}\vec{w} = \mathbf{B}\vec{0} = \vec{0}$$

Therefore $\vec{w} \in \mathrm{Null}(\mathbf{B}A)$.

(d) (4 points) We also want to examine how a matrix $\mathbf{B}$ alters the column space of matrix $\mathbf{A}$. In this part, consider generic matrices $\mathbf{A}$ and $\mathbf{B}$, unrelated to the earlier parts.

$$\dim(\text{Column space}(\mathbf{BA})) \geq \dim(\text{Column space}(\mathbf{A}))$$

Is the above statement true or false? If true, prove it, if false, provide a counter example (i.e. an example of A and B where the inequality does not hold). *Hint: There are no restrictions on what* $\mathbf{A}$ *or* $\mathbf{B}$ *can be.*

Solution: False, show counter example. Let $\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The dimension of the column space of $\mathbf{A}$ is two. The dimension of the column space of $\mathbf{BA}$ is one. The statement is false.

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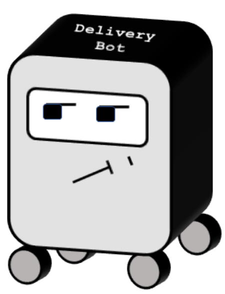
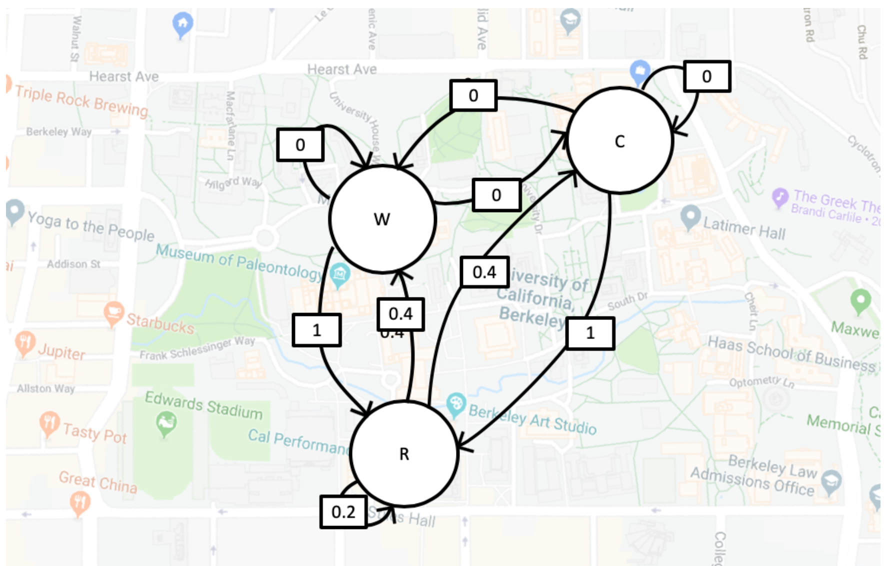
8. The DeliveryBot (13 points)

DeliveryBot is a service opening soon to deliver food from a restaurant to Wheeler Hall and Cory Hall, each represented by the R, W, and C nodes on the state transition diagram below, respectively. The number of DeliveryBots at the restaurant, Wheeler Hall, and Cory Hall are $x_R[t]$, $x_W[t]$, and $x_C[t]$, respectively. The owner of the company has asked for your expertise to track the number of DeliveryBots at each location!

Based on market research, the owner gives you predictions about the movement of DeliveryBots:



a) A DeliveryBot!



b) Delivery state transition diagram.

(a) (3 points) Let $\vec{x}[t] = \begin{bmatrix} x_R[t] \\ x_W[t] \\ x_C[t] \end{bmatrix}$. Write the transition matrix $\mathbf{S}$, where $\vec{x}[t+1] = \mathbf{S}\vec{x}[t]$.

Solution: Writing out the system of equations,

$$\begin{aligned} x_R[t+1] &= 0.2x_R[t] + 1x_W[t] + 1x_C[t] \\ x_W[t+1] &= 0.4x_R[t] \\ x_C[t+1] &= 0.4x_R[t] \end{aligned}$$

It helps to write these out so you don't accidentally use the transpose!
Converting the equations from above to matrix form:

$$\mathbf{S} = \begin{bmatrix} 0.2 & 1 & 1 \\ 0.4 & 0 & 0 \\ 0.4 & 0 & 0 \end{bmatrix}$$

(b) (3 points) For this part of the problem you may assume the state transition matrix is $\mathbf{S} = \begin{bmatrix} 0.4 & 0 & 0 \\ 0.2 & 0 & 0 \\ 0.4 & 1 & 1 \end{bmatrix}$, unrelated to part (a).

The owner counts the number of bots at each location at $t = 2$ and would like to infer the number of bots at time $t = 1$ in each location.

Can you help the owner compute the number of bots in each location at time $t = 1$? If this is possible, write $\vec{x}[1]$ in terms of $\vec{x}[2]$. If this is not possible, justify why.

Solution: No. Since the columns of $\mathbf{S}$ are linearly dependent, we know that there does not exist an inverse to matrix $\mathbf{S}$. As proved in lecture and discussion, there is no unique solution to $\mathbf{S}\vec{x}[1] = \vec{x}[2]$.

(c) (7 points) Unexpectedly, some squirrels have been attacking the DeliveryBots in order to access any potential food inside! When a squirrel attacks a DeliveryBot, the DeliveryBot goes to the new S node (secret squirrel node) on the state transition diagram in Figure 8.2.

The number of DeliveryBots at node S are $x_S[t]$. Let $\vec{z}[t] = \begin{bmatrix} x_R[t] \\ x_W[t] \\ x_C[t] \\ x_S[t] \end{bmatrix}$ represent the number of DeliveryBots at each of nodes in the state transition diagram. The state transition matrix for this new scenario is $\mathbf{T} = \begin{bmatrix} 0.2 & 0.5 & 0.5 & 0 \\ 0.2 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0 \\ 0.4 & 0.5 & 0.5 & 1 \end{bmatrix}$. The eigenvalues and eigenvectors corresponding to this matrix $\mathbf{T}$ are given by: $\lambda_1 = 1, \vec{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\lambda_2 = 0.558, \vec{v}_2 = \begin{bmatrix} 2.79 \\ 1 \\ 1 \\ -4.79 \end{bmatrix}$, $\lambda_3 = 0, \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$, $\lambda_4 = -0.358, \vec{v}_4 = \begin{bmatrix} -1.79 \\ 1 \\ 1 \\ -0.21 \end{bmatrix}$.

Furthermore, $\begin{bmatrix} 20 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 20\vec{v}_1 + 4.367\vec{v}_2 - 4.367\vec{v}_4$.

The restaurant owner starts out with 20 DeliveryBots at the restaurant. **At steady state ($t \to \infty$), how many bots are in each of the locations?**

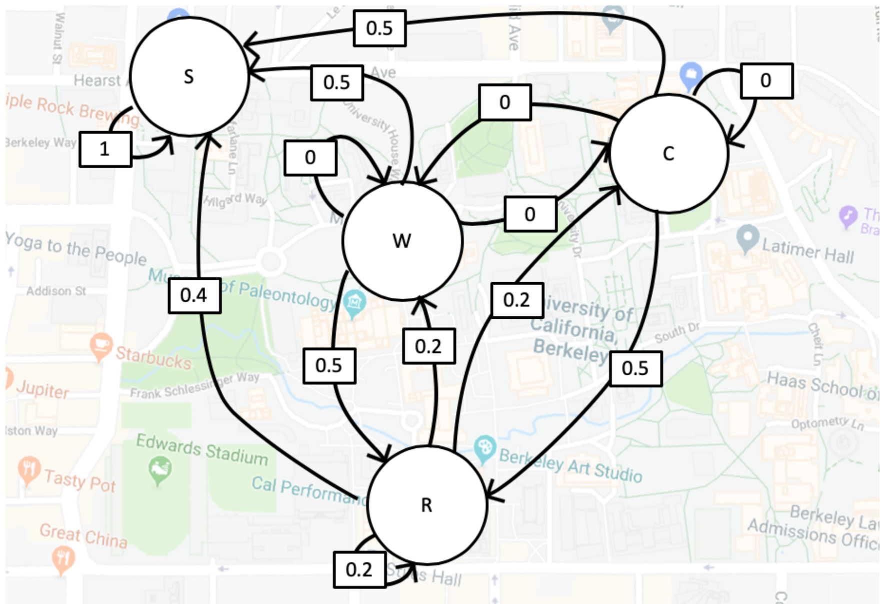


Figure 8.2: New DeliveryBot state transition diagram.

Solution: All 20 of the bots will be broken and end up at node S. The state vector will be $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 20 \end{bmatrix}$.

Applying the $\mathbf{T}$ matrix n times to $\vec{z}[0]$ yields $\lambda_1^n 20\vec{v}_1 + \lambda_2^n 4.367\vec{v}_2 - \lambda_4^n 4.367\vec{v}_4$. As n approaches infinity, the only component of $\vec{z}[0]$ that remains is $20\vec{v}_1$. This is because λ_2, λ_3, λ_4 whose magnitudes are less than one.

From a more intuitive approach, we can first note that the system is conservative. This means that the total number of bots stays the same at every timestep. At each timestep 0.4 of the bots at the restaurant, 0.5 of the bots at Wheeler, and 0.5 of the bots at Cory all go to node S. Once the bots are at node S, they cannot leave. In other words, a bot has a non-zero probability of being broken, but zero probability of getting fixed. Eventually as t becomes large, all the bots will be stuck at node S.

PRINT your name and student ID: ____

9. Proof (9 points)

Consider a square matrix **A**. Prove that if **A** has a non-trivial nullspace, i.e. if the nullspace of **A** contains more than just $\vec{0}$, then matrix **A** is not invertible.

Justify every step. Proofs that are not properly justified will not receive full credit. Simply invoking a theorem such as the "Invertible Matrix Theorem" will receive no credit.

Solution: We are given that the nullspace of **A** contains a vector other than $\vec{0}$. Let such a vector be $\vec{y} \neq \vec{0}$, where $A\vec{y} = \vec{0}$. Imagine, for the sake of contradiction, that **A** had an inverse $\mathbf{A}^{-1}$. Then we find that

$$\begin{aligned} & A\vec{y} = \vec{0} \\ \implies \quad & (A^{-1}A)\vec{y} = A^{-1}\vec{0} \\ \implies \quad & \vec{y} = \vec{0}, \end{aligned}$$

since by the definition of an inverse, $A^{-1}A = \mathbf{I}$.

But we said that $\vec{y} \neq \vec{0}$, so this is a contradiction! Therefore, our original hypothesis must have been false, so **A** cannot have an inverse.

Thus, the matrix **A** is not invertible.