EECS 16A Designing Information Devices and Systems I
Summer 2020 Final Exam Instructions

Read the following instructions before the exam.

Format & How to Submit Answers

There are 16 problems (4 introductory questions and 12 exam questions) of varying numbers of points. The problems are of varying difficulty, so pace yourself accordingly and avoid spending too much time on any one question until you have gotten all of the other points you can. Don't get bogged down in calculations; if you are having trouble with one problem, there may be easier points available later in the exam!

All answers will be submitted to the Gradescope "Final Exam" Assignment (`https://www.gradescope.com/courses/137582/assignments/593619`). Questions are multiple choice, short answer, or free response. **There are 210 points possible on the exam, but your final score will be taken out of 195 points**. This means that a score of 160/210, normally 76%, will be bumped up to 160/195, or 82%. You cannot score more than 100% on this exam.

Partial credit may be given for certain incorrect answer choices for some multiple choice problems. There is no penalty for incorrect answers. Partial credit will be available for most free response questions.

Post any content or clarifying questions privately on Piazza. There will be no exam clarifications; if we find a bug on the exam, that sub-question will be omitted from grading.

Free Response Questions: You will scan and submit 9 free response subparts to the "Final Exam" assignment as you would a homework assignment. You should plan to have a stable internet connection during the three-hour exam period, as it will be necessary to save and submit your work. You are not required to print anything out to take this exam; you are required to have something to write on, something to write with, and a way to scan written work to a PDF or image file on a desktop or laptop computer.

If you'd like, you can do all of the free response questions on the same PDF and upload it (with all free response answers) to each of the free response subparts - just make sure each subpart is clearly labeled. If you're having trouble uploading your answers to Gradescope, you can also email them to eecs16a@berkeley.edu, before the exam period (see below) is over.

We recommend you test your system for uploading a handwritten image before the Final Exam begins. Please refer to this YouTube video for instructions on uploading your PDF or image files. `https://youtu.be/j_kha2UkeMY?t=144`

Timing & Penalties

You have 180 minutes for the exam, with a 5 minute grace period. After the 5 minute grace period ends, exam scores will be penalized exponentially as follows: an exam that is submitted N minutes after the end of the grace period will lose 2^N points. **The exam will become available at your personalized link at 7:10 pm PT; the grace period will expire at 10:15 pm PT.** If your submission is timestamped at 10:16 pm PT, you will lose 2 points; if it is timestamped at 10:18 pm PT, you will lose 8 points.

We will count the latest time at which you submit any question as your exam timestamp. Do NOT edit or resubmit your answers after the deadline. We recommend having all of your answers input and submitted by 10:10 pm; it is your responsibility to submit the exam on time.

If you cannot access your exam at your link by 7:15 pm, please email eecs16a@berkeley.edu. If you are having technical difficulties submitting your exam, you can email your answers (either typed or scanned) to eecs16a@berkeley.edu.

Academic Honesty

This is an open-note, open-book, open-internet, and **closed-neighbor** exam. You may use any calculator or calculation software that you wish, including Wolfram-Alpha and Mathematica. **No collaboration is allowed, and do not attempt to cheat in any way. Cheating will not be tolerated.**

We have zero tolerance against violation of the Berkeley Honor Code. Given supporting evidence of cheating, we reserve the right to automatically fail all students involved and report the instance to the student conduct committee. We reserve the right to **audit students with oral exams** after the midterm to ensure academic honesty. Feel free to report suspicious activity through this form. (`https://forms.gle/ZzXLksZEmx9bn1mj7`).

Our advice to you: if you can't solve a particular problem, move on to another, or state and solve a simpler one that captures at least some of its essence. You will perhaps find yourself on a path to the solution.

Good luck!

EECS 16A Designing Information Devices and Systems I
Summer 2020 Midterm 2

1. **Pledge of Academic Integrity (2 points)**
   By my honor, I affirm that:

   (1) this document, which I will produce for the evaluation of my performance, will reflect my original, bona fide work;

   (2) as a member of the UC Berkeley community, I have acted and will act with honesty, integrity, and respect for others;

   (3) I have not violated—nor aided or abetted anyone else to violate—nor will I—the instructions for this exam given by the course staff, including, but not limited to, those on the cover page of this document; and

   (4) I have not committed, nor will I commit, any act that violates—nor aided or abetted anyone else to violate—the UC Berkeley Code of Student Conduct.

   Write your name and the current date as an acknowledgement of the above. (See Gradescope)

2. **Administrivia (1 point)**
   I know that I will lose 2^n points for every n minutes I submit after the exam submission grace period is over.

   For example, if the exam becomes available at my personalized link at 7:10 p.m. PT; the grace period will expire at 10:15 p.m. PT. If my submission is timestamped at 10:16 p.m. PT, I will lose 2 points; if it is timestamped at 10:18 p.m. PT, I will lose 8 points.

   ○ Yes

3. **What have you enjoyed most about EECS16A? (2 points)**

4. **Tell us about something you're looking forward to! (2 points)**

5. Least Squares Fitting (9 points)

In an upward career move, you join the starship USS Enterprise as a data scientist. One morning the Chief Science Officer, Mr. Spock, hands you some data for the position (y) of a newly discovered particle at different times (t). The data has four points and **contains some noise**:

$$(t = 0, y = 5), \quad (t = 1, y = 7.1), \quad (t = 2, y = 12.5), \quad (t = 3, y = 19)$$

Your research shows that the path of the particle is represented by the function:

$$y = w_1 \, \mathrm{e}^t + w_2 \, \mathrm{e}^{-2t} + \sin(t) + w_3 t \tag{1}$$

You decide to fit the collected data to the function in Equation (1) using the Least Squares method.

(a) (3 points) You need to find the coefficients w_1, w_2, and w_3 that *minimize the squared error* between the fitted curve and the collected data points. So you set up a system of linear equations, $\mathbf{A}\vec{\hat{\alpha}} \approx \vec{b}$ in order to find the approximate value of $\vec{\hat{\alpha}} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$. Select the correct setup.

(A) $\begin{bmatrix} \mathrm{e}^0 & \mathrm{e}^0 & \sin(0)+0 \\ \mathrm{e}^1 & \mathrm{e}^{-2} & \sin(1)+1 \\ \mathrm{e}^2 & \mathrm{e}^{-4} & \sin(2)+2 \\ \mathrm{e}^3 & \mathrm{e}^{-6} & \sin(3)+3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \approx \begin{bmatrix} 5 \\ 7.1 \\ 12.5 \\ 19 \end{bmatrix}$

(B) None of these options are correct.

(C) $\begin{bmatrix} \mathrm{e}^0 & \mathrm{e}^0 & \sin(0) & 0 \\ \mathrm{e}^1 & \mathrm{e}^{-2} & \sin(1) & 1 \\ \mathrm{e}^2 & \mathrm{e}^{-4} & \sin(2) & 2 \\ \mathrm{e}^3 & \mathrm{e}^{-6} & \sin(3) & 3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \approx \begin{bmatrix} 5 \\ 7.1 \\ 12.5 \\ 19 \end{bmatrix}$

(D) $\begin{bmatrix} \mathrm{e}^0 & \mathrm{e}^0 & 0 \\ \mathrm{e}^1 & \mathrm{e}^{-2} & 1 \\ \mathrm{e}^2 & \mathrm{e}^{-4} & 2 \\ \mathrm{e}^3 & \mathrm{e}^{-6} & 3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \approx \begin{bmatrix} 5 \\ 7.1 \\ 12.5 \\ 19 \end{bmatrix}$

(E) $\begin{bmatrix} \mathrm{e}^0 & \mathrm{e}^0 & 0 \\ \mathrm{e}^1 & \mathrm{e}^{-2} & 1 \\ \mathrm{e}^2 & \mathrm{e}^{-4} & 2 \\ \mathrm{e}^3 & \mathrm{e}^{-6} & 3 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \approx \begin{bmatrix} 5-\sin(0) \\ 7.1-\sin(1) \\ 12.5-\sin(2) \\ 19-\sin(3) \end{bmatrix}$

(b) (3 points) Mr. Spock thinks one of the data points is wrong and asks you to redo the fit with only three data points. What will happen to the norm of the error, $\|\vec{e}\| = \left\|\vec{b} - \mathbf{A}\hat{\vec{\alpha}}\right\|$?

(A) There will be an infinite number of possible fits.

(B) $\|\vec{e}\|$ will be zero.

(C) $\|\vec{e}\|$ will decrease, but will have a non-zero value.

(D) The data cannot be fit.

(E) $\|\vec{e}\|$ will increase.

(F) $\|\vec{e}\|$ might increase or decrease. There is not enough information to tell.

(c) (3 points) Your colleague tries to repeat your fitting process with the same four data points in part (a), but they misread the equation relating t and y, i.e. they use the following function (which is **different than part (a)**):

$$y = w_1 \mathrm{e}^t + w_2 \mathrm{e}^t + \sin(t) + w_3 t \tag{2}$$

Your colleague tries to find w_1, w_2 and w_3 by setting up a system of equations $\mathbf{A}\hat{\vec{\alpha}} \approx \vec{b}$ and utilizing the equation:

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \hat{\vec{\alpha}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{b}. \tag{3}$$

What is the most likely outcome of Equation (3)?

(A) They won't find $\hat{\vec{\alpha}}$, because the new equation does not correspond to the data.

(B) They will find $\hat{\vec{\alpha}}$, and $\|\vec{e}\|$ will be zero.

(C) They won't find $\hat{\vec{\alpha}}$, because $\mathbf{A}$ has linearly dependent columns.

(D) They will find $\hat{\vec{\alpha}}$, but $\|\vec{e}\|$ will be larger than part (a).

(E) There is not enough information to decide the outcome.

6. Project Projections (9 points)

(a) (3 points) Find $\text{proj}_{\mathbb{S}}(\vec{v})$, the projection of vector $\vec{v}$ onto the subspace $\mathbb{S}$.

$$\vec{v} = \begin{bmatrix} 3 \\ 9 \\ 0 \end{bmatrix} \qquad \mathbb{S} = \text{span}\left\{ \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \right\}$$

(b) (3 points) Let $\hat{\vec{v}}$ be an approximation of $\vec{v}$. We compute different approximations using the expressions below:

$$\vec{v} = \begin{bmatrix} 6 \\ 10 \\ -7 \end{bmatrix}, \quad \mathbb{S}_1 = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ -8 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \right\}, \quad \mathbb{S}_2 = \text{span}\left\{ \begin{bmatrix} 3 \\ x \\ y \end{bmatrix}, \begin{bmatrix} z \\ -9 \\ 5 \end{bmatrix} \right\}, \quad \mathbf{A} = \begin{bmatrix} 3 & z \\ x & -9 \\ y & 5 \end{bmatrix}$$

$$\hat{\vec{v}}_{\mathbb{S}_1} := \text{proj}_{\mathbb{S}_1}(\vec{v}) \qquad \hat{\vec{v}}_{\mathbb{S}_2} := \text{proj}_{\mathbb{S}_2}(\vec{v}) \qquad \hat{\vec{v}}_{\mathbf{A}} := \mathbf{A}(\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\vec{v}$$

The error vectors for each of these approximations are

$$\vec{e}_{\mathbb{S}_1} := \vec{v} - \hat{\vec{v}}_{\mathbb{S}_1} \qquad \vec{e}_{\mathbb{S}_2} := \vec{v} - \hat{\vec{v}}_{\mathbb{S}_2} \qquad \vec{e}_{\mathbf{A}} := \vec{v} - \hat{\vec{v}}_{\mathbf{A}}$$

Rank the *norms* of the error vectors, from smallest to largest. Assume that x, y, z are values such that $\vec{v} \notin \mathbb{S}_2$.

(c) (3 points) Let α and β be scalars, and let $\vec{p}$ be a vector such that $\text{proj}_{\vec{q}}(\vec{p}) = \vec{w}$. Find an expression for

7. Essay Plagiarism Catcher (12 points)

Your lab TA, Rohan, has decided to quit engineering and teach an English course instead, and he assigns an essay for the first homework. He decides to put his EECS 16A knowledge to use by building a plagiarism detection system using linear algebra.

(a) (3 points) Rohan would like to find out if any of the students copied part of an example essay that he wrote. He uses an online tool "Essay2Vec" to convert the example essay to a vector:

$$\vec{v}_{\text{example}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

He also uses "Essay2Vec" to convert the students' essays into vectors, $\vec{v}_{\text{student}}$.
Rohan determines that if $\vec{v}_{\text{student}}$ makes an angle of less than 30° with $\vec{v}_{\text{example}}$, then he should check the essay for plagiarism by hand. By this criteria, which of the following submitted essays should Rohan check? Select all that apply.
Hint: $\cos 0^\circ = 1$, $\cos 90^\circ = 0$, $\cos 30^\circ = \frac{\sqrt{3}}{2} = 0.866$.

(1) $\vec{v}_{\text{student}} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

(2) $\vec{v}_{\text{student}} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$

(3) None of these $\vec{v}_{\text{student}}$ essays need to be checked.

(4) $\vec{v}_{\text{student}} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

(b) (3 points) Rohan next calculates the difference, $\vec{d}$, between the student essays and his example essay:

$$\vec{d} = \vec{v}_{\text{student}} - \vec{v}_{\text{example}}$$

Rohan expects $\vec{d}$ to have a small magnitude for a plagiarized essay. Let

$$\vec{v}_{\text{example}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{v}_{\text{student}} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

If θ is the angle between $\vec{v}_{\text{student}}$ and $\vec{v}_{\text{example}}$, find the relationship between $\|\vec{d}\|^2$ and θ.

(A) $\|\vec{d}\|^2 = \sqrt{3} + \sqrt{6} - 2\sqrt{18}\cos\theta$

(B) $\|\vec{d}\|^2 = 9 - 2\sqrt{18}\cos\theta$

(C) $\|\vec{d}\|^2 = \sqrt{3} + \sqrt{6} - \sqrt{18}\cos\theta$

(D) $\|\vec{d}\|^2 = 9 + \sqrt{18}\cos\theta$

(E) $\|\vec{d}\|^2 = 9 - \sqrt{18}\cos\theta$

(F) $\|\vec{d}\|^2 = 9 + 2\sqrt{18}\cos\theta$

(c) (3 points) Rohan considers the set $\mathbb{S}$ of **all possible essay vectors** $\vec{v}_{\text{student}}$ with $\|\vec{d}\| < \sqrt{10}$. In other words,

$$\mathbb{S} = \left\{\vec{v}_{\text{student}}\right\}, \quad \text{such that} \quad \|\vec{v}_{\text{student}} - \vec{v}_{\text{example}}\| < \sqrt{10}$$

Is $\mathbb{S}$ a subspace of $\mathbb{R}^3$? Assume $\vec{v}_{\text{example}}$ is the same as the previous two parts.

(A) No

(B) There is not enough information to determine.

(C) Yes

(d) (3 points) Rohan knows a student included a paragraph from the textbook in their essay, but he doesn't know which page of the essay it's on.

He assigns the letters A-Z to integers 1-26 and creates a vector $\vec{s} \in \mathbb{R}^{10,000}$ representing the student essay and a vector $\vec{p} \in \mathbb{R}^{500}$ representing the paragraph from the textbook. He uses Python to calculate $\text{corr}_{\vec{s}}(\vec{p})$ and searches for a peak in the cross-correlation.

Will the index of the highest peak definitely correspond to the location of the textbook paragraph in the essay? Why or why not?

(A) Yes, because the highest peak is where the angle between $\vec{s}$ and $\vec{p}$ is maximized.

(B) No, because the method used above will generally give higher peaks where the student essay contains many letters towards the end of the alphabet.

(C) Yes, because Rohan knows that $\vec{p}$ is somewhere in $\vec{s}$, so he is guaranteed to find one peak corresponding to the correct location.

(D) No, because the greatest similarity occurs at the greatest magnitude of the dot product, which could be a negative value.

(E) Yes, because the highest peak indicates the maximum dot product, which means the vectors are the most similar at that index.

(F) No, because the method used above will generally give higher peaks where the student essay contains many letters towards the beginning of the alphabet.

8. Determinants (15 points)

For any square matrix $\mathbf{A}$, we can compute $\det(\mathbf{A} - \lambda\mathbf{I})$. Parts (a) - (e) below each give a different expression for $\det(\mathbf{A} - \lambda\mathbf{I})$, and your job is to determine which of seven possible statements must be true, given this information. Select all that apply:

(a) (2 points) $\det(\mathbf{A} - \lambda\mathbf{I}) = (3-\lambda)^2$

(1) $\mathbf{A}$ is a 2×2 matrix
(2) $\mathbf{A}$ is a 3×3 matrix
(3) $\mathbf{A}$ has a nontrivial nullspace
(4) $\mathbf{A}$ is invertible
(5) The dimension of at least one of the eigenspaces cannot be determined from the given information
(6) $\mathbf{A}$ has complex eigenvalues (with nonzero imaginary component)
(7) $\mathbf{A}$ is the zero matrix

(b) (2 points) $\det(\mathbf{A} - \lambda\mathbf{I}) = -\lambda(5-\lambda)^2$

(1) $\mathbf{A}$ is a 2×2 matrix
(2) $\mathbf{A}$ is a 3×3 matrix
(3) $\mathbf{A}$ has a nontrivial nullspace
(4) $\mathbf{A}$ is invertible
(5) The dimension of at least one of the eigenspaces cannot be determined from the given information
(6) $\mathbf{A}$ has complex eigenvalues (with nonzero imaginary component)
(7) $\mathbf{A}$ is the zero matrix

(c) (2 points) $\det(\mathbf{A} - \lambda\mathbf{I}) = (14-\lambda)(3-\lambda)(9-\lambda)$

(1) $\mathbf{A}$ is a 2×2 matrix
(2) $\mathbf{A}$ is a 3×3 matrix
(3) $\mathbf{A}$ has a nontrivial nullspace
(4) $\mathbf{A}$ is invertible
(5) The dimension of at least one of the eigenspaces cannot be determined from the given information
(6) $\mathbf{A}$ has complex eigenvalues (with nonzero imaginary component)
(7) $\mathbf{A}$ is the zero matrix

(d) (2 points) $\det(\mathbf{A} - \lambda\mathbf{I}) = -\lambda(\lambda^2 + \lambda + 9)$

(1) **A** is a 2×2 matrix
(2) **A** is a 3×3 matrix
(3) **A** has a nontrivial nullspace
(4) **A** is invertible
(5) The dimension of at least one of the eigenspaces cannot be determined from the given information
(6) **A** has complex eigenvalues (with nonzero imaginary component)
(7) **A** is the zero matrix

(e) (2 points) $\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2$

(1) **A** is a 2×2 matrix
(2) **A** is a 3×3 matrix
(3) **A** has a nontrivial nullspace
(4) **A** is invertible
(5) The dimension of at least one of the eigenspaces cannot be determined from the given information
(6) **A** has complex eigenvalues (with nonzero imaginary component)
(7) **A** is the zero matrix

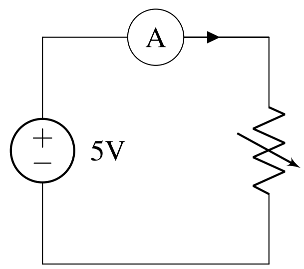
(f) (5 points) Let **A** be a 2×2 matrix. Prove that if the columns of **A** are linearly dependent, then det(**A**) = 0. Start with the following:

$$\mathbf{A} = \begin{bmatrix} | & | \\ \vec{x} & \vec{y} \\ | & | \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

9. Straining to Understand (18 points)

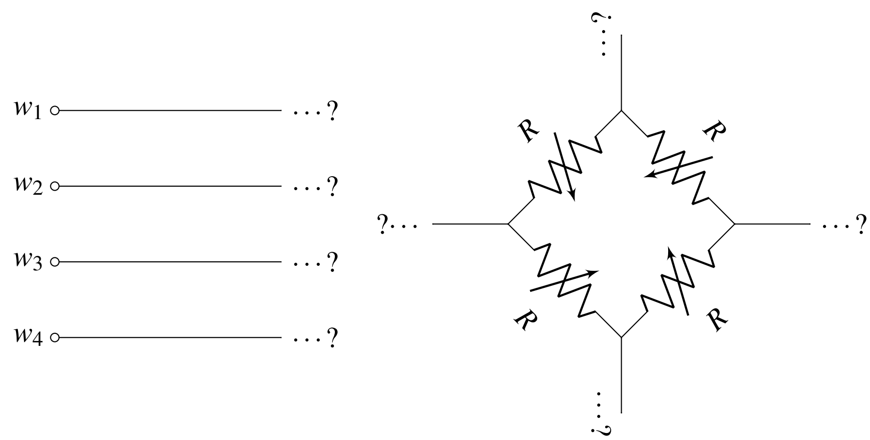
Your friend in mechanical engineering wants your help fixing their device which measures mechanical forces and material strength. (The device is called a *strain gauge sensor*.)

(a) (4 points) Your friend has made a test setup to measure how the resistance of their device changes as a function of force. The device datasheet specifies two parameters, R_0, the resistance without force, measured in Ω's, and ΔR, the change in resistance per pound of force, measured in $\frac{\Omega}{lb}$'s. The test setup appears as follows:



When no force is applied, your ammeter reads 10 *mA*. When 5 *lbs* are applied, your ammeter reads 8 *mA*. What are the parameters, R_0 and ΔR of the strain gauge?

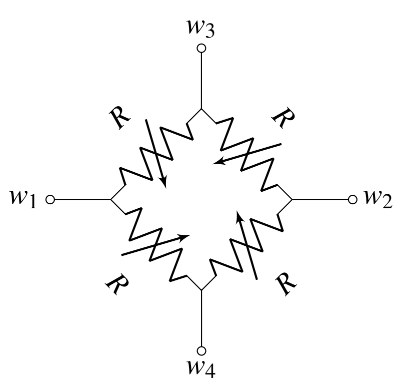
(b) (3 points) Your friend is glad to see how precise the resistors are. They assemble a full test setup with four of these devices. While testing this new circuit, they manage to destroy the connector going to the device. Thankfully, the wires labeled w_1, w_2, w_3, and w_4 are still intact, but they no longer know which wire goes to which node in the circuit, depicted in the figure below. All resistors have a value of $R = 680\Omega$.



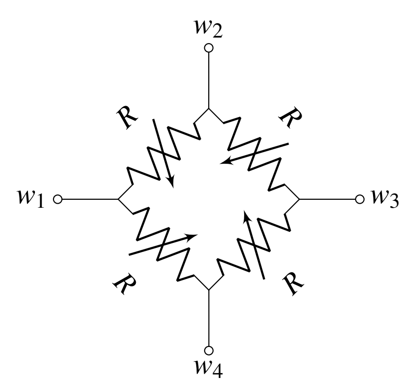
In an attempt to figure out which terminal might be which, using your multimeter you measure the equivalent resistance seen between pairs of wires and get the following reading:

$R_{w_1-w_2}$	$R_{w_1-w_3}$	$R_{w_1-w_4}$	$R_{w_2-w_3}$	$R_{w_2-w_4}$	$R_{w_3-w_4}$
680.0 Ω	510.0 Ω	510.0 Ω	510.0 Ω	510.0 Ω	680.0 Ω

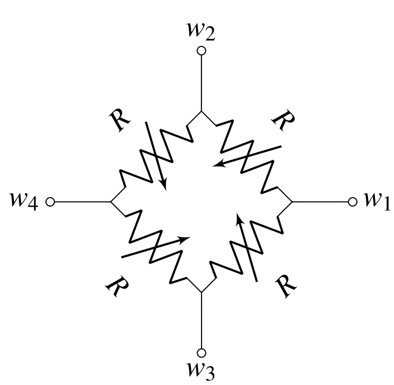
Select the circuit that represents the connections:



(A)



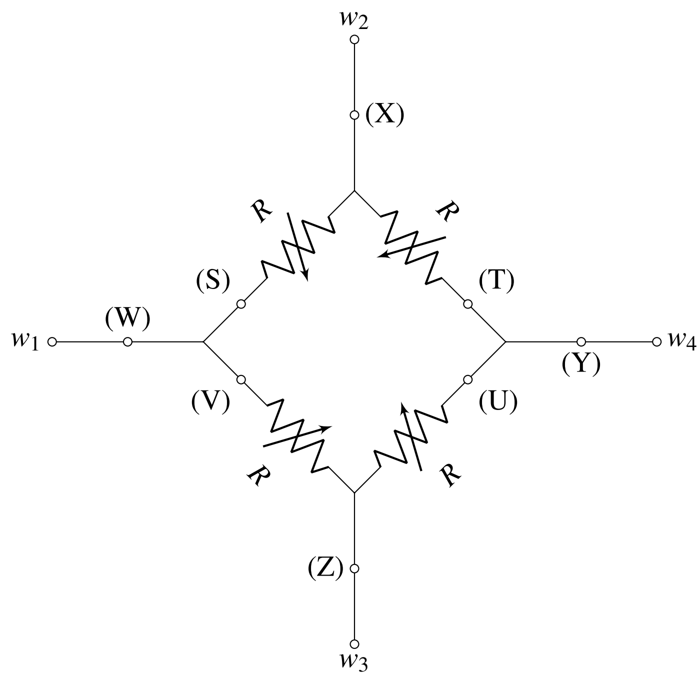
(B)



(C)

(D) None of these options are correct.

(c) (3 points) Congrats! You've identified the terminals and informed your friend how to figure out the configuration if the issue happens again. A week later, they come back to you with the following circuit, which may or may not be the correct answer to part (b). After a round of destructive testing, they found a single open in the circuit somewhere below (at one of these points: (S), (T), (U), (V), (W), (X), (Y), (Z)), but they're not quite sure where. All resistors have a value of $R = 680\Omega$, as before.



They want your help determining the location of the open circuit. You make the following equivalent resistance measurements. Your multimeter reads O.L. when it detects an open circuit.

$R_{w_1-w_2}(\Omega)$	$R_{w_1-w_3}(\Omega)$	$R_{w_1-w_4}(\Omega)$	$R_{w_2-w_3}(\Omega)$	$R_{w_2-w_4}(\Omega)$	$R_{w_3-w_4}(\Omega)$
O.L.	O.L.	O.L.	680.0	510.0	510.0

Where is the open circuit, i.e. where is the break located? Select the node with the break.

(d) (8 points) Now assume that the resistors used depend on the applied pressure through the following affine function:

$$R = R_0 + \alpha P,$$

where R is the total resistance, P is the applied pressure you are trying to detect, R_0 is the resistance when no pressure is applied, and α is a real valued coefficient.

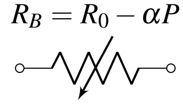
Your friend has **2** resistors with positive α's and **2** resistors with negative α's, denoted R_A and R_B respectively. Both resistor pairs have the same R_0 but **equal and opposite** α's. Using the configuration below, how should they connect the four resistors if they want to get an output voltage, V_{out}, that will be a linear (not affine) function of pressure?

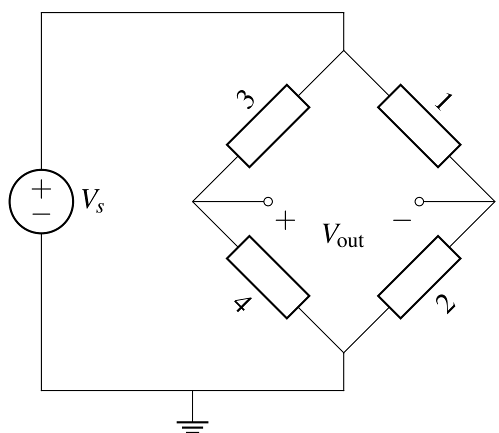
Redraw the circuit, replacing boxes 1 - 4 with the resistors R_A and R_B, in the proper locations in order to achieve this goal. **Calculate the absolute value** of the output voltage of the circuit you have come up with to show that it is indeed a linear function of pressure.

Note: there is more than one correct solution to this problem.

$R_A = R_0 + \alpha P$

$R_B = R_0 - \alpha P$





10. One Ambitious Crossover (10 points) Pumped up by a summer full of 16A knowledge, you open up a freelancer business called Odd Jobs, along with your trusty sidekicks Specs and Tiny. Your very first client wants you to fix their antique AM radio!

(a) (3 points) You need a circuit producing an output voltage V_{out}, which is a quadratic function of the input voltage V_{in}. Specs, the circuitry wiz in your team, brings in a non-linear device, as shown in Figure 1. The current through this device I_D depends on the voltage V_D across its terminals in the following manner:

$$I_D = c_1 V_D^2 + c_2 V_D + c_3$$

Here c_1, c_2 and c_3 are scalars.

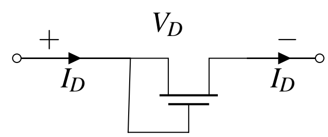


Figure 1: Non-linear device showing I_D and V_D

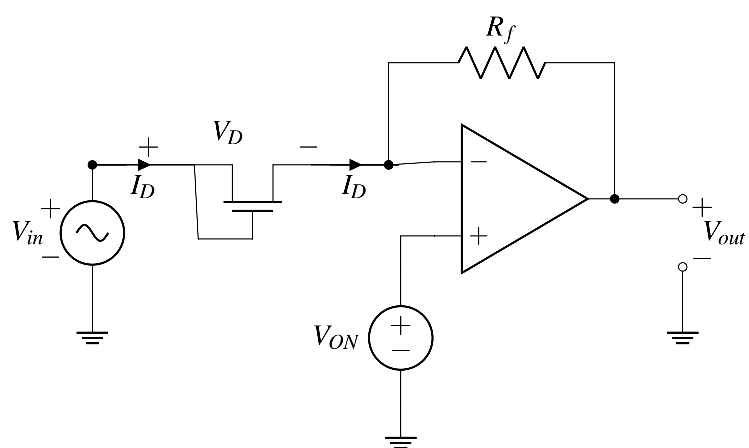


Figure 2: Circuit with an op-amp and the non-linear device, showing V_{in} and V_{out}

Specs constructs the circuit shown in Figure 2. V_{ON} is -0.5V for this part. Find the expression for V_{out} for this circuit.

(b) (4 points) For this part, let us assume that we have a (possibly different) circuit where $V_{out} = uV_{in}^2 + vV_{in} + w$ and u, v and w are scalars that are unknown. Tiny measures the output voltage V_{out} for a few values of V_{in}.

V_{in}	0V	1V	2V	3V
V_{out}	$v_{out,0}$	$v_{out,1}$	$v_{out,2}$	$v_{out,3}$

However, it turns out Tiny introduced some noise in the V_{out} readings during measurement. The readings $v_{out,0}$, $v_{out,1}$, $v_{out,2}$ and $v_{out,3}$ are scalars. You utilize the values of V_{in} and V_{out} to write a system of linear equations, which you need to solve for u, v and w:

$$Q \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \vec{e} = \begin{bmatrix} v_{out,0} \\ v_{out,1} \\ v_{out,2} \\ v_{out,3} \end{bmatrix}$$

Here $\vec{e}$ is the error introduced due to noise. You use the least squares method to reach the following solution:

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -1.0 \\ 0.4 \\ -1.1 \end{bmatrix}$$

Find the projection of $\begin{bmatrix} v_{out,0} \\ v_{out,1} \\ v_{out,2} \\ v_{out,3} \end{bmatrix}$ on the columnspace of matrix Q. (Hint: *Start by finding matrix Q!*)

(c) (3 points) Now let us assume that the answer for part (b) is given by: $\begin{bmatrix} \alpha \\ \beta \\ \delta \\ \gamma \end{bmatrix}$, i.e. the projection of $\begin{bmatrix} v_{out,0} \\ v_{out,1} \\ v_{out,2} \\ v_{out,3} \end{bmatrix}$ on the columnspace of matrix Q is $\begin{bmatrix} \alpha \\ \beta \\ \delta \\ \gamma \end{bmatrix}$. Now Tiny writes a system of linear equations as follows:

$$Q \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ \delta \\ \gamma \end{bmatrix}.$$

If you use Gaussian elimination on this system, what will be the result?

(A) There is no solution for $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$

(B) Infinite solutions exist for $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$

(C) There is not enough information

(D) Unique solution exists for $\begin{bmatrix} u \\ v \\ w \end{bmatrix}$

11. Denoising with Orthogonal Matching Pursuit (21 points)

Suppose we have a noisy measurement $\vec{y}_{\text{noisy}}$ such that

$$\vec{y}_{\text{noisy}} = \vec{y}_{\text{clean}} + \vec{\eta},$$

where $\vec{y}_{\text{clean}}$ is the measurement with no noise and $\vec{\eta}$ is an unknown vector of noise. We'd like to remove the noise from $\vec{y}_{\text{noisy}}$, but this is in general a hard problem because we don't know $\vec{y}_{\text{clean}}$ or $\vec{\eta}$ so there are infinite possible pairs that satisfy the above equation.

Suppose we additionally know that

$$\vec{y}_{\text{clean}} = \mathbf{A}\vec{x}$$

where $\vec{x}$ is a sparse vector and $\mathbf{A}$ is an $m \times 4$ matrix with columns $\vec{a}_1 \ldots \vec{a}_4$. You go to your favorite TA Moses and he suggests you can use OMP to determine $\vec{x}$ and find $\vec{y}_{\text{clean}}$!

He gives you the following table of inner products:

$< \cdot, \cdot >$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	$\vec{\eta}$
$\vec{a}_1$	1	-0.2	0.1	0	-0.2
$\vec{a}_2$		1	0.2	0.3	0
$\vec{a}_3$			1	0.3	-0.2
$\vec{a}_4$				1	γ

For example, the inner product $< \vec{a}_1, \vec{a}_2 >= -0.2$ and $< \vec{a}_1, \vec{a}_3 >= 0.1$.

(a) (3 points) Select all of the following that are guaranteed to be true based on the table above.

(1) If $\gamma = 1$, then $\vec{\eta} = \vec{a}_4$
(2) $||\vec{a}_3|| = ||\vec{a}_2||$
(3) $\text{proj}_{\vec{a}_2}(\vec{\eta}) = \vec{0}$
(4) $||\vec{\eta}|| = 1$
(5) $\vec{a}_1$ and $\vec{a}_2$ are orthogonal

(b) (6 points) Moses reminds you that the first step of OMP is to calculate the inner product of your measurement vector with each column of **A**.
Let $\vec{x} = \begin{bmatrix} 0 & 0 & 8 & 0 \end{bmatrix}^T$. Calculate $< \vec{a}_1, \vec{y}_{\text{noisy}} >$. Show your work. (*Hint: Find an expression for the unknown inner product in terms of $\vec{x}$ and the inner products in the table.*)

(c) (2 points) Moses helps you calculate the remaining inner products:

$$< \vec{a}_2, \vec{y}_{\text{noisy}} >= 1.6 \qquad < \vec{a}_3, \vec{y}_{\text{noisy}} >= 7.8 \qquad < \vec{a}_4, \vec{y}_{\text{noisy}} >= \gamma + 2.4$$

Find the range of values of γ for which the first iteration of OMP will select 3 as the index of the nonzero entry of $\vec{x}$. (*Note: the first entry in $\vec{x}$ has index "1" and the last entry has index "4."*)

(d) (3 points) Now assume we have a new noisy measurement with different $\vec{y}_{\text{clean}}$, $\vec{y}_{\text{noisy}}$, and $\vec{x}$ than the prior parts. The matrix **A** is unchanged and the table is still valid. We repeat the table here for your convenience:

$< \cdot, \cdot >$	$\vec{a}_1$	$\vec{a}_2$	$\vec{a}_3$	$\vec{a}_4$	$\vec{\eta}$
$\vec{a}_1$	1	-0.2	0.1	0	-0.2
$\vec{a}_2$		1	0.2	0.3	0
$\vec{a}_3$			1	0.3	-0.2
$\vec{a}_4$				1	γ

Using the new noisy measurement, you run two iterations of OMP and get the following calculations:

- Iteration 1: You find index $i = 2$ maximizes $| < \vec{e}, \vec{a}_i > |$ and you calculate $< \vec{a}_2, \vec{y}_{\text{noisy}} >= 6$.
- Iteration 2: You find index $i = 1$ maximizes $| < \vec{e}, \vec{a}_i > |$ and you calculate $< \vec{a}_1, \vec{y}_{\text{noisy}} >= -9$.

What is $\hat{\vec{x}}$, the OMP estimate of $\vec{x}$, after these two iterations? Let $\hat{\vec{x}} = \begin{bmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 & \hat{x}_4 \end{bmatrix}^T$.

(e) (3 points) Let $\hat{\vec{y}} = \mathbf{A}\hat{\vec{x}}$ be the best estimate of $\vec{y}_{\text{noisy}}$ after the two iterations of OMP described in part (d). (Note, $\hat{\vec{y}}$ is also your estimate of $\vec{y}_{\text{clean}}$.) You want to run a third iteration of OMP, so you calculate the error, $\vec{e} = \vec{y}_{\text{noisy}} - \hat{\vec{y}}$.
What is the inner product of $\vec{e}$ with each of the four columns, $\vec{a}_1 \ldots \vec{a}_4$?

(A) $<\vec{a}_i, \vec{e}> = 0$ for $i = 1,2$. There isn't enough information to calculate $<\vec{a}_i, \vec{e}>$ for $i = 3,4$.
(B) There isn't enough information to calculate $<\vec{a}_i, \vec{e}>$ for any i.
(C) $<\vec{a}_i, \vec{e}> = 0$ for all i.
(D) $<\vec{a}_1, \vec{e}> = -0.2, \quad <\vec{a}_2, \vec{e}> = 0, \quad <\vec{a}_3, \vec{e}> = -0.2, \quad <\vec{a}_4, \vec{e}> = \gamma$
(E) $<\vec{a}_i, \vec{e}> = 0$ for $i = 3,4$. There isn't enough information to calculate $<\vec{a}_i, \vec{e}>$ for $i = 1,2$.

(f) (4 points) Now, let $\mathbf{B} \in \mathbb{R}^{9\times 10}$. We use OMP to solve

$$\vec{v} = \mathbf{B}\vec{u}.$$

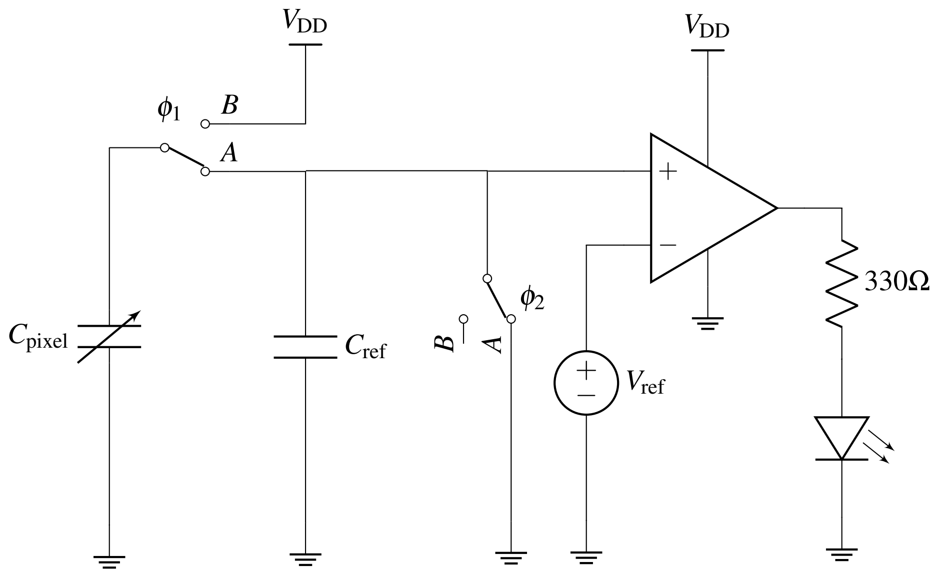
After 8 iterations of OMP, we get $\hat{\vec{u}}$ as our best estimate of $\vec{u}$, and we find that the error $\vec{v} - \mathbf{B}\hat{\vec{u}} = \vec{0}$. Which of the following statements must be true? Choose all that apply. Assume $\vec{v} \neq \vec{0}$.

(1) There is at least one column of $\mathbf{B}$ that is orthogonal to $\vec{v}$.
(2) $\text{rank}(\mathbf{B}) \leq 8$
(3) There is at least one column of $\mathbf{B}$ that is not orthogonal to $\vec{v}$.
(4) $\vec{v} \in C(\mathbf{B})$, where $C(\mathbf{B})$ is the columnspace of $\mathbf{B}$.
(5) $\text{rank}(\mathbf{B}) = 8$
(6) $\text{rank}(\mathbf{B}) \geq 8$

12. Capacitive Touchscreen 2: Electric Boogaloo (15 points)

As an expert on the capacitive touchscreen, you have been recruited to redesign the circuit from lab to be used in a professional setting.

(a) (3 points) At first, you revisit the capacitive touchscreen circuit below, in which C_{pixel} varies depending on whether or not a touch has occurred, and the switches ϕ_1, ϕ_2 are sequentially configured to create three different states: the **clean** state, during which C_{ref} is reset to 0, the **charge** state, during which C_{pixel} is charged up, and the **read** state, during which the charge is being redistributed (shared), and read out. These states are cyclically repeated at a certain rate, providing a constantly updated reading of whether or not a touch has been detected.



$$V_{\text{DD}} = 3\text{V}, C_{\text{pixel}} = \begin{cases} 44\text{pF} & \text{no touch} \\ 88\text{pF} & \text{touch} \end{cases}, C_{\text{ref}} = 44\text{pF}, t_{\text{switch}} = 1\text{ms}$$

Which of the following demonstrates a correct switching order? (i.e. a switching order that can lead to detecting a touch by properly cycling through the clean, charge, and read states).
Each step will be notated as follows: (ϕ_1 = [A or B], ϕ_2 = [A or B]) where A and B correspond to the labeled switch connections. Only one switch can be modified per step. Select all that apply.

(1) $(\phi_1 = A, \phi_2 = A)$, $(\phi_1 = A, \phi_2 = B)$, $(\phi_1 = B, \phi_2 = B)$, $(\phi_1 = A, \phi_2 = B)$

(2) $(\phi_1 = A, \phi_2 = A)$, $(\phi_1 = A, \phi_2 = B)$, $(\phi_1 = A, \phi_2 = A)$, $(\phi_1 = A, \phi_2 = A)$

(3) $(\phi_1 = A, \phi_2 = A)$, $(\phi_1 = A, \phi_2 = B)$, $(\phi_1 = B, \phi_2 = B)$, $(\phi_1 = B, \phi_2 = A)$

(4) $(\phi_1 = A, \phi_2 = A)$, $(\phi_1 = B, \phi_2 = A)$, $(\phi_1 = B, \phi_2 = B)$, $(\phi_1 = A, \phi_2 = B)$

(5) $(\phi_1 = A, \phi_2 = A)$, $(\phi_1 = B, \phi_2 = A)$, $(\phi_1 = A, \phi_2 = A)$, $(\phi_1 = \text{A}, \phi_2 = B)$

(b) (3 points) What is the amount of charge, $Q_{\text{no-touch}}$, that V_{DD} has to provide in every clean-charge-read cycle of operation, assuming there is **no touch**? Is the amount of charge, Q_{touch}, any different when **there is a touch**?

Hint: Do not add up the amount of charge in each step, think about conservation of charge and what that means in the context of calculating total charge in a cycle.

Note: Do not consider the charge required for the comparator to work in your calculations.

(A) $Q_{no-touch} = 132\text{pC}$, No, $Q_{touch} = Q_{no-touch}$

(B) $Q_{no-touch} = 264\text{pC}$, No, $Q_{touch} = Q_{no-touch}$

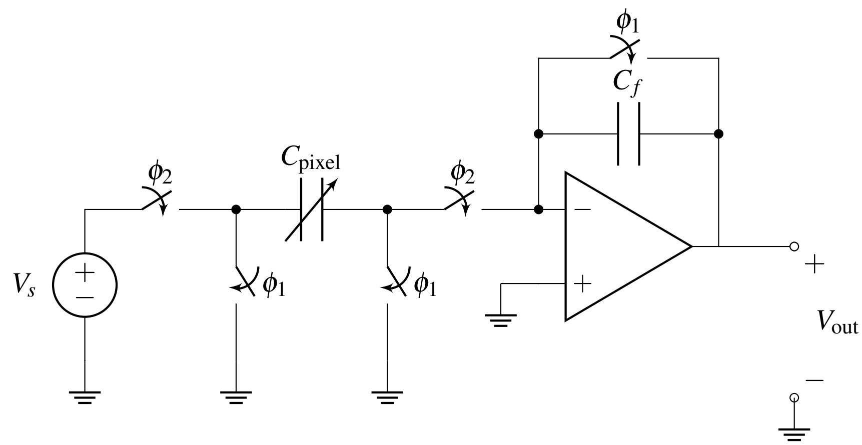
(C) $Q_{no-touch} = 264\text{pC}$, Yes: $Q_{touch} = \frac{Q_{no-touch}}{2}$

(D) $Q_{no-touch} = 66\text{pC}$, Yes: $Q_{touch} = 2Q_{no-touch}$

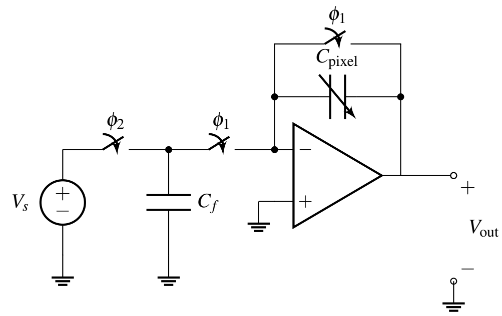
(E) $Q_{no-touch} = 66\text{pC}$, Yes: $Q_{touch} = \frac{Q_{no-touch}}{2}$

(F) $Q_{no-touch} = 132\text{pC}$, Yes: $Q_{touch} = 2Q_{no-touch}$

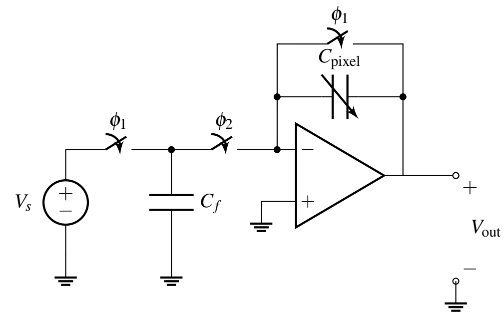
(c) (6 points) After analyzing the circuit used in your lab you are given a new design to look at. This design has the property that the output is not a binary function of touch vs no-touch, but a linear function of the variable capacitance C_{pixel}. It operates in 2 phases: phase 1, where switches ϕ_1 are closed and switches ϕ_2 are open, and phase 2, where switches ϕ_2 are closed and switches ϕ_1 are open. Calculate V_{out} during **phase 2**, as a function of the voltage source, V_s, and the capacitors, C_{pixel}, C_f. **Show your work.**



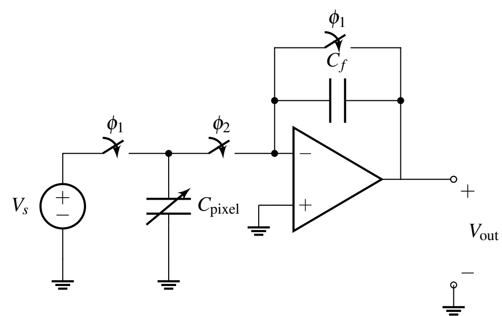
(d) (3 points) Your mentor points out that the circuit from part (c) can be further simplified and two of the switches can be removed while maintaining the same functionality. Which of the following circuits achieves this goal?



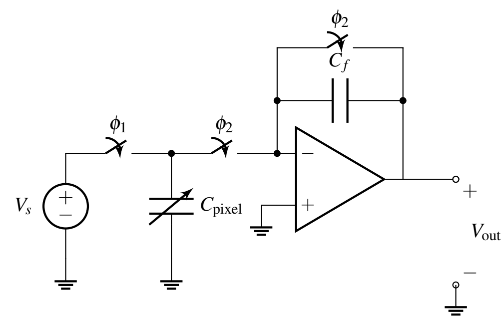
(A)



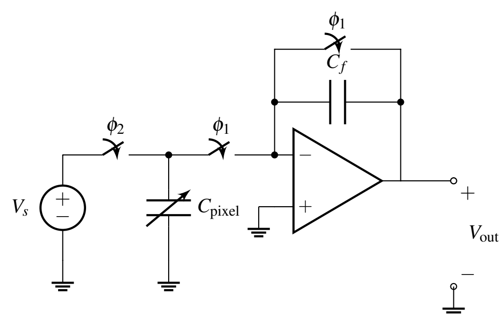
(B)



(C)



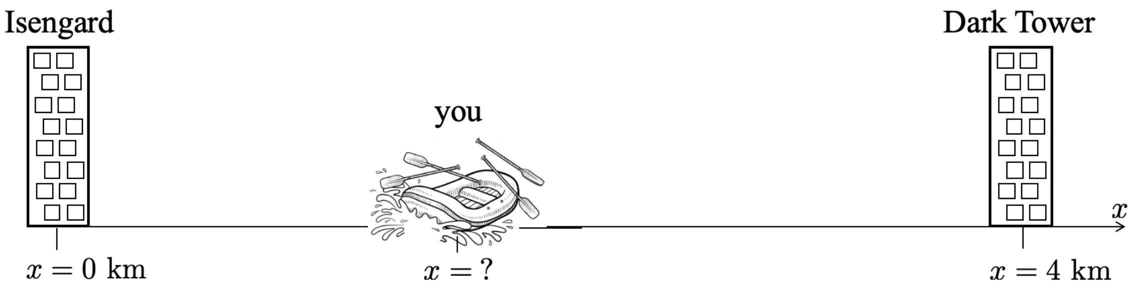
(D)



(E)

13. One does not simply raft into Mordor (23 points)

You've decided to go rafting to celebrate the end of your summer classes! Unfortunately, an hour into your trip you realize that there are no landmarks that look familiar, so you're not sure how far you are from your starting point. You remember from your studies of the area that there are two towers, Isengard at the position $x = 0$ km and the Dark Tower at $x = 4$ km. You know you are between the two towers, as shown below:



You know that each tower emits a sound signal once a day at midday. Specifically, Isengard will emit $\vec{b}_1$ and the Dark Tower will emit $\vec{b}_2$:

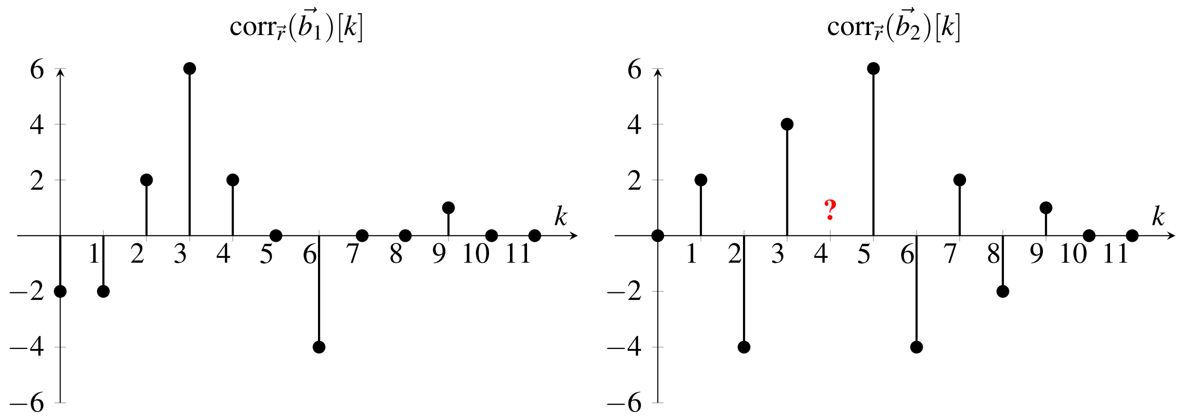
$$\vec{b}_1 = \begin{bmatrix} -1 & -1 & -1 & 1 & 1 \end{bmatrix}^T \qquad \vec{b}_2 = \begin{bmatrix} -1 & 1 & -1 & 1 & -1 \end{bmatrix}^T$$

Both signals are emitted at a rate of 2 samples per seconds (i.e. sample interval is 0.5 sec), and the signals are emitted only once.

It's only a few minutes from midday so you decide to wait. You use an app on your phone to record the incoming signal (the app also records at 2 samples per second). You start recording at exactly 12:00 PM and receive the following:

$$\vec{r} = \begin{bmatrix} 0 & 0 & 0 & -1 & -1 & -2 & 2 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}^T$$

(a) (2 points) Your first step is to calculate a linear cross-correlation between $\vec{r}$ and each known tower signature. The cross-correlations are plotted below. Calculate the missing value, which is denoted with a question mark.



(b) (3 points) Recall that the signals were emitted from each building at the same time (12:00 PM). How many seconds after 12:00 PM did it take for the signal from Isengard to reach you? What about the signal from the Dark Tower? Assume that environmental noise (besides the tower-emitted signals) is minimal.

(c) (3 points) Now, assume that you received Isengard's signal 8 seconds after it was sent and you received the Dark Tower's signal 2 seconds after it was sent. Can you determine your exact position x? If yes, calculate your position. If not, explain why not. Assume sound travels at 340 m/s.

(d) (3 points) You see a giant eagle, so you get out of your raft to follow it. But you soon realize that you don't know your x position *or* your y position! Luckily, you have a phone app which tells you that are:

- d_1 km away from Isengard which is located at $x = 0$ km, $y = 0$ km
- d_2 km away from the Dark Tower which is located at $x = 4$ km, $y = 0$ km
- d_3 km away from Minas Tirith which is located at $x = 1$ km, $y = 3$ km

Write a system of linear equations of the form $\mathbf{A}\vec{x} = \vec{b}$ that you can solve to find your position.

Let $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ where x, y have units of kilometers (km).

Unfortunately, your app hasn't been updated in a while, and there is a bug in the app. Rather than telling you the true $\vec{b}$, the app gives you $\vec{b}_0 = \vec{b} + \vec{e}$ where $\vec{e}$ is the error from the bug. Nevertheless, you attempt to find your position by solving

$$\mathbf{A}\vec{x} = \vec{b}_0,$$

where $\vec{x} = \begin{bmatrix} x & y \end{bmatrix}^T$ as before. Suppose the matrix $\mathbf{A}$ has two unique non-zero eigenvalues λ_1, λ_2 with associated eigenvectors, $\vec{v}_1$, $\vec{v}_2$, where both eigenvectors are normalized. You can write the error vector, $\vec{e}$, as a linear combination of the eigenvectors of $\mathbf{A}$ as follows:

$$\vec{e} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2$$

(e) (8 points) Show that

$$\Delta = \left\| \frac{\alpha_1}{\lambda_1} \vec{v}_1 + \frac{\alpha_2}{\lambda_2} \vec{v}_2 \right\|,$$

where Δ is the distance between your calculated position ($\vec{x}$ found using $\vec{b}_0$) and your true position ($\vec{x}$ found using $\vec{b}$). Clearly justify each step.

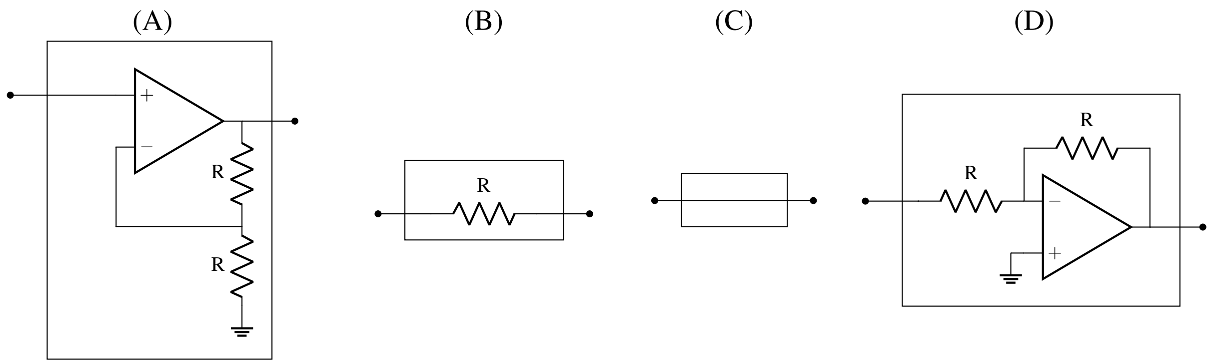
(f) (4 points) Let $\lambda_1 = 2$, $\lambda_2 = 1$, $\alpha_1 = 4$, and $\alpha_2 = -1$. Find an upper bound on the distance Δ. In other words, find a constant, c, such that Δ is always less than c, regardless of the values of $\vec{v}_1$, $\vec{v}_2$. Choose the smallest value of c such that the expression $\Delta \leq c$ is always true. (Your answer should not depend on $\vec{v}_1$, $\vec{v}_2$.)

14. Op Amp Blocks (12 points)

For each subpart below you will be given a formula for $V_{out} = \alpha V_1 + \beta V_2$, where α and β will be real numbers, and V_{out}, V_1, V_2 are denoted in the Outline Schematic. Place the given circuit blocks in the outline schematic to produce the desired V_{out}.

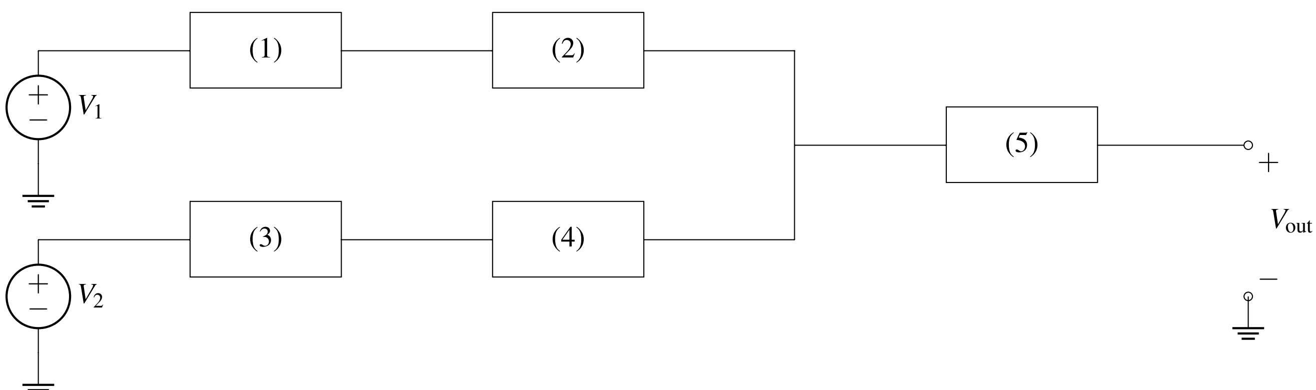
Note: There is more than one solutions to each subpart; you only need to find one of them per subpart.

Circuit Blocks:

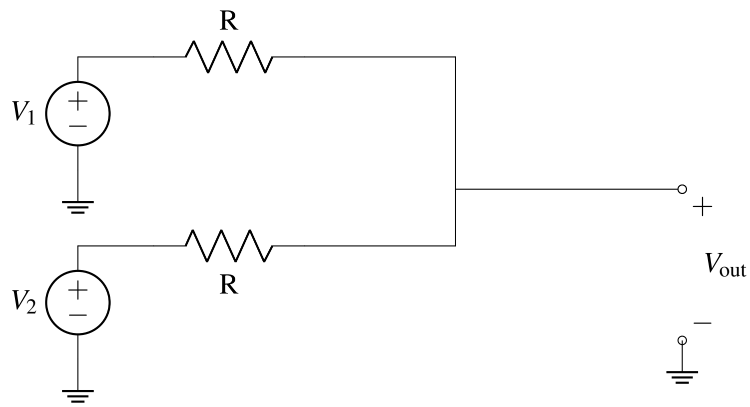


Important Design Constraints: All resistors have the same value, R. You can use the resistor and wire blocks more than once for each operation, but each op amp configuration can be used at most once for each operation.

Outline Schematic:



For example: $V_{out} = \frac{1}{2}V_1 + \frac{1}{2}V_2$



In this case, two correct answers would be BCBCC and CBCBC (there are more, but we only want you to find one).

Fill in the outline schematic for the following 3 expressions:

(a) (4 points) $V_{out} = V_1 + V_2$

(b) (4 points) $V_{out} = V_1 - V_2$

(c) (4 points) $V_{out} = \frac{2}{3}V_1 + \frac{4}{3}V_2$

15. Orthogonality (26 points)

(a) (4 points) Let $\mathbf{W} = \vec{x}\vec{y}^T$ where $\vec{x} \in \mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^m$ are orthogonal to one another.

i. What is the dimension of $C(\mathbf{W})$, the columnspace of $\mathbf{W}$?

(A) 0
(B) 1
(C) n
(D) $n-1$
(E) m
(F) $m-1$
(G) Not enough information

ii. What is the dimension of $N(\mathbf{W})$, the nullspace of $\mathbf{W}$?

(A) 0
(B) 1
(C) n
(D) $n-1$
(E) m
(F) $m-1$
(G) Not enough information

iii. Is $\mathbf{W}$ invertible?

(A) Yes
(B) No

iv. The *trace* of a square matrix is the sum of the entires along the diagonal. (For example, $\mathbf{S} = \begin{bmatrix} s_{11} & s_{12} & \dots \\ s_{21} & s_{22} & \dots \\ \dots & \dots & s_{nn} \end{bmatrix}$ has a trace $= s_{11} + s_{22} + \ldots + s_{nn}$.) If $m = n$, what is the trace of $\mathbf{W}$?

(A) $||\vec{x}||$
(B) $||\vec{y}||$
(C) $||\vec{x}|| \cdot ||\vec{y}||$
(D) 0
(E) Not enough information

(b) (4 points) Recall that an orthonormal matrix is a matrix that satisfies the following two properties:

- All columns have norm of 1

- All columns are orthogonal to one another

Let $\vec{e}_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $\vec{e}_2 = \begin{bmatrix}0\\1\end{bmatrix}$, $\vec{v} = \begin{bmatrix}1\\-1\end{bmatrix}$, $\vec{u} = \begin{bmatrix}-3\\4\end{bmatrix}$.
Which of the following matrices are orthonormal? Choose all that apply.

(1) $\begin{bmatrix} | & | \\ \text{proj}_{\vec{e}_1}(\vec{v}) & \text{proj}_{\vec{e}_2}(\vec{v}) \\ | & | \end{bmatrix}$ (2) $\begin{bmatrix} | & | \\ \text{proj}_{\vec{e}_1}(\vec{u}) & \text{proj}_{\vec{e}_2}(\vec{u}) \\ | & | \end{bmatrix}$ (3) $\begin{bmatrix} | & | \\ \frac{1}{5}\vec{u} & \frac{1}{\sqrt{2}}\vec{v} \\ | & | \end{bmatrix}$

(4) $\begin{bmatrix} | & | \\ \text{proj}_{\vec{v}}(\vec{e}_1) & \text{proj}_{\vec{v}}(\vec{e}_2) \\ | & | \end{bmatrix}$ (5) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ (6) $\begin{bmatrix} | & | \\ \frac{1}{\sqrt{2}}\vec{v} & \vec{e}_2 \\ | & | \end{bmatrix}$

(c) (6 points) Prove that multiplying by an orthonormal matrix does not change the inner product:

$$< \mathbf{M}\vec{u}, \mathbf{M}\vec{v} > = < \vec{u}, \vec{v} >,$$

where $\mathbf{M} \in \mathbb{R}^{n\times n}$ is an orthonormal matrix with columns $\vec{m}_1 \dots \vec{m}_n$. Finish the proof by filling in the numbered boxes with one of the corresponding multiple-choice options. Some boxes are intentionally left blank.

$$\begin{aligned} < \mathbf{M}\vec{u}, \mathbf{M}\vec{v} > &= \boxed{}(\mathbf{M}\vec{v}) \\ &= \boxed{(1)}\mathbf{M}\vec{v} \qquad (*) \end{aligned}$$

Let $(\mathbf{M}^T\mathbf{M})_{ij}$ be the element in the i^{th} row and j^{th} column of $\mathbf{M}^T\mathbf{M}$. $(\mathbf{M}^T\mathbf{M})_{ij}$ is equal to the inner product of the i^{th} row of $\mathbf{M}^T$ and the j^{th} column of $\boxed{(2)}$. By the definition of $\boxed{\quad(3)\quad}$, the i^{th} row of $\mathbf{M}^T$ is equal to $\boxed{(4)}$. Therefore:

$$\begin{aligned} (\mathbf{M}^T\mathbf{M})_{ij} &= \boxed{\quad(5)\quad} \\ \text{If } i = j, \text{ then } (\mathbf{M}^T\mathbf{M})_{ij} &= \boxed{(6)} \text{ because } \boxed{\qquad(7)\qquad} \\ \text{If } i \neq j, \text{ then } (\mathbf{M}^T\mathbf{M})_{ij} &= \boxed{(8)} \text{ because } \boxed{\qquad(9)\qquad} \end{aligned}$$

Therefore, $\mathbf{M}^T\mathbf{M} = \boxed{}$. Going back to Eq. (*) and plugging this in yields:

$$\begin{aligned} < \mathbf{M}\vec{u}, \mathbf{M}\vec{v} > &= \vec{u}^T I\vec{v} \\ &= \vec{u}^T\vec{v} \\ &= < \vec{u}, \vec{v} > \end{aligned}$$

(1)
- (A) $\mathbf{M}^T\vec{u}^T$
- (B) $\vec{u}^T\mathbf{M}^T$
- (C) $\mathbf{M}^T$
- (D) $\vec{u}\mathbf{M}^T$

(2)
- (A) $\vec{u}^T\mathbf{M}^T$
- (B) $\mathbf{M}\vec{v}$
- (C) $\mathbf{M}^T$
- (D) $\mathbf{M}$

(3)
- (A) inner product
- (B) matrix multiplication
- (C) transpose
- (D) othogonality

(4)
- (A) $\vec{m}_i$
- (B) $\vec{v}$
- (C) $\vec{u}$
- (D) $\vec{m}_j$

(5)
- (A) $< \mathbf{M}\vec{u}, \mathbf{M}\vec{v} >$
- (B) $< \vec{m}_i, \vec{m}_j >$
- (C) $< \vec{m}_i^T\vec{u}, \vec{m}_j^T\vec{v} >$
- (D) $< \vec{u}, \vec{v} >$

(6)
- (A) 0
- (B) i
- (C) j
- (D) 1

(7)
- (A) $\mathbf{M}\mathbf{M}^T = I$
- (B) $||\vec{m}_k||^2 = 1$ for $1 \leq k \leq n$
- (C) the columns of $\mathbf{M}$ are orthogonal
- (D) $\mathbf{M}$ is a square matrix

(8)
- (A) i
- (B) 0
- (C) 1
- (D) j

(9)
- (A) $||\vec{m}_k||^2 = 1$ for $1 \leq k \leq n$
- (B) $\mathbf{M}\mathbf{M}^T = I$
- (C) $\mathbf{M}$ is a square matrix
- (D) the columns of $\mathbf{M}$ are orthogonal

(d) (4 points) Let $\mathbf{V}$ and $\mathbf{U}$ be orthonormal matrices in $\mathbb{R}^{n\times n}$ with columns $\vec{v}_1 \dots \vec{v}_n$ and $\vec{u}_1 \dots \vec{u}_n$, respectively. Let $\mathbf{D}$ be a diagonal matrix in $\mathbb{R}^{n\times n}$ with non-zero scalars $d_1 \dots d_n$ along the diagonal and zeros everywhere else, i.e. $\mathbf{D} = \begin{bmatrix} d_1 & 0 & ... \\ 0 & d_2 & ... \\ ... & ... & ... \end{bmatrix}$. If

$$\mathbf{W} = \mathbf{U}\mathbf{D}\mathbf{V}^T,$$

which of the following must be true? Select all that apply. Assume $1 \leq i \leq n$.

(1) $\mathbf{D} = \mathbf{V}\mathbf{W}\mathbf{U}^T$

(2) $\vec{u}_i^T\mathbf{W} = d_i\vec{v}_i^T$

(3) $\mathbf{W}\vec{u}_i = d_i\vec{v}_i$

(4) $\mathbf{W}\vec{v}_i = d_i\vec{u}_i$

(5) $\mathbf{W}^{-1} = \mathbf{V}\mathbf{D}^{-1}\mathbf{U}^T$

.

(e) (8 points) Let $\mathbf{W}$ be a square, symmetric $n \times n$ matrix ($\mathbf{W} = \mathbf{W}^T$). Prove that the columnspace of $\mathbf{W}$ is orthogonal to the nullspace of $\mathbf{W}$. In other words, prove that any vector, $\vec{v}$, in the columnspace of $\mathbf{W}$ is orthogonal to any vector, $\vec{u}$, in the nullspace of $\mathbf{W}$. Justify every step of your proof.

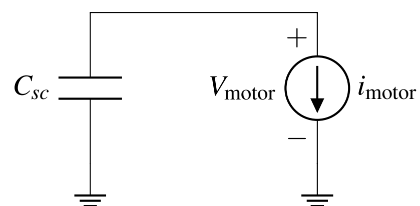
16. Feedback Formula (33 points)

Formula One (F1) is widely considered as the pinnacle of motorsport. Recently its competitiveness has come into question, since the same franchises have dominated the sport over the past decades.

(a) (3 points) In order to increase overtaking and thus competition, F1 is introducing electric motors that provide an instant boost to drivers if they are within a short distance of the driver ahead of them.

This instant boost will be coming from none other than a super-capacitor. During boost operation, the motor requires a constant current, $i_{\text{motor}} = 4\text{A}$. It also has a nominal voltage of $V_{\text{motor, nom}} = 10\text{V}$ but can operate with voltages as low as $V_{\text{motor, min}} = 8\text{V}$. When the voltage drops below $V_{\text{motor, min}}$, all extra acceleration is lost and the car resumes at its previous speed.

In order for the car to have a good chance to successfully overtake, the boost needs to be maintained for at least $t_{\text{overtake}} = 5\text{s}$. What is the minimum value of the super-capacitor C_{sc} that allows the boost to be maintained for at least t_{overtake}?



(b) (2 points) Given that super-capacitors can be discharged pretty quickly you come up with a backup acceleration plan: you add one more motor to your system. Unlike the previous motor that was modeled as a constant current source, this motor can be thought of as a load resistor, whose rotational speed increases as the voltage across it increases.

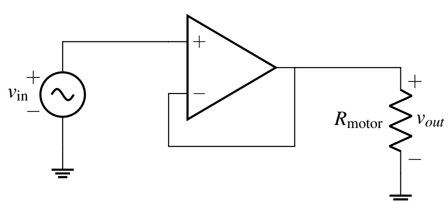
You want to design a system that **increases the voltage provided to the motor as the distance between a given driver and the driver leading them decreases**, making it maximum when the distance between them goes to zero ($V_{R_{\text{motor}},\,\text{max}} = 10\text{V}$). At the same time, you don't want to be spending extra energy if you are far away from the leading driver, so the acceleration motor should shut down (i.e. $V_{R_{\text{motor}},\,\text{min}} = 0\text{V}$) when the distance is larger than 15m.

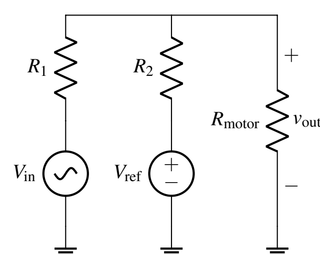
You are given the following components:

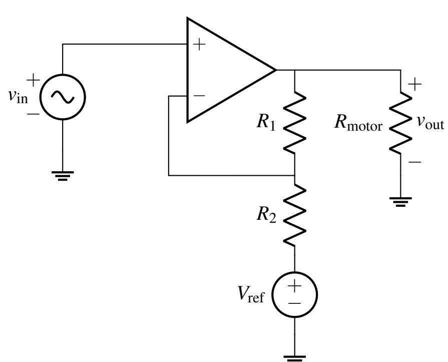
i. one Op Amp (assume that it comes with V_{DD} and V_{SS} that are sufficiently large so you don't need to worry about those)
ii. one motor that operates in the 0-10V range - modeled as R_{motor}
iii. one LIDAR sensor that returns a voltage in the 0-10V range directly proportional to distance. Specifically, when the distance is 0m the output of the sensor is 0V, and when the distance is larger that 15m the output of the sensor is "maxed-out" at 10V. The voltage grows linearly in all intermediate distances. This sensor is modeled as a variable voltage source v_{in}.
iv. one constant voltage source (V_{ref}) set at a voltage of your choice
v. 2 resistors (R_1 and R_2) of your choice

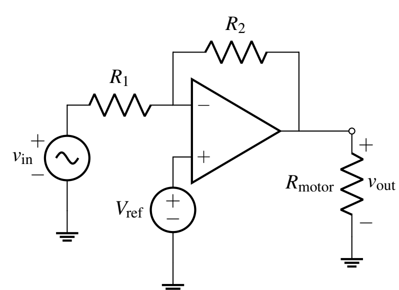
You may or may not need all of them.

Which of the following topologies should you pick to build this system?

(A)


(B)


(C)


(D)


(c) (8 points) **Redraw** your choice from part (b) and **explain** how your design satisfies the specifications in part (b). **Find numerical values** for R_1, R_2, and V_{ref} for the topology you picked in the previous part (or for the subset of these components in the topology you picked).

(d) (10 points) Safety regulations in F1 impose an upper speed limit, so you need to make sure the acceleration boost is cut-off before you reach that limit. However, your car only has an **accelerometer**. Fill in the boxes in the outline schematic to **build a circuit that can measure speed, (i.e. produce an output voltage proportional to speed)**, using the following components:

i. **one** Op Amp (assume that it comes with V_{DD} and V_{SS} that are sufficiently large so you don't need to worry about those)

ii. an accelerometer that returns a voltage (0-5V) directly proportional to acceleration - modeled as a variable voltage source v_{in}

iii. **one** resistor R of your choice

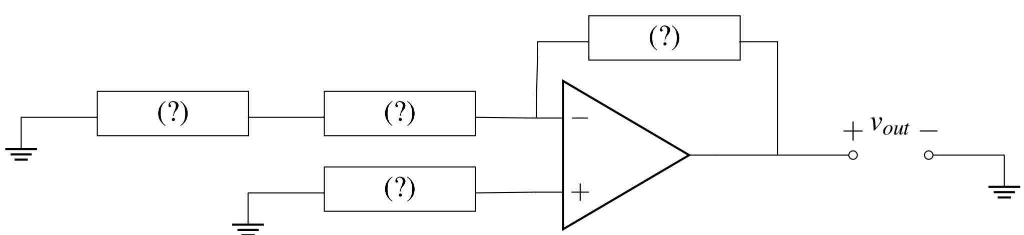
iv. **one** capacitor C of your choice

v. unlimited wires

Draw the design you end up with after filling the boxes and **explain** how it works. **Express** the output voltage v_{out} as a function of component values, R, C, and the input voltage v_{in}.
Hint: Remember, speed can be found as the integral of acceleration.
Note: The sign of the output does not matter.

The circuit blocks you have available are repeated below for your convenience.
Circuit Blocks:

Outline Schematic:



(e) (10 points) Alas! Some feisty wheel-to-wheel battles have broken your accelerometer. You are now stuck with only an **odometer** (a device that measures the distance you travel over time and converts it to a voltage). Use the exact same components to **build a backup speedometer**.

i. **one** Op Amp (assume that it comes with V_{DD} and V_{SS} that are sufficiently large so you don't need to worry about those)
ii. an odometer that returns a voltage (0-5V) directly proportional to distance traveled - modeled as a variable voltage source v_{in}
iii. **one** resistor R of your choice
iv. **one** capacitor C of your choice
v. unlimited wires

Draw the design you end up with after filling the boxes and **explain how it works**. **Express** the output voltage v_{out} as a function of component values, R, C, and the input voltage v_{in}.
Hint: Remember, speed can be found as the derivative of position. Which component that you have seen in class can be used to take a derivative of voltage and how can you use it in this design?
Note: The sign of the output, again does not matter.

The circuit blocks you have available are repeated below for your convenience.
Circuit Blocks:

Outline Schematic: