

Midterm 2 Solution

**1. HONOR CODE**

Please read the following statements of the honor code, and sign your name (you don't need to copy it).

*I will respect my classmates and the integrity of this exam by following this honor code. I affirm:*

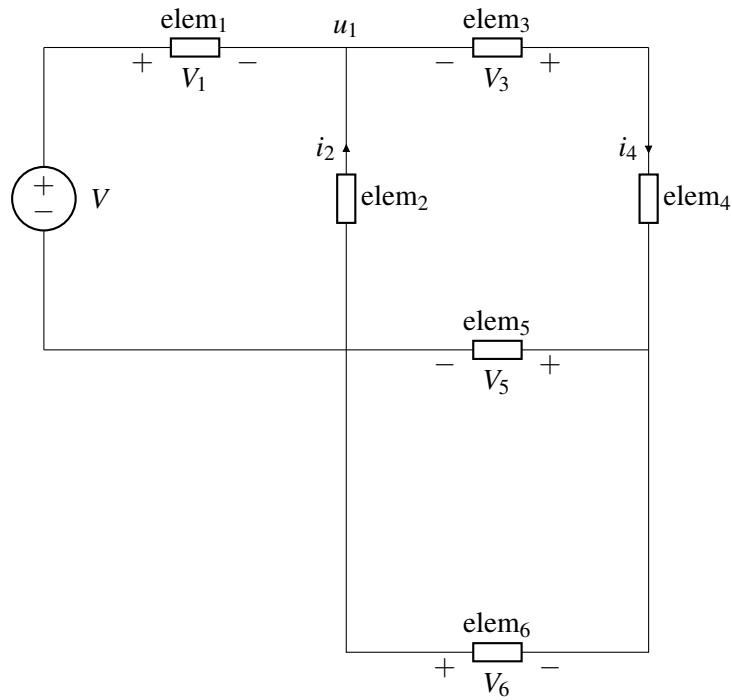
- *I have read the instructions for this exam. I understand them and will follow them.*
- *All of the work submitted here is my original work.*
- *I did not reference any sources other than my one reference cheat sheet.*
- *I did not collaborate with any other human being on this exam.*



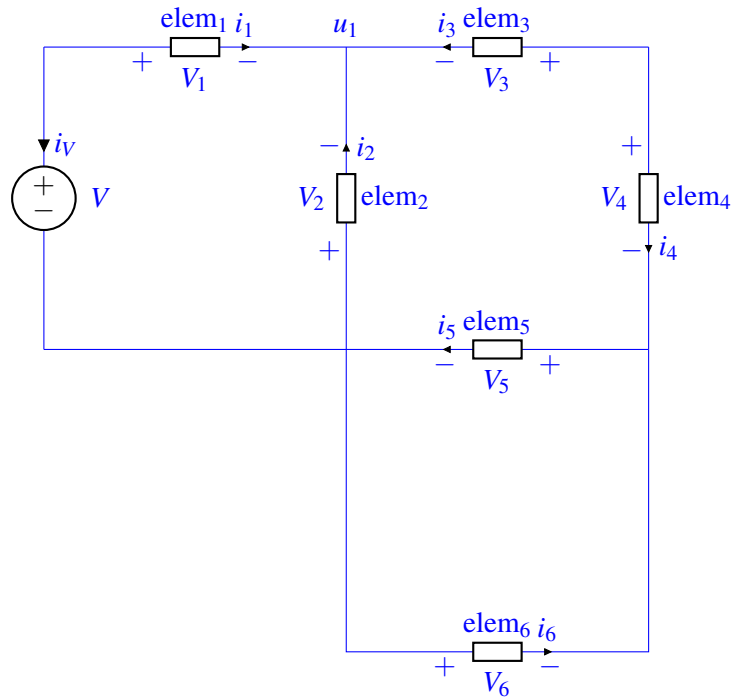
**2. Tell us about something that something you like about this class (1 point) All answers will be awarded full credit.**

### 3. KCL and KVL (3.5 points)

- (a) (1.5 points) Given the circuit below, label all the missing **element** voltages and currents using passive sign convention. You do not need to label the voltage source or node voltages.



**Solution:**



(b) (1 point) Using your labeled voltages/currents, write the KCL equation for node  $u_1$ .

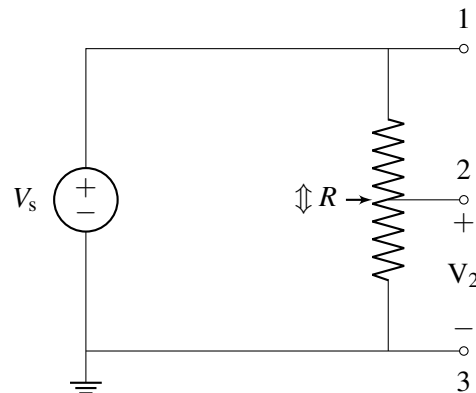
**Solution:**  $i_1 + i_2 + i_3 = 0$

(c) (1 point) Using your labeled voltages/currents, write the KVL equation for the loop containing  $\text{elem}_2$ ,  $\text{elem}_3$ ,  $\text{elem}_4$  and  $\text{elem}_5$ .

**Solution:**  $V_3 - V_2 - V_4 - V_5 = 0$

#### 4. Potentiometers (10 Points)

- (a) (2 points) A potentiometer is a component that can provide variable resistance. As shown in the figure, terminals 1 and 3 of the potentiometer are fixed, and the resistance between them is always equal to  $R$ . By moving the wiper at terminal 2 up and down, the length between terminals 1 & 2 and 2 & 3 vary, changing  $R_{12}$  (the resistance between terminals 1,2) and  $R_{23}$  (The resistance between terminals 2,3).



In the figure, the potentiometer is connected to a voltage source. Express the output voltage,  $V_2$  (the voltage at terminal 2), as a function of the source voltage  $V_s$ ,  $R_{12}$ , and  $R_{23}$ .

**Solution:** We can see that this circuit is essentially a voltage divider. The total resistance of the potentiometer remains constant, but is split up differently. Plugging in the appropriate resistances gives us:

$$V_2 = \frac{R_{23}}{R_{12} + R_{23}} V_s$$

- (b) (3 points) Now you are given a different type of variable resistor  $R_{var}$  with initial length  $L_0$  (mm), area  $A_0$  ( $\text{mm}^2$ ), and resistivity  $\rho$  ( $\Omega \cdot \text{mm}$ ). By squeezing the variable resistor, you decrease its length and increase its area: each squeeze of 1 mm will cause the *length* to decrease by 1 mm and the *area* to increase by 1/10 of  $A_0$ . Suppose the variable resistor is squeezed by  $d$  (mm), express the resistance  $R_{var}$  as a function of  $d$  and other variables in the problem.

**Solution:** Recall the general formula for resistance:

$$R = \rho \frac{L}{A}$$

Notice that, as a function of the 'squeezed' distance, the length varies as follows:

$$L(d) = L_0 - d$$

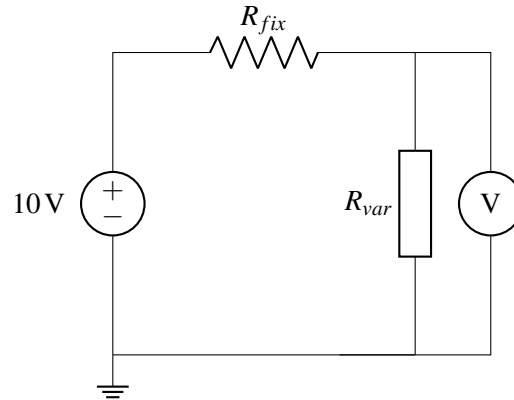
And similarly, the area varies as:

$$A(d) = A_0 + d(1/10)A_0$$

Plugging these in:

$$R(d) = \rho \frac{L_0 - d}{A_0 + dA_0/10}$$

- (c) (5 points) You would like to use a variable resistor as a pressure sensor. To do so you must choose an  $R_{fix}$  and setup the following circuit to measure the voltage change across  $R_{var}$  with a voltmeter. Assume initially  $R_{var}$  is  $2k\Omega$ . When the pressure increases,  $R_{var}$  becomes  $3k\Omega$ . You only have resistors of three values:  $2\Omega$ ,  $2k\Omega$ ,  $2M\Omega$  to choose for  $R_{fix}$ . If your voltmeter can only detect changes larger than  $0.5V$ , which one of these resistors would you choose? Justify your answer mathematically.



**Solution:** The circuit is a simple voltage divider, the voltmeter should measure

$$V_s \frac{R_{var}}{R_{fix} + R_{var}} = 10V \times \frac{2k\Omega}{R_{fix} + 2k\Omega}$$

After the pressure change, the difference in voltage that the voltmeter senses is

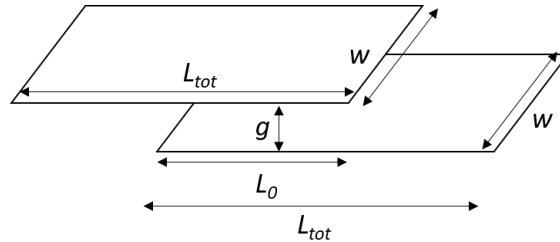
$$V_{diff} = V_s \left( \frac{R_{var,2}}{R_{fix} + R_{var,2}} - \frac{R_{var}}{R_{fix} + R_{var}} \right)$$

With  $R_{fix}$  respectively equal to  $2k\Omega$ ,  $2\Omega$  and  $2M\Omega$  the voltage differences due to pressure are  $1V$ ,  $3.3mV$  and  $4.98mV$ . Therefore, in order to have a detectable voltage ( $> 0.5V$ ), we must choose  $R_{fix} = 2k\Omega$

Intuitively, we must choose an  $R_{fix}$  similar to the value of the variable resistor to have the largest change of voltage in the voltage divider.

### 5. Physical Capacitors (5.5 points)

- (a) (1.5 point) A parallel plate capacitor with area,  $A$ , separated by a gap,  $g$ , and contains an insulating layer with permittivity,  $\epsilon$ , is shown below. The plates are perfectly aligned in the  $w$  direction. Express the capacitance with the given variables.

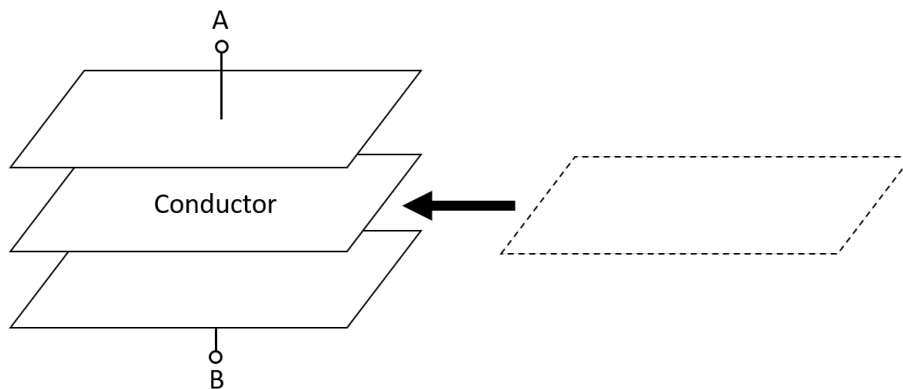


**Solution:**

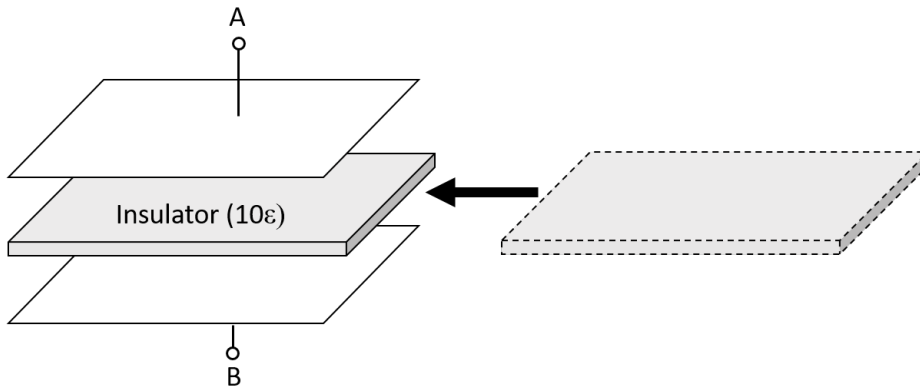
$$C = \epsilon \frac{w(L_0)}{g}$$

- (b) (4 points) For the capacitors in the following questions, assume air is between the plates, and permittivity of air is  $\epsilon$ . For each of the following statements, decide if it is true or false. Circle your answer and justify (intuitive arguments are valid) with a sentence or two.

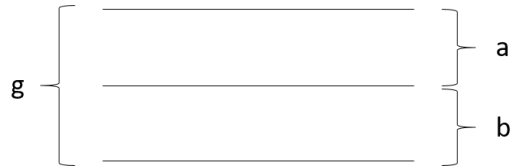
- i. Inserting an **infinitely thin, uncharged, conductive** plate between a parallel plate capacitor changes the overall capacitance between A and B.



- ii. Inserting a **finitely thick, uncharged, insulating** plate between a parallel plate capacitor changes the overall capacitance between A and B.



**Solution:** For the conductive plate, the capacitance does not change. A quick calculation shows this:



The initial capacitance without the middle plate is

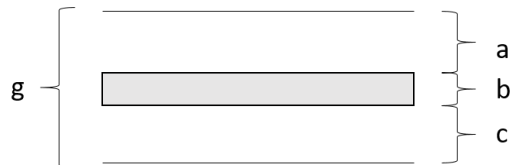
$$C = \epsilon \frac{A}{g}$$

The total capacitance with the middle plate is

$$\begin{aligned} \frac{1}{C} &= \frac{1}{\epsilon \frac{A}{a}} + \frac{1}{\epsilon \frac{A}{b}} \\ \frac{1}{C} &= \frac{a}{\epsilon A} + \frac{b}{\epsilon A} \\ \frac{1}{C} &= \frac{a+b}{\epsilon A} \\ C &= \frac{\epsilon A}{a+b} = \frac{\epsilon A}{g} \end{aligned}$$

Thus, the total capacitance is unchanged.

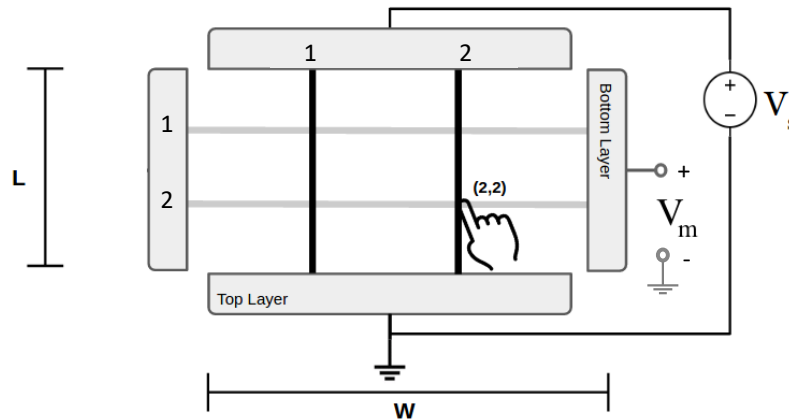
For the insulator, the capacitance does change. Intuitively, adding an insulator with a different permittivity than the original material changes how the electric field is distributed between the plates.



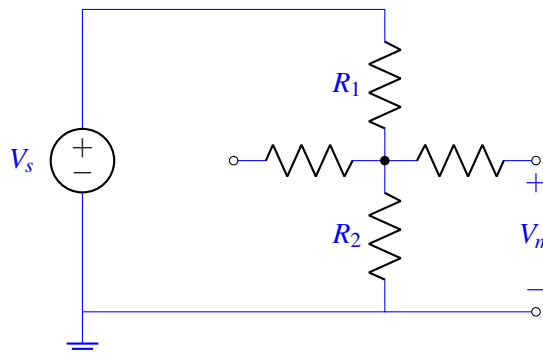
Clearly, if the permittivity inside the plates changes, the capacitance changes.

### 6. 2D Resistive Touch (11 points)

- (a) (3 points) Consider the 2x2 touchscreen below where the strips are equally spaced and connected by an ideal conducting plate on each end. There is no contact being made between overlapping strips except where the finger is touching. Draw an equivalent circuit diagram for when touch occurs at (2,2). Be sure to include  $V_s$ ,  $V_m$  in your diagram. You don't have to label the equivalent resistors.



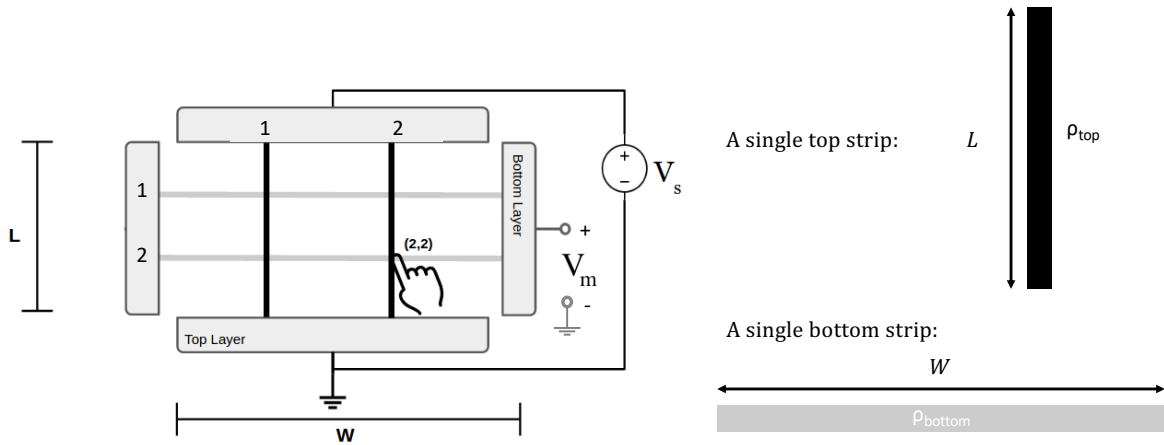
**Solution:** A general circuit diagram looks like this:



It is acceptable to exclude the horizontal resistor components as long as there is a connection to  $V_m$ . It is also not necessary to label the resistors at this point.

- (b) (3 points) In this resistive touchscreen, the top strips are made of a material with resistivity  $\rho_{top}$  and the bottom strips are made of a material with resistivity  $\rho_{bottom}$ .





What is  $V_m$  when touch occurs at (2,2)?

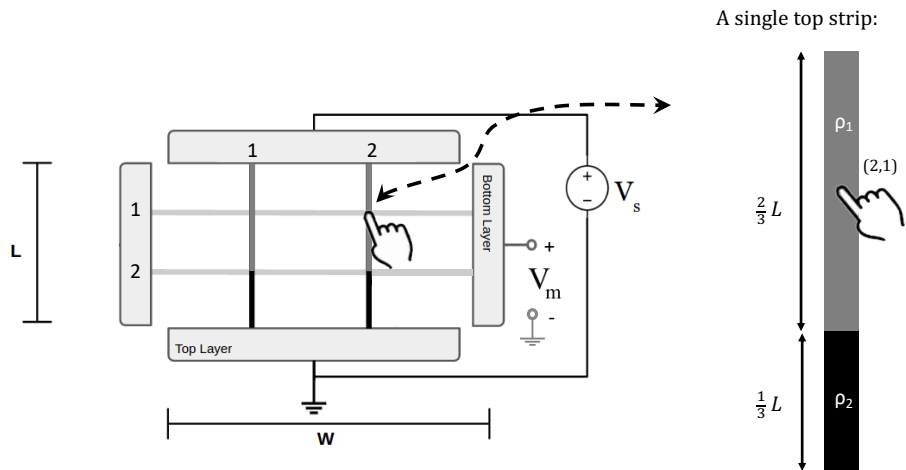
**Solution:** Looking at our circuit diagram in part a:

$$\begin{aligned}
 V_m &= \frac{R_2}{R_1 + R_2} \\
 &= \frac{\frac{\rho \frac{1}{3}L}{A}}{\frac{\rho \frac{1}{3}L}{A} + \frac{\rho \frac{2}{3}L}{A}} \\
 &= \frac{V_s}{3}
 \end{aligned}$$

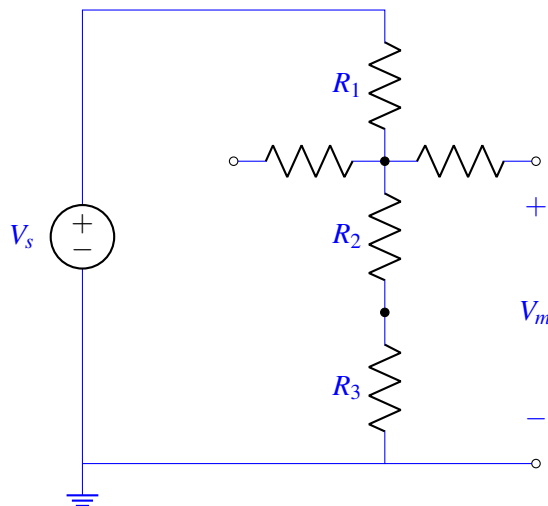
We see that the resistivity of the strip cancels out when calculating  $V_m$ , so having a different material for top and bottom strip has no effect on touch voltage. Thus,

$$V_m = \frac{V_s}{3}$$

- (c) (5 points) Now consider that the bottom strips are made from not one, but two materials such that the first two thirds of the strip has resistivity  $\rho_1$  and the last third of the strip has resistivity  $\rho_2$  as shown. What is  $V_m$  when touch occurs at (2,1)? You may express your answer in terms of the following variables:  $L$ ,  $W$ ,  $\rho_1$ ,  $\rho_2$  and  $V_s$ .



**Solution:** We can draw an equivalent circuit with three resistors:



We know the resistance is given by:

$$R_1 = \frac{\rho_1 \frac{1}{3}L}{A}$$

$$R_2 + R_3 = \frac{\rho_1 \frac{1}{3}L}{A} + \frac{\rho_2 \frac{1}{3}L}{A} = (\rho_1 + \rho_2) \frac{\frac{1}{3}L}{A}$$

We can use the same voltage divider method as before to derive  $V_m$ .

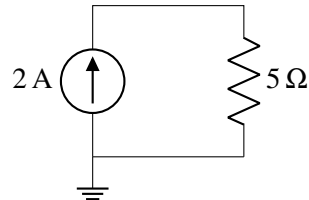
$$V_m = \frac{R_2 + R_3}{R_1 + R_2 + R_3} V_s$$

$$V_m = \frac{(\rho_1 + \rho_2) \frac{L}{3A}}{\frac{\rho_1 L}{3A} + (\rho_1 + \rho_2) \frac{L}{3A}} V_s$$

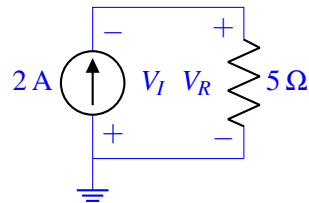
$$V_m = \frac{\rho_1 + \rho_2}{2\rho_1 + \rho_2} V_s$$

## 7. Non-Ideal Measurement (20 points)

- (a) (10 points) You are asked to find the power **supplied** by the current source in the circuit below. (Remember, the power supplied by an element to a circuit is the negative of the power dissipated by that element.)
- i. (2 points) Calculate the power supplied by the current source.

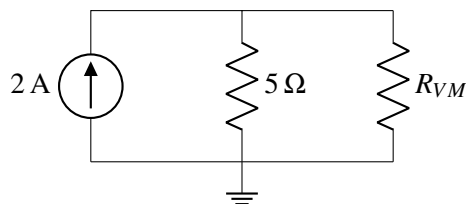


**Solution:** Recall that the power dissipated by a circuit element is equal to the element current times the element voltage, following passive sign convention. To apply this, we will label the circuit. One possible labeling is shown below:

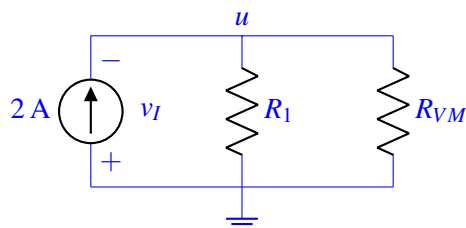


We know the current through the current source is 2 A. By KCL, the current through the resistor is also 2 A, and by Ohm's law the voltage across it is  $V_R = IR = 2A \times 5\Omega = 10V$ . By KVL, the voltage across the current source is  $V_I = -10V$ . Therefore, the power dissipated by the current source is  $P = IV = 2A \times -10V = -20W$ , so the power supplied by the source is 20 W.

- ii. (3 points) You would like to verify your calculation by measuring the voltage across the resistor. Unfortunately, you only have a **non-ideal** voltmeter. We will model the voltmeter as a resistor with resistance  $R_{VM}$ . If  $R_{VM}$  is  $5\Omega$ , how much power will the current source supply to the circuit?



**Solution:** Once we add the voltmeter, we can notice that  $R_{VM}$  is in parallel with  $R_1$ , so we can simplify the resistors to a single resistance  $\frac{R_1 R_{VM}}{R_1 + R_{VM}}$ . The equivalent resistance between node  $u$  (shown below) and ground is now  $R_1 || R_{VM}$ , which is  $\frac{5\Omega \cdot 5\Omega}{5\Omega + 5\Omega} = 2.5\Omega$ .



As above, we can use Ohm's law to find that the voltage at node  $u$  will be equal to  $2\text{ A} \times 2.5\ \Omega = 5\text{ V}$ . From this, we can derive that  $V_I = -u = -5\text{ V}$ , so the power dissipated by the source will be  $P = 2\text{ A} \times -5\text{ V} = -10\text{ W}$ . Finally, the power supplied by the source will be  $10\text{ W}$ .

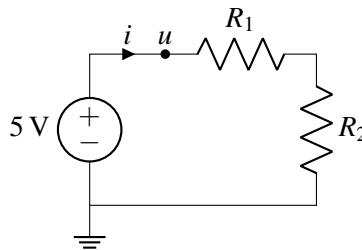
- iii. (5 points) Let  $P_0$  be the power supplied by the source if the voltmeter is not present in the circuit (i.e. your answer from part (i)), and  $P_I$  the power supplied after adding the non-ideal voltmeter (as shown in part (ii)). We would like to ensure  $P_I \geq 0.8 * P_0$ . You may again assume that the voltmeter can be modeled as a resistance  $R_{VM}$ . What constraint should you put on the value of  $R_{VM}$ ?

**Solution:** As before, we can apply Ohm's law to find that  $u = (2\text{ A}) * \frac{5\ \Omega * R_{VM}}{5\ \Omega + R_{VM}} = \frac{10VR_{VM}}{5\ \Omega + R_{VM}}$ , and  $P_I = V * I = (2\text{ A}) * u = 2\text{ A} * \frac{10V * R_{VM}}{5\ \Omega + R_{VM}}$ . This expression will decrease as  $R_{VM}$  decreases, so if we want this value to be greater than a certain threshold we need to set a lower bound on  $R_{VM}$ . To find the lower bound, we can set  $P_I$  equal to  $0.8P_0$  and solve for  $R_{VM}$ .

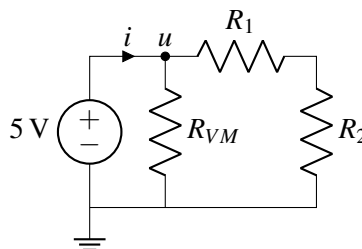
$$\begin{aligned} P_I &= 0.8P_0 \\ \frac{20\text{ W} * R_{VM}}{5\ \Omega + R_{VM}} &= 0.8 * 20\text{ W} \\ \frac{R_{VM}}{5\ \Omega + R_{VM}} &= 0.8 \\ R_{VM} &= 0.8(5\ \Omega + R_{VM}) \\ 0.2R_{VM} &= 0.8 * 5\ \Omega = 4\ \Omega \\ R_{VM} &= 20\ \Omega \end{aligned}$$

Therefore, in order to ensure  $P_I \geq 0.8P_0$ , we must set the constraint  $R_{VM} \geq 20\ \Omega$ .

- (b) (4 points) You want to measure the voltage at node  $u$  in the following circuit:



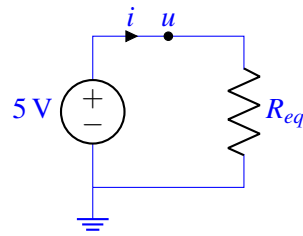
You decide to attach a **non-ideal** voltmeter into the circuit below.



How will adding the voltmeter change the voltage at node  $u$  and the current  $i$ ? Circle your answer, and justify with math or a brief written explanation.

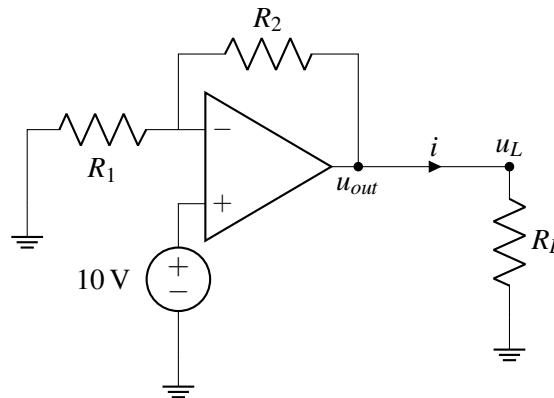
**Solution:** The voltage at  $u$  is being set by the voltage source directly, so it will always be equal to  $5\text{ V}$ . Therefore,  $u$  will not change.

$i$  is equal to the current supplied by the voltage source. To see how the current  $i$  changes, we can simplify all the resistors into a single equivalent resistor as shown below.

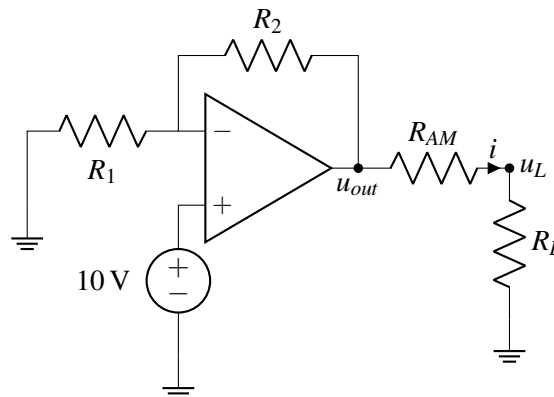


Without the voltmeter, we just have two resistors in series, so  $R_{eq} = (R_1 + R_2)$ . Adding the voltmeter is equivalent to adding a resistor in parallel, which will reduce the total equivalent resistance of the circuit to  $R_{VM} || (R_1 + R_2)$ . By Ohm's law, if the overall resistance decreases and the voltage across  $R_{eq}$  stays the same, the current supplied to  $R_{eq}$  must increase. Therefore,  $i$  will increase.

(c) (6 points) Now you want to measure the current in the following circuit:



You attach a **non-ideal** ammeter, which can be modeled as a resistor with some finite non-zero resistance  $R_{AM}$ , and the resulting circuit is shown below:



How will adding the ammeter change the following quantities? You may assume the op amp is ideal. Circle your answer and justify with math or a brief written explanation.

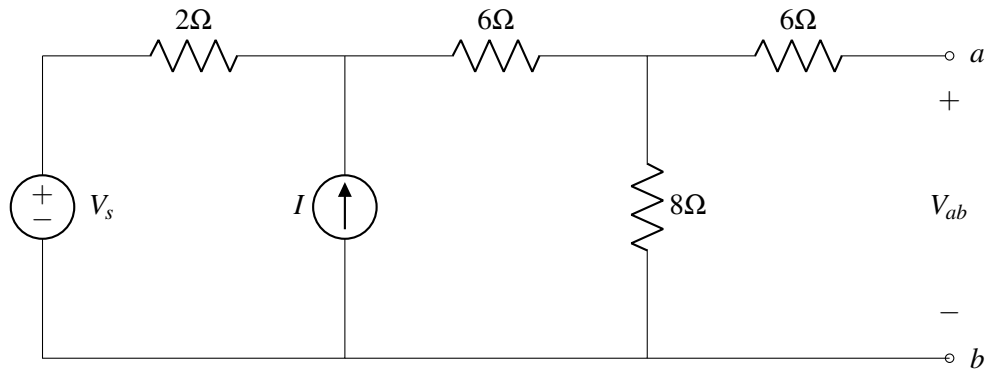
**Solution:** Remember that an ideal op amp can be modeled as a dependent voltage source. Because the ammeter is attached after the output of the op amp, it won't affect the voltage supplied by the dependent source. Therefore,  $u_{out}$  will stay the same.

Before adding the ammeter,  $u_L$  will be equal to  $u_{out}$ , because they are labeling the same node. However, after adding the ammeter, we notice that  $u_L$  is the output of a voltage divider with  $u_{out}$  as the input voltage, so the magnitude of  $u_L$  will be less than the magnitude of  $u_{out}$ . Therefore,  $u_L$  will decrease.

By adding  $R_{AM}$  in series, we are essentially increasing the equivalent resistance of the load at the output of the op amp. As shown above, the voltage  $u_{out}$  supplied to the load will remain the same, so by Ohm's law the current through the load must decrease. Therefore,  $i$  will decrease.

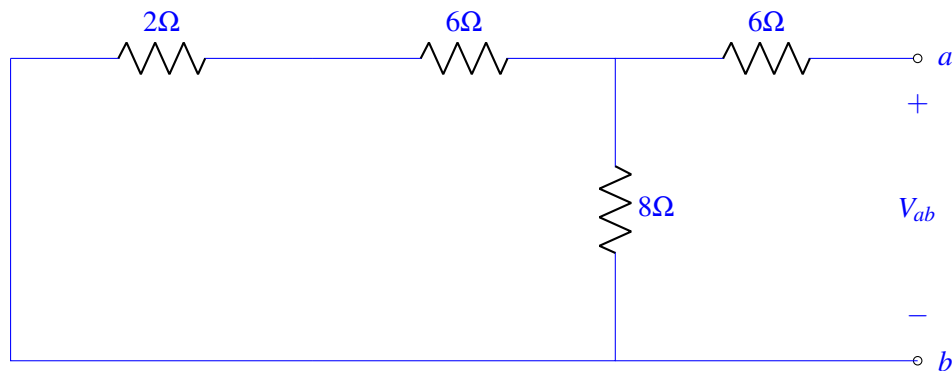
### 8. Equivalence and Superposition (15 points)

(a) (12 points)



i. (4 points) Find the Thevenin equivalent resistance between terminals  $a$  and  $b$ .

**Solution:** We first need to zero-out any independent sources:



From here we can calculate the Thevenin equivalent resistance as:

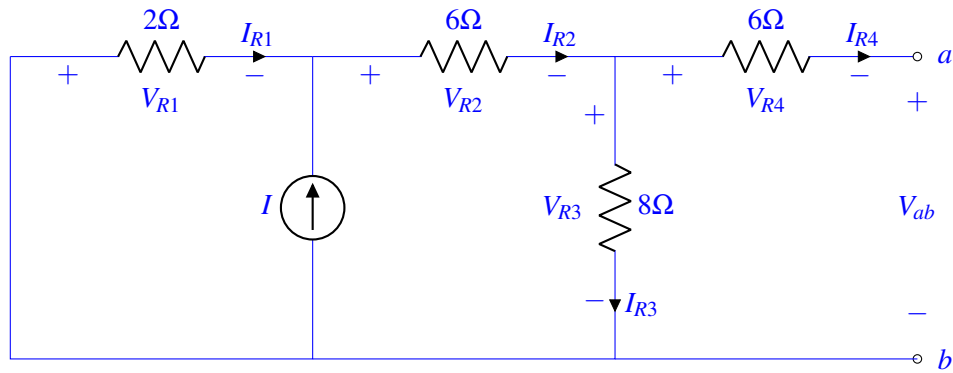
$$R_{th} = (8 \parallel (2 + 6))\Omega + 6\Omega = \frac{8 \cdot 8}{8 + 8}\Omega + 6\Omega = 4\Omega + 6\Omega = 10\Omega$$

$$R_{th} = 10\Omega$$

ii. (8 points) Use superposition to find the Thevenin equivalent voltage between terminals  $a$ ,  $b$  as a function of  $I$  and  $V_s$ .

**Solution:**

i. Nulling all sources except  $I$ :



Using KCL we can write the following equations:

$$I_{R2} = I + I_{R1}$$

$$I_{R4} = 0$$

$$I_{R2} = I_{R3}$$

Next, we can use IV relationships and NVA to write:

$$V_{ab} = V_{R3} = 8\Omega \cdot I_{R3}$$

$$V_{R2} + V_{R3} = -V_{R1}$$

$$6\Omega \cdot I_{R2} + 8\Omega \cdot I_{R3} = -2\Omega \cdot I_{R1}$$

Combining the two sets of equations:

$$14\Omega \cdot I_{R2} = -2\Omega \cdot I_{R1}$$

$$14\Omega \cdot I_{R2} = -2\Omega \cdot I_{R2} + 2\Omega \cdot I$$

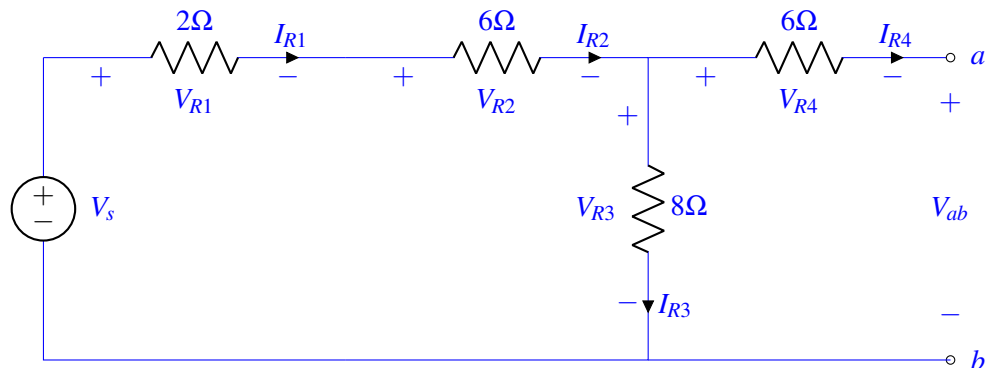
$$16\Omega \cdot I_{R2} = 2\Omega \cdot I$$

$$I_{R2} = \frac{I}{8} = I_{R3}$$

Which gives us the final result:

$$V_{ab} = 8\Omega \cdot I_{R3} = 8\Omega \cdot \frac{I}{8} A = I \text{ Volts}$$

ii. Nulling all sources except  $V_s$ :





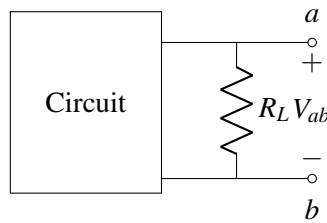
From a voltage divider equation we can write:

$$V_{ab} = \frac{8}{8+2+6}V_s = \frac{1}{2}V_s$$

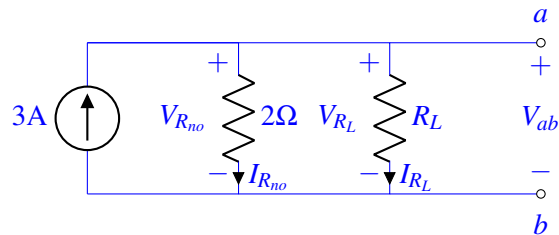
iii. By combining the results we obtained from analyzing each source separately we can write:

$$V_{th} = I + \frac{1}{2}V_s$$

- (b) (3 points) Now you are given a new mystery circuit! You connect a load resistor  $R_L$  to your circuit, as shown below. You know that the Norton equivalent circuit from nodes  $a$  and  $b$  has  $I_{no} = 3A$  and  $R_{no} = 2\Omega$ . Find the value of  $R_L$  such that  $V_{ab} = 4V$ .



**Solution:** The norton equivalent circuit looks like:



$$V_{ab} = 3A \cdot (2 \parallel R_L)\Omega = 3A \cdot \frac{2R_L}{2+R_L}\Omega = 4V$$

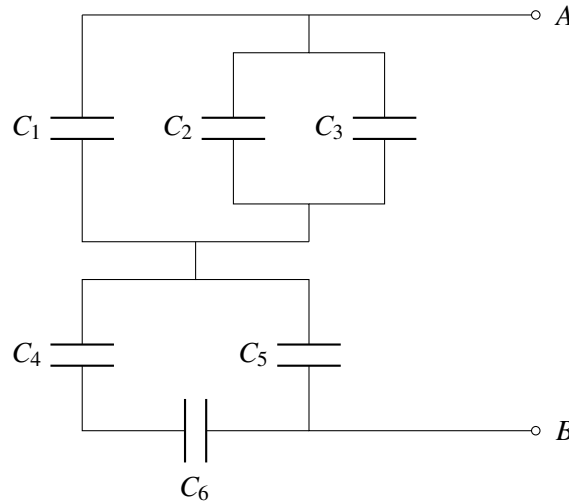
$$6R_L = 8 + 4R_L$$

$$2R_L = 8\Omega$$

$$R_L = 4\Omega$$

### 9. Capacitors in the Wild (10 points)

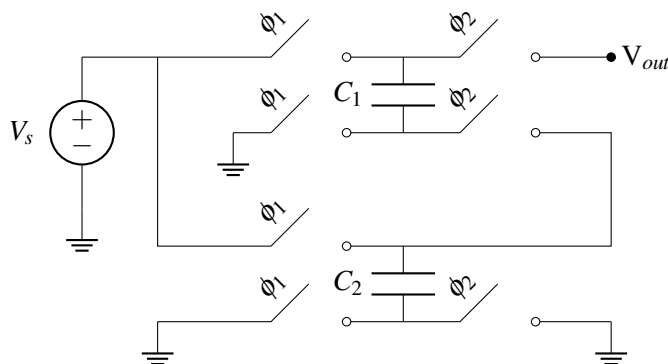
- (a) (2 points) Find the equivalent capacitance between nodes  $A$  and  $B$  in terms of each of the capacitors,  $C_i$ . You can use the parallel operator ( $\parallel$ ) for simplification.



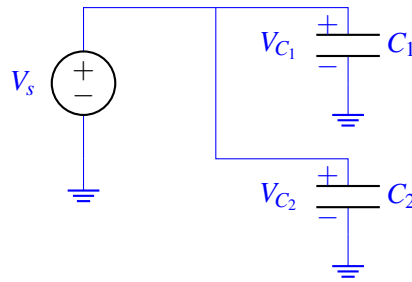
**Solution:**

$$C_{eq} = (C_1 + C_2 + C_3) \parallel (C_4 \parallel C_6 + C_5)$$

- (b) (8 points) In the following circuit, switches labeled  $\phi_1$  are closed during phase 1 and open during phase 2; switches labeled  $\phi_2$  are closed during phase 2 and open during phase 1. Assume before phase 1 both capacitors are fully discharged. Each phase is long enough that the circuit reaches steady-state. Find  $V_{out}$  in phase 2 in terms of  $V_s$ ,  $C_1$  and  $C_2$ .



**Solution:** First we draw the equivalent circuit diagrams in both phases. Phase 1:

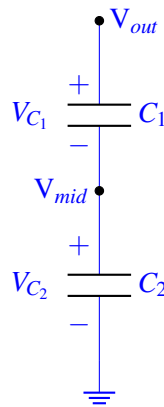


In this circuit,  $C_1$  and  $C_2$  are in parallel, and the voltage across both capacitors is  $V_s$ . Therefore, the magnitude of charges during phase 1 on both capacitors are

$$Q_{1,1} = C_1 \times V_s$$

$$Q_{2,1} = C_2 \times V_s$$

During phase 2, the circuit looks like the following:



Now, the voltage across  $C_1$  is  $V_{out} - V_{mid}$ , and the voltage across  $C_2$  is  $V_{mid}$ . Therefore, calculating the magnitude of charge on both capacitors during phase 2:

$$Q_{1,2} = C_1 \times (V_{out} - V_{mid})$$

$$Q_{2,2} = C_2 \times V_{mid}$$

Notice that there are two floating nodes in phase 2:  $V_{out}$  and  $V_{mid}$ . Therefore, we write charge conservation for both nodes:

For  $V_{out}$ , we only have positive charges from  $C_1$ :

$$C_1 V_s = (V_{out} - V_{mid}) C_1 \quad (1)$$

For  $V_{mid}$ , negative charge on  $C_2$  and positive charge on  $C_1$  are conserved:

$$-V_s C_1 + V_s C_2 = -(V_{out} - V_{mid}) C_1 + V_{mid} C_2 \quad (2)$$

By substituting equation (1) into equation (2), we find

$$-(V_{out} - V_{mid}) C_1 + V_s C_2 = -(V_{out} - V_{mid}) C_1 + V_{mid} C_2$$

$$V_s C_2 = V_{mid} C_2$$

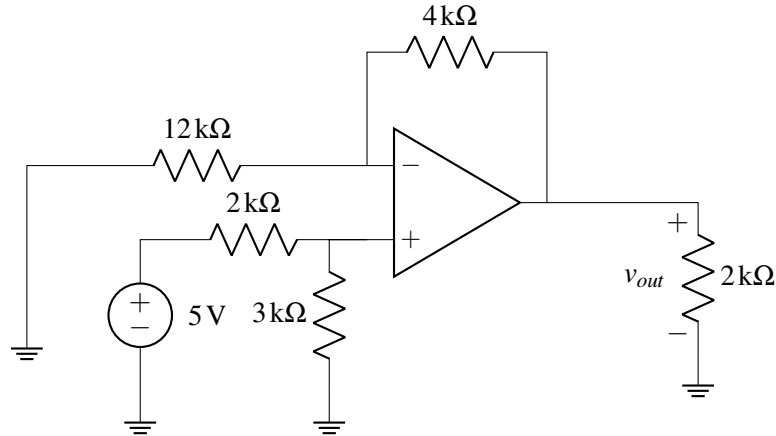
Plugging back into equation (1):

$$C_1 V_s = (V_{out} - V_s) C_1$$

$$V_{out} = 2V_s$$

### 10. Op-amps and Comparators (9 points)

- (a) (3 points) You are given the following op-amp in negative feedback. Find  $v_{out}$ .



**Solution:** First, we can find  $V^+$  using a voltage divider.

$$V^+ = 5V * \frac{3k\Omega}{2k\Omega + 3k\Omega}$$

$$V^+ = 3V$$

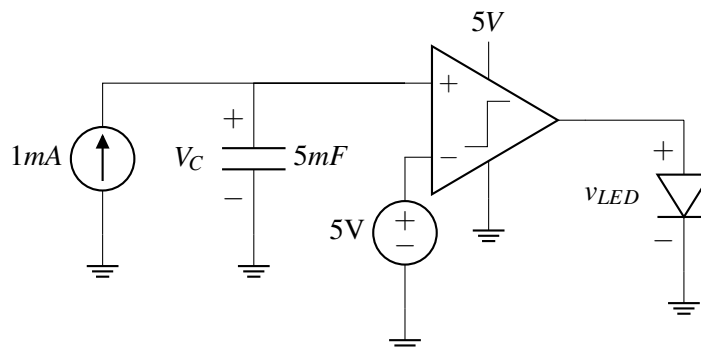
As op-amp is in negative feedback and it is an ideal opamp, by golden rules,  $V^- = 3V$  too.

Next, using node voltage analysis at  $v^-$ , we can write:

$$\frac{V^- - 0}{12k\Omega} + \frac{V^- - v_{out}}{4k\Omega} = 0$$

Using  $V^- = 3V$ , we solve that  $v_{out} = 4V$ .

- (b) (3 points) You are given the circuit below. The capacitor is initially uncharged. At time  $t = 0$ , the current source is turned on. Find  $V_C(t)$ .



**Solution:** We know that the IV relationship of a capacitor is  $I = C \frac{dV}{dt}$ . By integrating both sides, we can derive that

$$V_C(t) = \frac{I * t}{C} + V(0)$$

In this case, we know  $V(0) = 0V$  because it is initially uncharged. Thus,  $V(t) = \frac{1mA}{5mF}t = 0.2t$ .

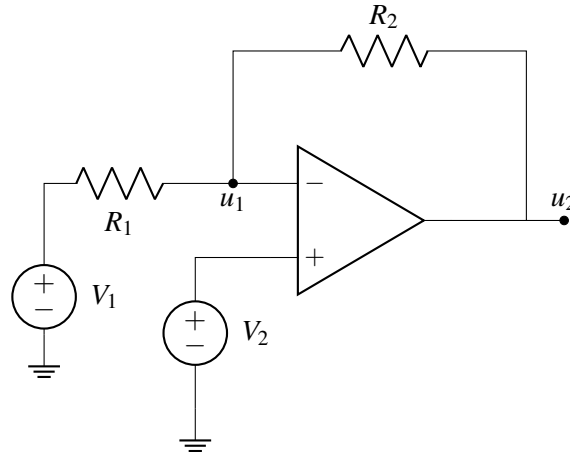
- (c) (3 points) The LED turns on when the voltage across it is greater than 3.3V. Using the same setup as part (b), at what time  $t$  does the LED turn on?

**Solution:** When  $V_+$  is greater than  $V_-$  for the comparator, the op-amp will output  $V_{dd}$  (5V), which will turn on the LED. In order to have  $V_+ > V_-$ ,  $V(t) > 5V$ . This happens when  $V_C = 5V$ ,  $0.2t = 5$ , so  $t = 25s$ .

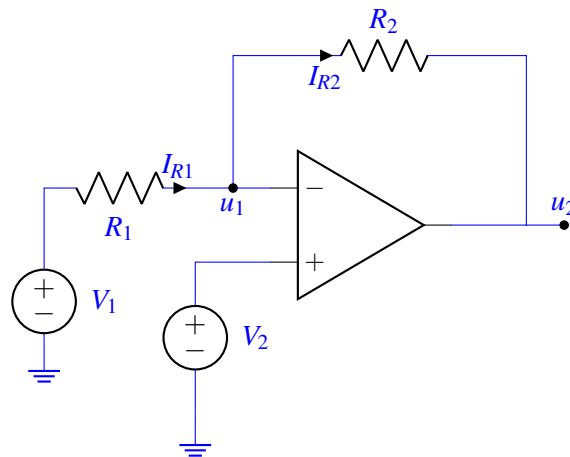
### 11. Eigen Circuits (19 points)

(a) (7 points) You are given the following op-amp circuit in negative feedback.

i. (5 points) Express  $u_1$  and  $u_2$  in terms of  $V_1$ ,  $V_2$ ,  $R_1$  and  $R_2$ . Assume the op-amp is ideal.



**Solution:** First, we label node voltages and element currents



By properties of an ideal op-amp in negative feedback (Golden rules), the positive and negative terminals have the same voltage. Therefore,

$$u_1 = V_2$$

Writing out KCL at node  $u_1$ , we get

$$I_{R2} = I_{R1}$$

$$\frac{V_1 - u_1}{R_1} = \frac{u_1 - u_2}{R_2}$$

replacing  $u_1$  with  $V_2$

$$\frac{V_1 - V_2}{R_1} = \frac{V_2 - u_2}{R_2}$$

$$\frac{V_1}{R_1} - \frac{V_2}{R_1} = \frac{V_2}{R_2} - \frac{u_2}{R_2}$$

Rearranging,

$$\begin{aligned}\frac{u_2}{R_2} &= \frac{V_2}{R_2} + \frac{V_2}{R_1} - \frac{V_1}{R_2} \\ u_2 &= V_2 + V_2\left(\frac{R_2}{R_1}\right) - V_1\left(\frac{R_2}{R_1}\right) \\ u_2 &= V_2\left(\frac{R_1 + R_2}{R_1}\right) - V_1\left(\frac{R_2}{R_1}\right)\end{aligned}$$

- ii. (2 points) Write the above equations in the form of  $A\vec{x} = \vec{b}$ .  $\vec{x}$  is  $\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$  and  $\vec{b}$  is  $\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ .

**Solution:** From the above equations for  $u_1$  and  $u_2$  based on  $V_1$  and  $V_2$ , we can get

$$\begin{bmatrix} 0 & 1 \\ -\frac{R_2}{R_1} & \frac{R_1 + R_2}{R_2} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad (1)$$

- (b) (6 points) Assume that after plugging in values for  $R_1$  and  $R_2$ , the matrix  $\mathbf{A}$  you get is  $\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$ , find the eigenvalue(s) and the eigenvector(s) of this matrix.

**Solution:** To solve for eigenvalues, we find the  $\lambda$ 's such that determinant of  $(A - \lambda I)$  is 0.

$\begin{bmatrix} 0 - \lambda & 1 \\ -2 & 3 - \lambda \end{bmatrix}$  has determinant  $-\lambda(3 - \lambda) + 2$ . Setting  $-\lambda(3 - \lambda) + 2 = 0$  gives us  $\lambda_1 = 1$ ,  $\lambda_2 = 2$ .

When  $\lambda_1 = 1$

$$\left[ \begin{array}{cc|c} -1 & 1 & 0 \\ -2 & 2 & 0 \end{array} \right] \Rightarrow \left[ \begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

The eigenvector is  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ .

When  $\lambda_1 = 2$

$$\left[ \begin{array}{cc|c} -2 & 1 & 0 \\ -2 & 1 & 0 \end{array} \right] \Rightarrow \left[ \begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

The eigenvector is  $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ .

- (c) (6 points) Assume you are given another linear circuit with inputs  $V_1$  and  $V_2$  and outputs  $u_1$  and  $u_2$ . The eigenvalues for this circuit are  $\lambda_1 = 1$  and  $\lambda_2 = 2$ . The corresponding eigenvectors are  $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,

$\vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ . If we want  $u_1 = 3V$ ,  $u_2 = 7V$ , what would the input values be?

**Solution:** Since  $\vec{v}_1$  and  $\vec{v}_2$  are linearly independent,  $\text{span}\{v_1, v_2\} = \mathbb{R}^2$ , any input  $\vec{x}$  can be represented as a linear combination of  $\vec{v}_1$  and  $\vec{v}_2$ , i.e.  $\vec{x} = \alpha\vec{v}_1 + \beta\vec{v}_2$ . By definition of eigen value and eigen vectors, we can write

$$\begin{aligned}A\vec{x} &= \lambda\vec{x} \\A(\alpha\vec{v}_1 + \beta\vec{v}_2) &= \alpha A\vec{v}_1 + \beta A\vec{v}_2 \\&= \alpha\lambda_1\vec{v}_1 + \beta\lambda_2\vec{v}_2 \\&= \alpha \cdot 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \cdot 2 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} \\&= \alpha \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \cdot \begin{bmatrix} 2 \\ 6 \end{bmatrix}\end{aligned}$$

For output  $\begin{bmatrix} 3 \\ 7 \end{bmatrix}$  we pick  $\alpha = 1$ ,  $\beta = 1$ ,  $\vec{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ .